ON WEAKLY RIGID RINGS

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Abstract. Let $R$ be a ring with a monomorphism $\alpha$ and an $\alpha$-derivation $\delta$. We introduce $(\alpha, \delta)$-weakly rigid rings which are a generalisation of $\alpha$-rigid rings and investigate their properties. Every prime ring $R$ is $(\alpha, \delta)$-weakly rigid for any automorphism $\alpha$ and $\alpha$-derivation $\delta$. It is proved that for any $n$, a ring $R$ is $(\alpha, \delta)$-weakly rigid if and only if the $n$-by-$n$ upper triangular matrix ring $T_n(R)$ is $(\overline{\alpha}, \overline{\delta})$-weakly rigid if and only if $M_n(R)$ is $(\overline{\alpha}, \overline{\delta})$-weakly rigid. Moreover, various classes of $(\alpha, \delta)$-weakly rigid rings are provided. Examples to illustrate and delimit the theory are provided.

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1. Introduction. Throughout this paper $R$ denotes an associative ring with unity; $\alpha$ is a monomorphism of $R$ which is not assumed to be surjective; and $\delta$ an $\alpha$-derivation of $R$, that is $\delta$ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for all $a, b \in R$.

According to Krempa [18], a monomorphism $\alpha$ of a ring $R$ is called to be rigid if $aa(a) = 0$ implies $a = 0$ for $a \in R$. A ring $R$ is said to be $\alpha$-rigid if there exists a rigid monomorphism $\alpha$ of $R$.

The second author and E. Hashemi in [12] defined a ring $R$ with a monomorphism $\alpha$ and an $\alpha$-derivation $\delta$, to be called $(\alpha, \delta)$-compatible if for each $a, b \in R$, $ab = 0$ implies $a\delta(b) = 0$, and $ab = 0$ if and only if $aa(b) = 0$.

We say a ring $R$ with a monomorphism $\alpha$ and $\alpha$-derivation $\delta$, to be called $(\alpha, \delta)$-weakly rigid if for each $a, b \in R$, $aRb = 0$ implies $a\delta(b) = 0$, and $aRb = 0$ if and only if $aa(Rb) = 0$.

By [12], a ring $R$ is $\alpha$-rigid if and only if it is $(\alpha, \delta)$-compatible and reduced. Notice that the class of $\alpha$-rigid rings and $(\alpha, \delta)$-compatible rings is a narrow class of rings, and it is easy to see that every $(\alpha, \delta)$-compatible ring is $(\alpha, \delta)$-weakly rigid; but there are various classes of $(\alpha, \delta)$-weakly rigid rings which are not $(\alpha, \delta)$-compatible, as we will see in Section 2.

It is clear that every prime ring $R$ is $(\alpha, \delta)$-weakly rigid for any automorphism $\alpha$ and $\alpha$-derivation $\delta$. In this paper we prove that for any positive integer $n$, a ring $R$ is $(\alpha, \delta)$-weakly rigid if and only if the $n$-by-$n$ upper triangular matrix ring $T_n(R)$ is $(\overline{\alpha}, \overline{\delta})$-weakly rigid if and only if the matrix ring $M_n(R)$ is $(\overline{\alpha}, \overline{\delta})$-weakly rigid.
We also show that if $R$ is a semiprime $(\alpha, \delta)$-weakly rigid ring, then the ring of polynomials $R[X]$, for $X$ an arbitrary non-empty set of indeterminates, is a semiprime $(\bar{\alpha}, \bar{\delta})$-weakly rigid ring. If $R$ is an $\alpha$-rigid ring, then $R[x]/(x^n)$ is an $(\bar{\alpha}, \bar{\delta})$-weakly rigid ring, for any $n \geq 2$, where $(x^n)$ is the ideal generated by $x^n$.

Suppose that $R$ is a ring with a monomorphism $\alpha$ and $\alpha$-derivation $\delta$. We show that when $R$ has a classical quotient ring $Q$ and $R$ is $(\alpha, \delta)$-weakly rigid, $Q$ is also $(\bar{\alpha}, \bar{\delta})$-weakly rigid.

Recall from [9, 17] that a ring $R$ is called (quasi-) Baer if the right annihilator of every (ideal) non-empty subset of $R$ is generated, as a right ideal, by an idempotent of $R$. Recall from [6] that a ring is called left (resp. right) principally quasi-Baer (or simply left (resp. right) p.q.-Baer) if the left annihilator of a principal left (resp. right) ideal of $R$ is generated by an idempotent.

Armendariz [1] has shown that a reduced ring $R$ (i.e. having no non-zero nilpotent elements) is Baer if and only if $R[x]$ is Baer.

The Ore extensions of quasi-Baer and p.q.-Baer rings have been investigated by many authors [1, 5, 7, 11, 12, 14, 15, 21, 22, 24]. Most of these have worked either with the case $\delta = 0$ and $\alpha$ an automorphism or with the case in which $\alpha$ is the identity.

Birkenmeier et al. in [5, Theorem 1.2] show that if $R$ is a quasi-Baer ring and $\alpha$ is an automorphism of $R$, then $R[x; \alpha]$ is a quasi-Baer ring. They also provided an example of a quasi-Baer ring $R$ with an endomorphism $\alpha$ such that $R[x; \alpha]$ is not quasi-Baer. In [7], they also proved that a ring $R$ is right p.q.-Baer if and only if the polynomial ring $R[x]$ is right p.q.-Baer.

Hong et al. in [14] have shown that if $R$ is an $\alpha$-rigid ring, then $R$ is Baer if and only if $R[x; \alpha, \delta]$ is a Baer ring if and only if the skew power series ring $R[[x; \alpha]]$ is a Baer ring. By [5, Lemma 1.9], a reduced (and hence $\alpha$-rigid) ring is Baer if and only if it is quasi-Baer, and by [14] a ring $R$ is $\alpha$-rigid if and only if the Ore extension $R[x; \alpha, \delta]$ is reduced. The second author and E. Hashemi in [12], extended Hong et al.’s results of [14]. Note also that there is a commutative reduced p.q.-Baer ring which the power series ring is not a p.q.-Baer ring [20].

Although the class of $\alpha$-rigid (or $(\alpha, \delta)$-compatible) quasi-Baer rings is too narrow, we show that there are many rich classes of $(\alpha, \delta)$-weakly rigid quasi-Baer rings. For every prime ring $R$ and any automorphism $\alpha$ and $\alpha$-derivation $\delta$, the rings $M_n(R)$, $T_n(R)$, $R[X]$ and power series ring $R[[X]]$ are $(\bar{\alpha}, \bar{\delta})$-weakly rigid quasi-Baer rings.

For an $(\alpha, \delta)$-weakly rigid ring $R$, the relationship between $R$, the skew polynomial ring $R[x; \alpha, \delta]$, skew Laurent polynomial ring $R[x; x^{-1}; \alpha]$, skew power series ring $R[[x; \alpha]]$ and skew Laurent power series ring $R[[x; x^{-1}; \alpha]]$ is studied, and we show that strong connections exist between these rings and their various properties. Known results relating to $\alpha$-rigid rings can be obtained as corollaries of our results. Among applications, we show that a number of interesting properties of an $(\alpha, \delta)$-weakly rigid ring $R$ such as the quasi-Baer property and the principally quasi-Baer property transfer to its extensions and vice versa.

We provide examples which show that, in general, the quasi-Baerness (or p.q.-Baerness) of $R$ and the aforementioned extensions do not depend on each other. As a consequence we extend and unify several known results.

2. Weakly rigid rings. In this section the notion of $(\alpha, \delta)$-weakly rigid rings is introduced, and a number of properties of this generalisation are established. We give a good supply of examples of $(\alpha, \delta)$-weakly rigid rings.
For a non-empty subset $X$ of a ring $R$, $r_R(X) = \{ c \in R \mid Xc = 0 \}$ (respectively $\ell_R(X) = \{ c \in R \mid cX = 0 \}$) is called the right (respectively left) annihilator of $X$ in $R$.

**Definition 2.1.** A ring $R$ with a monomorphism $\alpha$, is called $\alpha$-weakly rigid if for each $a, b \in R$, $aRb = 0$ if and only if $aa(Rb) = 0$.

A ring $R$ with a derivation $\delta$ is called $\delta$-weakly rigid if for each $a, b \in R$, $aRb = 0$ implies $\delta(b) = 0$.

A ring $R$ with a monomorphism $\alpha$ and $\alpha$-derivation $\delta$, is called $(\alpha, \delta)$-weakly rigid if it is both $\alpha$-weakly rigid and $\delta$-weakly rigid.

Every $\alpha$-compatible ring is $\alpha$-weakly rigid, and $(\alpha, \delta)$-compatible rings are clearly $(\alpha, \delta)$-weakly rigid; but there are various classes of $(\alpha, \delta)$-weakly rigid rings which are not $(\alpha, \delta)$-compatible (and hence not $\alpha$-rigid), as we will see in this section.

Let $R$ be a ring, $\alpha$ an endomorphism and $\delta$ an $\alpha$-derivation of $R$. It is easy to see that for any subring $S$ of the full matrix ring $M_n(R)$, $\tilde{\alpha} : S \to S$, given by $\tilde{\alpha}((a_{ij})) = (\alpha(a_{ij}))$, is a homomorphism, and $\tilde{\delta} : S \to S$, given by $\tilde{\delta}((a_{ij})) = (\delta(a_{ij}))$, is an $\tilde{\alpha}$-derivation. We shall denote the $(i, j)$-th entry of a matrix $A \in M_n(R)$ by $A_{ij}$.

In the following example we see that for an $\alpha$-rigid ring $R$, $M_n(R)$ or $T_n(R)$ is not necessarily $\tilde{\alpha}$-compatible (and hence not $\tilde{\alpha}$-rigid).

**Example 2.2.** Let $D$ be a domain and $\alpha$ be the automorphism of the polynomial ring $R := D[x_1, x_2, \ldots, x_m]$, with indeterminates $x_1, x_2, \ldots, x_m$, given by $\alpha(x_i) = x_{i+1}$ for $1 \leq i \leq m - 1$ and $\alpha(x_m) = x_1$. Then $R$ is an $\alpha$-rigid ring. Take $a = E_{11}x_1 + E_{12}x_2$ and $b = E_{12}x_2 - E_{22}x_1$, where $E_{ij}$ denotes the matrix unit. We have $a, b \in T_n(R) \subseteq M_n(R)$. It is seen that $ab = 0$ but $\tilde{\alpha}a(b) \neq 0$. Hence neither $M_n(R)$ nor $T_n(R)$ is $(\tilde{\alpha}, \tilde{\delta})$-compatible.

Although the class of $\alpha$-rigid (or $(\alpha, \delta)$-compatible) rings do not pass to matrix rings by the above example, we show that the weak rigid property overcomes these shortfalls.

**Theorem 2.3.** Let $R$ be a ring and $\alpha$ an endomorphism of $R$. Then the following are equivalent:

(i) $R$ is an $\alpha$-weakly rigid ring;
(ii) $M_n(R)$ is an $\tilde{\alpha}$-weakly rigid ring for every positive integer $n$;
(iii) $M_n(R)$ is an $\tilde{\alpha}$-weakly rigid ring for some positive integer $n$.

**Proof.** (i) $\Rightarrow$ (ii). Suppose that $R$ is $\alpha$-weakly rigid and $AM_n(R)B = 0$, with $A = (a_{ij}), B = (b_{ij}) \in M_n(R)$. To prove that $A\tilde{\alpha}(M_n(R)B) = 0$, it is enough to show that, for each $r \in R$ and $1 \leq i, j, \leq n$, $A\tilde{\alpha}(rE_{ij}B) = 0$. To do this, we show that $(A\tilde{\alpha}(rE_{ij}B))_{tk} = 0$, for each $1 \leq t, k \leq n$, where $(A\tilde{\alpha}(rE_{ij}B))_{tk}$ is the $(t, k)$-th entry of the matrix $A\tilde{\alpha}(rE_{ij}B)$. Now we have $rE_{ij}B = rE_{ij}E_{11} + rE_{ij}E_{12} + \cdots + rE_{ij}E_{kn}$. So $\tilde{\alpha}(rE_{ij}B) = \alpha(rb_{ij}E_{11} + \alpha(rb_{ij})E_{12} + \cdots + \alpha(rb_{ij})E_{kn})$. Thus $(A\tilde{\alpha}(rE_{ij}B))_{tk} = (\alpha(a_{11} + a_{12}E_{12} + \cdots + a_{nk}E_{kn}))(rb_{ij}E_{1k})_{tk}$. Therefore it is enough to show that $a_{ij}\alpha(rb_{jk}) = 0$, for each $r \in R$ and $1 \leq i, j, k \leq n$. But we have $AM_n(R)B = 0$, so $(ArE_{ij})_{tk} = 0$, and hence $a_{ij}rE_{jk} = 0$ for each $r \in R$ and $1 \leq i, j, k \leq n$. So $a_{ij}rE_{jk} = 0$, and hence $a_{ij}\alpha(rb_{jk}) = 0$, since $R$ is $\alpha$-weakly rigid. Thus $a_{ij}\alpha(rb_{jk}) = 0$, for each $r \in R$ and $1 \leq i, j, k \leq n$, and hence $A\tilde{\alpha}(M_n(R)B) = 0$. Next assume that $A\tilde{\alpha}(M_n(R)B) = 0$, with $A = (a_{ij}), B = (b_{ij}) \in M_n(R)$. To prove that $AM_n(R)B = 0$, it is enough to show that $ArE_{ij}B = 0$, for each $r \in R$ and $1 \leq i, j, k \leq n$. To do this, we show that $(ArE_{ij}B)_{tk} = a_{ij}rE_{jk}E_{tk} = 0$, for each $1 \leq t, k \leq n$. Since $A\tilde{\alpha}(M_n(R)B) = 0$, we get $(A\tilde{\alpha}(rE_{ij}B))_{tk} = 0$, so $a_{ij}\alpha(rb_{jk}) = 0$, for each $1 \leq i, j, k \leq n$. Thus $a_{ij}\alpha(rb_{jk}) = 0$, and
hence \( a_{ij} R b_{jk} = 0 \) for each \( 1 \le i, j, t, k \le n \). Consequently \( AM_n(R)B = 0 \), and so \( M_n(R) \) is \( \overline{\alpha} \)-weakly rigid. (ii) \( \Rightarrow \) (iii). Is trivial.

(iii) \( \Rightarrow \) (i). Suppose that for some \( n \), \( M_n(R) \) is an \( \overline{\alpha} \)-weakly rigid ring and that \( aRb = 0 \) with \( a, b \in R \). It is easy to see that \( aE_{i1}M_n(R)bE_{11} = 0 \), and hence \( aE_{i1}E_{11} = 0 \), since \( M_n(R) \) is \( \overline{\alpha} \)-weakly rigid. So \( aE_{i1}E_{11} = 0 \), and hence \( aO(r)E_{11} = 0 \) for each \( r \in R \); consequently \( aO(Rb) = 0 \). Next assume that \( aO(Rb) = 0 \), so \( aE_{i1}E_{11} = 0 \). Thus \( aE_{i1}M_n(R)bE_{11} = 0 \), since \( M_n(R) \) is \( \overline{\alpha} \)-weakly rigid. Therefore for each \( r \in R \), \( aE_{i1}rE_{11}bE_{11} = 0 \) so \( aRb = 0 \), whence \( R \) is \( \overline{\alpha} \)-weakly rigid.

**Theorem 2.4.** Let \( R \) be a ring and \( \delta \) a derivation of \( R \). Then the following are equivalent:

(i) \( R \) is a \( \delta \)-weakly rigid ring;

(ii) \( M_n(R) \) is a \( \delta \)-weakly rigid ring for every positive integer \( n \);

(iii) \( M_n(R) \) is an \( \delta \)-weakly rigid ring for some positive integer \( n \).

**Proof.** (i) \( \Rightarrow \) (ii). Suppose that \( R \) is \( \delta \)-weakly rigid and \( AM_n(R)B = 0 \), with \( A = (a_{ij}), B = (b_{ij}) \in M_n(R) \). To prove that \( A\overline{\delta}(B) = 0 \), it is enough to show that, for each \( 1 \le i, j \le n \), \( (A\overline{\delta}(B))_{ij} = 0 \). Now for each \( 1 \le i, j \le n \), we have \( (A\overline{\delta}(B))_{ij} = (a_{i1}\overline{\delta}(b_{1j}) + a_{i2}\overline{\delta}(b_{2j}) + \cdots + a_{in}\overline{\delta}(b_{nj}))E_{ij} \). Since \( AM_n(R)B = 0 \), we get \( ARE_{ib} = 0 \) for each \( r \in R \) and \( 1 \le t \le n \). So \( (ARE_{ib})_{ij} = 0 \) for each \( 1 \le i, j \le n \). But \( (ARE_{ib})_{ij} = a_{i1}rE_{ij}b_{ij} \), so \( a_{i1}rE_{ij}b_{ij} = 0 \) for each \( r \in R \) and \( 1 \le i, j \le n \). Thus \( a_{i1}\overline{\delta}(b_{ij}) = 0 \), so \( (A\overline{\delta}(B))_{ij} = 0 \), and hence \( M_n(R) \) is \( \delta \)-weakly rigid.

(ii) \( \Rightarrow \) (iii). Is trivial.

(iii) \( \Rightarrow \) (i). Assume that \( M_n(R) \) is \( \delta \)-weakly rigid for some \( n \), and \( aRb = 0 \) with \( a, b \in R \). Then we have \( aE_{i1}M_n(R)bE_{11} = 0 \), and since \( M_n(R) \) is \( \delta \)-weakly rigid, \( aE_{i1}\overline{\delta}(bE_{11}) = 0 \). Thus \( a\overline{\delta}(b)E_{11} = 0 \), consequently \( R \) is \( \delta \)-weakly rigid.

**Corollary 2.5.** For any positive integer \( n \), a ring \( R \) is \( (\alpha, \delta) \)-weakly rigid if and only if \( M_n(R) \) is an \( (\overline{\alpha}, \overline{\delta}) \)-weakly rigid ring.

**Theorem 2.6.** Let \( R \) be a ring and \( \alpha \) an endomorphism of \( R \). Then the following are equivalent:

(i) \( R \) is an \( \alpha \)-weakly rigid ring;

(ii) \( T_n(R) \) is an \( \overline{\alpha} \)-weakly rigid ring for every positive integer \( n \);

(iii) \( T_n(R) \) is an \( \overline{\alpha} \)-weakly rigid ring for some positive integer \( n \).

**Proof.** The proof is similar to that of Theorem 2.3.

**Theorem 2.7.** Let \( R \) be a ring and \( \delta \) a derivation of \( R \). Then the following are equivalent:

(i) \( R \) is a \( \delta \)-weakly rigid ring;

(ii) \( T_n(R) \) is a \( \delta \)-weakly rigid ring for every positive integer \( n \);

(iii) \( T_n(R) \) is a \( \delta \)-weakly rigid ring for some positive integer \( n \).

**Proof.** The proof is similar to that of Theorem 2.4.

**Example 2.8.** Let \( D \) be a prime ring and \( \alpha \) be the automorphism of the polynomial ring \( R := D[x_1, x_2, \ldots, x_m] \), with indeterminates \( x_1, x_2, \ldots, x_m \), given by \( \alpha(x_i) = x_{i+1} \) for \( 1 \le i \le m \) and \( \alpha(x_m) = x_1 \). Then \( R \) is an \( \alpha \)-weakly rigid ring. As in Example 2.2, it is seen that neither \( M_n(R) \) nor \( T_n(R) \) is \( (\overline{\alpha}, \overline{\delta}) \)-compatible. However by Theorems 2.3 and 2.6, \( M_n(R) \) and \( T_n(R) \) are both \( \overline{\alpha} \)-weakly rigid rings.
Let $R$ be a ring. Define $R_n = RI_n + \sum_{i=1}^{n} \sum_{j=i+1}^{n} RE_{ij}$, for $n \geq 2$, where $E_{ij}$ is the matrix units for all $i, j$ and $I_n$ is the identity matrix. Note that $R_n$ is a subring of $T_n(R)$.

**Theorem 2.9.** If $R$ is an $\alpha$-rigid ring, then $R_n$ is an $(\bar{\alpha}, \bar{\delta})$-weakly rigid ring, for any $n$.

**Proof.** Assume $A, B \in R_n$ and $AR_n B = 0$. Since $R_n = RI_n + \sum_{i=1}^{n} \sum_{j=i+1}^{n} RE_{ij}$, in order to prove that $A\bar{\alpha}(R_n B) = 0$, it is enough to show that $A\bar{\alpha}(rI_n B) = A\bar{\alpha}(rE_{ij} B) = 0$, for each $r \in R$ and $1 \leq i \leq n$, $1 \leq j \leq n$. First suppose that $r \in R$ and $1 \leq i \leq n$, $1 \leq j \leq n$; then $(A\bar{\alpha}(rE_{ij} B))_{tk} = a_{ij} \alpha(rb_{jk}) E_{tk}$. Since $AR_n B = 0$, $(AR_e B)_{tk} = 0$, so $a_{ij}rb_{jk} = 0$, and hence $a_{ii}rb_{kk} = 0$. Since $R$ is $\alpha$-weakly rigid, $a_{ii} \alpha(rb_{kk}) = 0$, so $(A\bar{\alpha}(rE_{ij} B))_{tk} = 0$; hence $A\bar{\alpha}(rE_{ij} B) = 0$. Now, for each $1 \leq i \leq j \leq n$, $(A\bar{\alpha}(rI_n B))_{ij} = a_{ij} \alpha(r) \alpha(b_{ij}) E_{ij} + a_{i1+\bar{\alpha}}(r) \alpha(b_{i1+1,j}) E_{ij} + \cdots + a_{ji} \alpha(r) \alpha(b_{j1}) E_{ij}$. Since $AR_n B = 0$, we get $A\alpha(rI_n B) = 0$. Hence $(A\bar{\alpha}(I_n B))_{ij} = a_{ji} \alpha(r) b_{ij} E_{ij} + a_{i1+\bar{\alpha}}(r) b_{i1+1,j} E_{ij} + \cdots + a_{ji} \alpha(r) b_{j1} E_{ij} = 0$. (*)&n

Since $(AE_{i,j} B)_{ij} = 0$, $(AE_{i,1} B)_{ij} = 0$, and since $(AE_{i+1,j} B)_{ij} = 0$, $a_{ij} b_{jj} = 0$. By this way, after $j - i$ steps, we get $a_{ij} b_{jj} = 0$, since $(AE_{i,j} B)_{ij} = 0$. Now, for each $1 \leq i \leq j \leq n$, $(A\bar{\alpha}(rI_n B))_{ij} = a_{ij} \alpha(r) \alpha(b_{ij}) E_{ij} + a_{i1+\bar{\alpha}}(r) \alpha(b_{i1+1,j}) E_{ij} + \cdots + a_{ji} \alpha(r) \alpha(b_{j1}) E_{ij}$. Since $AR_n B = 0$, $A(rI_n B) = 0$, so $(A\bar{\alpha}(I_n B))_{ij} = 0$, and hence $(A\bar{\alpha}(rE_{ij} B))_{ij} = 0$, which implies $a_{ij} \alpha(r) E_{ij} = 0$, and hence $a_{ij} \alpha(rb_{ij}) = 0$, since $R$ is $\alpha$-rigid, $a_{ij} \alpha(rb_{ij}) = 0$. We have $(AE_{i,j} B)_{ij} = 0$, so $a_{ij} b_{jj} = 0$. Also $(A\bar{\alpha}(I_n B))_{ij} = 0$, so $(AE_{i+1,j} B)_{ij} = 0$, and hence $(A\bar{\alpha}(rE_{ij} B))_{ij} = 0$, which implies $a_{ij} \alpha(rb_{ij}) = 0$, and hence $a_{ij} \alpha(rb_{ij}) = 0$, since $R$ is $\alpha$-rigid. By this way, after $j - i$ steps, we get $(A\bar{\alpha}(I_n B))_{ij} = 0$, so $A\bar{\alpha}(rI_n B) = 0$, hence $A\bar{\alpha}(rE_{ij} B) = 0$. Now suppose that $A\bar{\alpha}(R_n B) = 0$. By a similar method as employed in the above argument we can show that $AR_n B = 0$, and hence $R_n$ is a $\bar{\alpha}$-weakly rigid. Next assume that $AR_n B = 0$, we then show that $A\bar{\delta}(B) = 0$. For each $1 \leq i \leq j \leq n$, $(A\bar{\delta}(B))_{ij} = a_{ij} \delta(b_{ij}) E_{ij} + a_{i1+\bar{\alpha}}(b_{i1+1,j}) E_{ij} + \cdots + a_{ji} \delta(b_{j1}) E_{ij}$. Since $AR_n B = 0$, $A(rI_n B) = 0$, so $(A\bar{\alpha}(I_n B))_{ij} = 0$, and hence $a_{ij} \alpha(rb_{ij}) E_{ij} + a_{i1+\bar{\alpha}}(b_{i1+1,j}) E_{ij} + \cdots + a_{ji} \alpha(rb_{ij}) E_{ij} = 0$. As we have seen in the first part of the proof, $a_{ij} b_{jj} = 0$, $a_{i1+\bar{\alpha}}(b_{i1+1,j}) = 0$, $\cdots$, $a_{ji} \alpha(rb_{ij}) = 0$. Since $R$ is $\alpha$-rigid, $a_{ij} \delta(b_{ij}) = 0$, $a_{i1+\bar{\alpha}}(b_{i1+1,j}) = 0$, $\cdots$, $a_{ji} \delta(b_{ij}) = 0$. This implies that $(A\bar{\delta}(B))_{ij} = 0$ for each $i, j$, and the result follows. $\square$

In [19], T. K. Lee and Y. Zhou defined $V_n = \sum_{i=1}^{n-1} E_{i,i+1}$, for $n \geq 2$, where $E_{ij}$ is the matrix units for all $i, j$.

For even integers $n = 2k \geq 2$, $A_n(R) := \sum_{i=1}^{k} \sum_{j=k+i}^{n} RE_{ij}$, so we have $A_n(R) = RI_n + RV_n + RV_n^2 + \cdots + RV_n^{k-1} + A_n(R)$.

For odd integers $n = 2k + 1 \geq 3$, $A_n(R) := \sum_{i=1}^{k+1} \sum_{j=k+i}^{n} RE_{ij}$, so we have $A_n(R) = RI_n + RV_n + RV_n^2 + \cdots + RV_n^{k-1} + A_n(R)$.

Note that $A_n(R)$ is a subring of $T_n(R)$.

**Theorem 2.10.** If $R$ is an $\alpha$-rigid ring, then $A_n(R)$ is an $(\bar{\alpha}, \bar{\delta})$-weakly rigid ring, for any $n \geq 2$.

**Proof.** The proof is similar to that of Theorem 2.9. $\square$

The trivial extension of $R$, which is denoted by $T(R, R) = \{([a, b] | a, b \in R)\}$, is a ring with matrix addition and multiplication. By Theorem 2.9, we see that if $R$ is $\alpha$-rigid, then $T(R, R)$ is $(\bar{\alpha}, \bar{\delta})$-weakly rigid.
Let $R$ be a ring, and let
\[
T(R, n) := \left\{ \begin{pmatrix}
    a_1 & a_2 & a_3 & \cdots & a_n \\
    0 & a_1 & a_2 & \cdots & a_{n-1} \\
    0 & 0 & a_1 & \cdots & a_{n-2} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & a_1 \\
\end{pmatrix} \mid a_i \in R \right\},
\]
with $n \geq 2$. Then $T(R, n)$ is a subring of the triangular matrix ring $T_n(R)$.

Observe that $T(R, n) \cong R[x]/(x^n)$, for any $n \geq 2$. A proof similar to that of Theorem 2.9, can be employed to prove that when $R$ is an $\alpha$-rigid ring, $T(R, n)$, and hence $R[x]/(x^n)$ is an $(\bar{\alpha}, \bar{\delta})$-weakly rigid ring, for any $n \geq 2$.

Let $R$ be a ring, $\alpha$ an endomorphism and $\delta$ an $\alpha$-derivation of $R$. Then $\bar{\alpha} : R[x] \to R[x]$, given by $\bar{\alpha}(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} \alpha(a_i) x^i$, and $\bar{\delta} : R[x] \to R[x]$, given by $\bar{\delta}(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} \delta(a_i) x^i$. Then $\bar{\alpha}$ is an endomorphism and $\bar{\delta}$ is an $\bar{\alpha}$-derivation of $R[x]$.

According to Hirano [13], a ring $R$ is called quasi-Armendariz if for polynomials $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, $g(x) = b_0 + b_1 x + \cdots + b_m x^m \in R[x]$, $f(x)R[x]g(x) = 0$ if and only if $a_i R b_j = 0$ for each $i, j$. In [13], Hirano studied some properties of quasi-Armendariz rings and proved that the condition quasi-Armendariz is a Morita invariant property and that every semiprime ring is quasi-Armendariz.

In [13, Theorem 3.16] Hirano shows that quasi-Armendariz conditions preserves by polynomial rings. Now we get the following:

**Theorem 2.11.** If $R$ is a quasi-Armendariz $(\alpha, \delta)$-weakly rigid ring, then $R[x]$ is a quasi-Armendariz $(\bar{\alpha}, \bar{\delta})$-weakly rigid ring.

Since semiprime rings are quasi-Armendariz by [13], we get the following:

**Corollary 2.12.** If $R$ is a semiprime $(\alpha, \delta)$-weakly rigid ring, then $R[x]$ is a semiprime $(\bar{\alpha}, \bar{\delta})$-weakly rigid ring.

Recall that an idempotent $e \in R$ is left (respectively right) semi-central in $R$ if $Re = eRe$ (respectively $eR = eRe$).

**Theorem 2.13.** If $R$ is an $(\alpha, \delta)$-weakly rigid ring and $e$ is a left semi-central idempotent of $R$, then $eR$, $Re$ and $eRe$ are also $(\alpha, \delta)$-weakly rigid rings.

**Proof.** Suppose that $er, es \in eR$ and $ereRes = 0$. Since $Re = eRe$, we have $erRes = 0$, so $era(Res) = 0$, $er\delta(es) = 0$, and hence $era(eRes) = 0$, as $R$ is $(\alpha, \delta)$-weakly rigid. Now, if $era(eRes) = 0$, then $era(Res) = 0$, and hence $erRes = 0$. Thus $ereRes = 0$, and the result follows. \(\square\)

Notice that in Theorem 2.13, in order to restrict $\alpha, \delta$ from $R$ to $eR$, we need to assume that $\alpha(e), \delta(e) \in eR$, and this condition is satisfied for semi-central idempotents, since $R$ is $(\alpha, \delta)$-weakly rigid.

We now show that there exists an example of a ring $R$ with an idempotent $e \in R$ and an endomorphism $\alpha : R \to R$ such that $\alpha(e) \notin eR$, so the condition semi-central in Theorem 2.13 is not superfluous.
Example 2.14. Let $\mathcal{Q}$ be the ring of rational numbers; then $M_2(\mathcal{Q})$ is a prime ring. Let $\alpha$ be the automorphism of $M_2(\mathcal{Q})$, given by $\alpha([a_{ij}]) = ([a_{ij}]^T)^{-1}$, for each $a_{ij}, b_{ij} \in \mathcal{Q}$. Since $M_2(\mathcal{Q})$ is a prime ring, $\alpha$ is an automorphism, $M_2(\mathcal{Q})$ is an $(\alpha, \delta)$-weakly rigid ring. Now $e = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is an idempotent. We have $\alpha(e) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and $\alpha(e) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = e$.

For an index set $I$, suppose that for each $i \in I$, $R_i$ is a ring, $\alpha_i : R_i \to R_i$ an endomorphism and $\delta_i : R_i \to R_i$ an $\alpha_i$-derivation of $R_i$. So $\tilde{\alpha} : \prod_{i \in I} R_i \to \prod_{i \in I} R_i$ given by $\tilde{\alpha}((r_i)_{i \in I}) = \{\alpha_i(r_i)\}_{i \in I}$ is an endomorphism, and $\tilde{\delta} : \prod_{i \in I} R_i \to \prod_{i \in I} R_i$ given by $\tilde{\delta}((r_i)_{i \in I}) = \{\delta_i(r_i)\}_{i \in I}$ is an $\tilde{\alpha}$-derivation of $\prod_{i \in I} R_i$.

It is easy to see that if for each $i \in I$, $R_i$ is an $(\alpha_i, \delta_i)$ weakly rigid ring, then $\prod_{i \in I} R_i$ is also an $(\tilde{\alpha}, \tilde{\delta})$-weakly rigid ring.

Now we concern the classical quotient rings of $(\alpha, \delta)$-weakly rigid rings. A ring $R$ is called right Ore if given $a, b \in R$ with $b$ regular there exist $a_1, b_1 \in R$ with $b_1$ regular such that $ab_1 = ba_1$. It is a well-known fact that $R$ is a right Ore ring if and only if there exists the classical right quotient ring of $R$.

Let $R$ be an Ore ring, $\alpha$ an endomorphism and $\delta$ an $\alpha$-derivation of $R$. Define $\tilde{\alpha} : Q \to Q$ given by $\tilde{\alpha}(rc^{-1}) = \alpha(r)\alpha(c)^{-1}$ and $\tilde{\delta} : Q \to Q$ given by $\tilde{\delta}(rc^{-1}) = (\delta(r) - rc^{-1}\delta(c))\alpha(c)^{-1}$. Then $\tilde{\alpha}$ is an endomorphism and $\tilde{\delta}$ an $\tilde{\alpha}$-derivation of the classical quotient ring $Q$ of $R$. The set of regular elements of $R$ is denoted by $C$.

Theorem 2.15. Suppose that there exists the classical quotient ring $Q$ of a ring $R$ with an endomorphism $\alpha$ and $\alpha$-derivation $\delta$. If $R$ is $(\alpha, \delta)$-weakly rigid, then $Q$ is $(\tilde{\alpha}, \tilde{\delta})$ weakly rigid.

Proof. Assume that $aQb = 0$, with $a = r_1c_1^{-1}, b = r_2c_2^{-1}$ for some $r_1, r_2 \in R$ and $c_1, c_2 \in C$. We first show that $\tilde{a}\tilde{a}(Qb) = 0$. For each $r_3c_3^{-1} \in Q$, it is enough to show that $r_1c_1^{-1}\alpha(r_3c_3^{-1})\alpha(r_2c_2^{-1}) = 0$. Since $R$ satisfies the Ore condition, there exist $r_4 \in R, c_4 \in C$ such that $r_3c_3^{-1} = c_4^{-1}r_4$. Hence $r_1c_1^{-1}\alpha(r_3)\alpha(c_3)^{-1}\alpha(r_2)c_2^{-1} = r_1c_1^{-1}\alpha(c_4r_4^{-1})\alpha(c_2^{-1}) = r_1c_1^{-1}\alpha(c_4)\alpha(r_4)c_2^{-1}$. Since $R$ satisfies the Ore condition, there exist $r_5 \in R, c_5 \in C$ such that $r_1(c_4r_4^{-1})^{-1} = c_5^{-1}r_5$. Hence $r_1c_1^{-1}\alpha(r_3)c_3^{-1}\alpha(r_2)c_2^{-1} = c_5^{-1}r_5\alpha(r_4)c_2^{-1}\alpha(r_2)c_2^{-1}$. Therefore it is enough to show that $r_5\alpha(r_4\alpha(r_2)c_2^{-1}) = 0$. Since $r_1c_1^{-1}Qr_2 = 0$ and $c_4^{-1}Qr_4 \subseteq Q$, we have $r_1c_1^{-1}c_4^{-1}Qr_4r_2 = 0$, and hence $c_5^{-1}r_5\alpha(r_4)c_2^{-1} = 0$. Thus $r_5\alpha(r_4)c_2^{-1} = 0$ so $r_5\alpha(r_4) = 0$, and since $R$ is $(\alpha, \delta)$-weakly rigid, $r_5\alpha(r_4)c_2^{-1} = 0$. So $r_5\alpha(r_4)c_2^{-1} = 0$; therefore $\tilde{a}\tilde{a}(Qb) = 0$.

Now we show that $\tilde{a}\tilde{b}(b) = 0$. We have $\tilde{a}\tilde{b}(b) = r_1c_1^{-1}(\delta(b) - r_2c_2^{-1}\delta(c_2))\alpha(c_2)^{-1}$. Since $r_1c_1^{-1}Qr_2 = 0$, we have $r_1c_1^{-1}r_2c_2^{-1}\delta(c_2)\alpha(c_2^{-1}) = 0$. So it is enough to show that $r_1c_1^{-1}\delta(b) = 0$. Since $R$ satisfies the Ore condition, there exist $r_3 \in R, c_3 \in C$ such that $r_1c_1^{-1} = c_3^{-1}r_3$. So it is enough to show that $r_3\delta(b) = 0$. Since $r_1c_1^{-1}Qr_2 = 0$, we have $c_3^{-1}r_3Qr_2 = 0$, so $r_3\delta(b) = 0$; therefore $\tilde{a}\tilde{b}(b) = 0$.

Next suppose that $\tilde{a}\tilde{a}(Qb) = 0$, with $a = r_1c_1^{-1}, b = r_2c_2^{-1}$ for some $r_1, r_2 \in R$ and $c_1, c_2 \in C$; we then show that $\tilde{a}\tilde{a}(Qb) = 0$. For each $r_3c_3^{-1} \in Q$, it is enough to show that $r_1c_1^{-1}r_3c_3^{-1}r_2c_2^{-1} = 0$. Since $R$ satisfies the Ore condition, there exist $r_4, r_5 \in R, c_4, c_5 \in C$ such that $r_1c_1^{-1} = c_4^{-1}r_4$ and $r_2c_2^{-1} = r_5c_5^{-1}$. So it is enough to show that $r_4r_5r_2 = 0$. Since $\tilde{a}\tilde{a}(Qb) = 0$, $c_4^{-1}r_4\tilde{\alpha}(r_3c_3^{-1}r_2c_2^{-1}) = 0$; so $r_4\alpha(r_3r_5) = 0$. Since $R$ is $(\alpha, \delta)$-weakly rigid, $r_4r_5r_2 = 0$, and the result follows. □
Example 2.2. However the notion of an $(\alpha, \delta)$-weakly rigid rings is not closed under extensions to matrix rings or triangular matrix rings, by [9]. Clark defined a quasi-Baer ring and used it to characterise when a finite-dimensional ring overcomes these shortfalls. Further work has appeared in [3–9, 13, 23]. The class of quasi-Baer rings is closed under direct products and Morita invariance. Further work has appeared in [3–9, 13, 23].

We provide examples which show that, in general, the quasi-Baerness (or $p.q.$-Baerness) of 
\[\prod_{i=0}^{n} R_i, \quad \prod_{i=0}^{n} T_{i} \] the Ore extension whose elements are the polynomials
\[\sum_{i=0}^{n} r_i x^i \in R, \quad r_i \in R,\] where the addition is defined as usual and the multiplication by
\[xb = \alpha(b)x + \delta(b)\] for any $b \in R$.

The skew Laurent polynomial ring $R[x, x^{-1}; \alpha]$, whose elements are finite sums of elements of the form $x^{-j}r x^i$, where $r \in R$ and $i, j$ are non-negative integers. Multiplication is subject to the condition $xr = \alpha(r)x$ for all $r \in R$.

We also denote $R[[x; \alpha]]$ the skew power series ring, whose elements are formal power series of the form $\sum_{i=0}^{\infty} r_i x^i$ with coefficients $r_i \in R$, where the addition is defined as usual and the multiplication subject to the condition $xb = \alpha(b)x$ for any $b \in R$. The set $\{x^i\}_{i \geq 0}$ is an Ore subset of $R[[x; \alpha]]$, so that one can localise $R[[x; \alpha]]$ and form the skew Laurent series ring $R[[x, x^{-1}; \alpha]]$. Elements of $R[[x, x^{-1}; \alpha]]$ are formal combinations of elements of the form $x^{-j}r x^i$, where $r \in R$ and $i, j$ are non-negative integers.

In this section we consider the relationship between the properties of being quasi-Baer and left $p.q.$-Baer of a ring $R$ and of the rings $R[x; \alpha], R[x, x^{-1}; \alpha], R[[x; \alpha]]$ and $R[[x, x^{-1}; \alpha]]$, respectively.

We provide examples which show that, in general, the quasi-Baerness (or $p.q.$-Baerness) of $R$ and the aforementioned extensions do not depend on each other.

We will begin by considering some properties of $(\alpha, \delta)$-weakly rigid rings.

**Lemma 3.1.** Suppose that $R$ is an $(\alpha, \delta)$-weakly rigid ring; then for each $a, b \in R$ and positive integers $i, j$, $aRb = 0$ if and only if $\alpha^i(a)Ra^j(b) = 0$.

**Proof.** Suppose that $aRb = 0$, so $\alpha^i(aRb) = 0$, and hence $\alpha^i(a)\alpha^j(Rb) = 0$. Since $R$ is $\alpha$-weakly rigid, $\alpha^i(a)Rb = 0$. So for each $r \in R$, $\alpha^i(a)rRb = 0$; hence $\alpha^i(a)\alpha^j(Rb) = 0$. Thus $\alpha^i(a)\alpha^j(b) = 0$ whence $\alpha^i(a)Ra^j(b) = 0$, for each $i, j$. Now assume that $\alpha^i(a)Ra^j(b) = 0$, for each $i, j$. Since $R$ is $\alpha$-weakly rigid, $\alpha^i(a)\alpha^j(Ra^j(b)) = 0$, so
\[ \alpha'(aRa_e(b)) = 0. \] Since \( \alpha \) is injective, \( aRa_e(b) = 0 \). So \( \alpha \beta'(Rb) = 0 \), since \( R \) is \( \alpha \)-weakly rigid \( aRb = 0 \), and the result follows. \( \square \)

**Lemma 3.2.** Suppose that \( R \) is \( (\alpha, \delta) \)-weakly rigid; then for each \( a, b \in R \) and positive integers \( i, j \), \( aRb = 0 \) implies \( \delta^i(a)R\delta^j(b) = 0 \).

**Proof.** Suppose that \( aRb = 0 \), so for each \( r \in R, arRb = 0 \). Since \( R \) is \( (\alpha, \delta) \)-weakly rigid, \( ar\delta^i(b) = 0 \) for each \( r \in R \), and hence \( aR\delta^i(b) = 0 \) for each positive integer \( j \). Now we show by induction that for each positive integer \( i \), \( \delta^i(a)R\delta^i(b) = 0 \). For \( i = 1 \), since \( aR\delta^i(b) = 0 \), for each \( r \in R \), \( \delta(aR\delta^i(b)) = 0 \) so \( \delta(a)r\delta^i(b) + a(\delta\delta^i(b)) = 0 \). Since \( aR\delta^i(b) = 0 \), we get \( a(\delta\delta^i(b)) = 0 \) by Lemma 3.1. So \( \alpha(\delta\delta^i(b)) = 0 \) for each \( r \in R \). Since \( R \) is \( (\alpha, \delta) \)-weakly rigid, \( \alpha(\delta\delta^i(b)) = 0 \), so \( \delta(\delta\delta^i(b)) = 0 \). Now assume that the result is true for each \( t < i \). So we have \( \delta^i(a)R\delta^i(b) = 0 \). Hence \( \delta^{i-1}(a)R\delta^i(b) = 0 \) for each \( r \in R \). So \( \delta^{i-1}(a)R\delta^i(b) + a(\delta^{i-1}(a))\delta^i(b) = 0 \). Since \( \delta^{i-1}(a)R\delta^i(b) = 0 \), we have \( \alpha(\delta^{i-1}(a))R\delta^i(b) = 0 \) by Lemma 3.1. Since \( R \) is \( (\alpha, \delta) \)-weakly rigid, \( \alpha(\delta^{i-1}(a))\delta^i(b) = 0 \) for each \( r \in R \). So \( \delta^i(a)\delta^i(b) = 0 \) for each \( r \in R \), and so the result follows. \( \square \)

In [22, Proposition 3.2], the authors proved that a right semi-central idempotent \( e \) of a ring \( R \) is a right semi-central idempotent of \( R[x; \alpha, \delta] \) if and only if \( e \in Ra_e(e) \).

**Corollary 3.3.** If \( R \) is an \( \alpha \)-weakly rigid ring, then each right semi-central idempotent of \( R \) is a right semi-central idempotent of \( R[x; \alpha, \delta] \).

**Proof.** Let \( e \) be a right semi-central idempotent of \( R \). Then \( eR = eRe, so eR(1-I) = 0 \). Since \( R \) is \( \alpha \)-weakly rigid, \( e\alpha(1-e) = 0 \), so \( e = e\alpha(e) \). Now the result follows by [22, Proposition 3.2]. \( \square \)

Recall that for a quasi-Baer ring \( R \), for each left ideal \( I \) of \( R \), \( \ell_R(I) = Re \) for some idempotent \( e \) of \( R \). Since \( \ell_{\beta}(I) \) is an ideal, \( e \) is right semi-central.

**Theorem 3.4.** Let \( R \) be an \( \alpha \)-weakly rigid ring. If \( R \) is a quasi-Baer ring, then \( R[x; \alpha, \delta] \) is a quasi-Baer ring.

**Proof.** Let \( I \) be an ideal of \( S = R[x; \alpha, \delta] \). Let \( I_0 \) be the set of all leading coefficients of elements of \( I \) together with \( 0_R \). Then \( I_0 \) is a left ideal of \( R \). So \( \ell_{\beta}(I_0) = Re \) for some right semi-central idempotent \( e \) of \( R \). We prove that \( \ell_S(I) = Se \). Let \( f = a_n + \cdots + a_0x^n \in I \) so \( ef = e^a_n + \cdots + e^a_{n-1}x^{n-1} \in I \), since \( ea_n = 0 \). So \( ea_{n-1} = I_0 \); hence \( e^a_{n-1} = eea_{n-1} = 0 \). Continuing in this way it implies that \( ea_i = 0 \) for each \( 0 \leq i \leq n \) and that \( ef = 0 \). So \( Se \subseteq \ell_S(I) \). Let \( g = b_0 + \cdots + b_mx^m \in \ell_S(I) \), so for each \( f = a_0 + \cdots + a_0x^n \in I \) and \( r \in R \), \( grf = 0 \). So \( b_m\alpha^m(ra_m) = 0 \) for each \( r \in R \), and hence \( b_m\alpha^m(ra_m) = 0 \). Since \( R \) is \( \alpha \)-weakly rigid, \( b_mRa_m = 0 \). So \( b_m = \ell_{\beta}(I_0) = Re \), and hence \( b_m = b_me \). On the other hand by Corollary 3.3, \( e \) is a right semi-central idempotent of \( S \). Thus \( b_me^m = b_me^{e}\ e = b_me^ne, and hence grf = (b_0 + \cdots + b_{m-1}x^{m-1})ef + b_mx^m e = 0 \). But since \( Se \subseteq \ell_S(I) \), we get \( b_mx^mef = 0 \) so \( (b_0 + \cdots + b_{m-1}x^{m-1})ef = 0 \). By a similar way we get \( b_{m-1} = b_{m-1}e \) and that \( b_{m-1}x^{m-1} = b_{m-1}x^{m-1}e; \) so after \( m \) steps, we can see that \( b_i = b_i \ e \) and that \( b_i = b_i \ e \) for each \( i \). Thus \( g = ge \), and hence \( \ell_S(I) \subseteq Se \), and the result follows. \( \square \)

In [22, Example 2.1] the authors show that there exists a reduced Baer ring \( R \) with a monomorphism \( \alpha \) of \( R \) such that \( R[x; \alpha] \) is not a p.q.-Baer ring. The example shows that \( \alpha \)-weakly rigid condition on \( R \), in Theorem 3.4, is not superfluous.

In the proof of Lemma 3.5 and Theorem 3.6, we adapt the method which has been employed by Y. Zhou in [24].
Lemma 3.5. Let $R$ be an $(\alpha, \delta)$-weakly rigid ring. Let $L = \{ \ell_R(RU) \mid U \subseteq R \}$, $M = \{ \ell_S(SU) \mid S \subseteq S = R[x; \alpha, \delta] \}$ and $\Phi : L \to M$, given by $\Phi(I) = I[x; \alpha, \delta]$ and $\Psi : M \to L$, given by $\Psi(J) = J \cap R$; then $\Psi \circ \Phi = \text{id}_L$.

Proof. We first show that, for $U \subseteq R$, $\ell_R(RU)[x; \alpha, \delta] = \ell_S(SU)$. Let $f = a_0 + \cdots + a_nx^n \in \ell_R(RU)[x; \alpha, \delta]$. So for each $i$, $a_i \in \ell_R(RU)$, and for each $u \in U$, $a_iRu = 0$. So by Lemmas 3.1 and 3.2, $a_iR\delta(u) = a_iR\alpha(\delta(u)) = 0$ for each $j \geq 0$. Thus $f \in \ell_S(SU)$, and hence $\ell_R(RU)[x; \alpha, \delta] \subseteq \ell_S(SU)$. Now assume that $g = b_0 + \cdots + b_m\alpha^n \in \ell_S(SU)$; then $(b_0 + \cdots + b_m\alpha^n)ru = 0$ for each $r \in R$ and $u \in U$. So $b_m\alpha^n(Ru) = 0$, and hence $b_m\alpha^n(Ru) = 0$. Since $R$ is $\alpha$-weakly rigid, $b_mRu = 0$, and hence $b_m \in \ell_R(RU)$. On the other hand, $0 = gru = (b_0 + \cdots + b_m\alpha^n - 1)ru + b_m\alpha^nRru$. Since $b_mRu = 0$, by Lemmas 3.1 and 3.2, $b_mR\alpha(u) = b_mR\alpha(\delta(u)) = 0$ for each $j \geq 0$. So $b_m\alpha^nRru = 0$; hence $(b_0 + \cdots + b_m\alpha^n - 1)ru = 0$. By the same way we conclude that for each $i$, $b_i \in \ell_R(RU)$. Thus $g \in \ell_R(RU)[x; \alpha, \delta]$, and hence $\ell_R(RU)[x; \alpha, \delta] = \ell_S(SU)$. Therefore $\Phi$ is well defined. Next assume that $V \subseteq R[x; \alpha, \delta]$ and $\Phi$. We show that $\ell_S(SV) \cap R = \ell_R(RCV)$, where $CV$ is the set of all coefficients of elements of $V$. If $f \in \ell_S(SV) \cap R$, then it is clear that $f \in \ell_R(RCV)$. Let $a \in \ell_R(RCV)$; then $aRb = 0$ for each $b \in CV$. So by Lemmas 3.1 and 3.2, $aR\alpha(b) = aR\alpha(\delta(b)) = 0$ for each $j \geq 0$. So $a \in \ell_S(SV) \cap R$, whence $\ell_S(SV) \cap R = \ell_R(RCV)$, and $\Psi$ is well defined. Therefore $\Psi \circ \Phi(\ell_R(RU)) = \Psi(\ell_R(RU)[x; \alpha, \delta]) = \ell_R(RU)[x; \alpha, \delta] \cap R = \ell_S(SU) \cap R = \ell_R(RCV) = \ell_R(RU)$, and the result follows. □

Theorem 3.6. Let $R$ be an $(\alpha, \delta)$-weakly rigid ring. If $S = R[x; \alpha, \delta]$ is quasi-Baer, then $R$ is quasi-Baer.

Proof. Let $I$ be an ideal of $R$. By Lemma 3.5, $\ell_R(I)[x; \alpha, \delta] = \ell_S(SI)$. Since $S$ is quasi-Baer, for some idempotent $f = a_0 + \cdots + a_nx^n \in S$, $\ell_S(SI) = SF$. But $\ell_S(SI) = \ell_R(I)[x; \alpha, \delta]$, so for each $0 \leq i \leq n$, $a_i \in \ell_R(I)$. On the other hand, by Lemma 3.5, $\ell_R(I) = \ell_R(I)[x; \alpha, \delta] \cap R = \ell_S(SI) \cap R = SF \cap R$. So for each $a \in \ell_R(I)$, $a = a^2$, so $a = aa_0$. Since $a_0 \in \ell_R(I)$, $a_0 = a_0^2$, and hence $\ell_R(I) = Ra_0$, so the result follows. □

Corollary 3.7. Let $R$ be an $(\alpha, \delta)$-weakly rigid ring. Then, $R$ is quasi-Baer if and only if $R[x; \alpha, \delta]$ is quasi-Baer.

In [21, Example 3.6] the authors provided some examples of quasi-Baer rings $R[x; \delta]$, such that $R$ is not p.q.-Baer. So the condition $\delta$-weakly rigid in Theorem 3.6 is not superfluous.

The following (see [14, Example 9]) is an example of a ring $R$ such that $R[x; \alpha]$ is quasi-Baer, but $R$ is not p.q.-Baer, so the condition $\alpha$-weakly rigid in Theorem 3.6 is not superfluous.

Example 3.8. Let $R = \{ (a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a \equiv b \mod 2 \}$; $\alpha : R \to R$ given by $\alpha(a, b) = (b, a)$. Then by [14, Example 9], $R[x; \alpha]$ is quasi-Baer, but $R$ is not p.q.-Baer.

Theorem 3.9. Let $R$ be an $\alpha$-weakly rigid ring. If $R$ is a left p.q.-Baer ring, then $R[x; \alpha]$ is a left p.q.-Baer ring.

Proof. Let $R$ be a left p.q.-Baer $\alpha$-weakly rigid ring. Let $f = a_0 + \cdots + a_nx^n \in S = R[x; \alpha, \delta]$. For each $0 \leq i \leq n$ we have $\ell_R(Ra_i) = Re_i$ for some right semi-central idempotent $e_i \in R$. Put $e = e_ne_{n-1} \cdots e_0$. Since for each $0 \leq i \leq n$, $e_i$ is a right semi-central idempotent of $R$, we get $e^2 = e_ne_{n-1} \cdots e_0e_ne_{n-1} \cdots e_0 = e_ne_{n-1} \cdots e_0 \cdots e_ne_{n-1} \cdots e_0 = \cdots = e_ne_{n-1} \cdots e_0 = e$. Now for each $r \in R$ we have $er = e_ne_{n-1} \cdots e_0e_ne_{n-1} \cdots e_0e_ne_{n-1} \cdots e_0 = ere$. Thus $e$
is a right semi-central idempotent in $S$. Now we claim that $\ell_S(Sf) = Se$. Since $e$ is right semi-central in $S$, we have $eSf = eSef = 0$. So $Se \subseteq \ell_S(Sf)$. Now suppose that $g = b_0 + b_1x + \cdots + b_mx^m \in \ell_S(Sf)$, so $gRf = 0$, and hence $b_m\alpha^m(Ra_n) = 0$. Since $R$ is $\alpha$-weakly rigid, $b_mRan = 0$, and hence $b_m \in \ell_R(Ran)$. Thus $b_m = b_men$. Since $gSf = 0$, $ge_nRf = 0$. But $e_nRe_n = e_nRe_n(a_0 + \cdots + a_nx^n) = 0$. So $ge_nRf = 0$. Therefore $g = ge_n$, and each $irxi$ is a right semi-central idempotent of $S$. Theorem 3.11 is not superfluous.

Let $R$ be an $(\alpha, \delta)$-weakly rigid ring. If $R[x; \alpha, \delta]$ is left p.q.-Baer, then $R$ is left p.q.-Baer.

Proof. The proof is similar to that of Theorem 3.6.

COROLLARY 3.12. Let $R$ be an $(\alpha, \delta)$-weakly rigid ring. Then $R$ is left p.q.-Baer if and only if $R[x; \alpha, \delta]$ is left p.q.-Baer.

Example 3.8 and [21, Example 3.6] show that the condition $(\alpha, \delta)$-weakly rigid in Theorem 3.11 is not superfluous.

The set $\{x^i\}_{i \geq 0}$ is easily seen to be a left Ore subset of $S = R[x; \alpha]$, so that one can localise $S$ and form the skew Laurent polynomial ring $T = R[x, x^{-1}; \alpha]$. Elements of $T$ are finite sums of elements of the form $x^{-i}rx^j$, where $r \in R$ and $i, j$ are non-negative integers. Multiplication is subject to $xr = \alpha(r)x$ for all $r \in R$. In the case in which $\alpha$ is an automorphism; elements of $T$ have the form $\sum_{i=m}^{n}r_ix^i$, where $r_i \in R$ and $m, n \in \mathbb{Z}$. We consider D. A. Jordan’s [16] construction of the ring $A(R, \alpha)$. Let $A(R, \alpha)$ or $A$ be the subset $\{x^{-i}rx^i | r \in R, i \geq 0\}$ of the skew Laurent polynomial ring $R[x, x^{-1}; \alpha]$. For each $j \geq 0$, $x^{-i}rx^i = x^{-i+j}\alpha^j(r)x^{i+j}$. It follows that the set of all such elements forms a subring of $R[x, x^{-1}; \alpha]$ with

$$x^{-i}rx^i + x^{-j}sx^j = x^{-i+j}(\alpha^j(r) + \alpha^i(s))x^{i+j}$$

and

$$(x^{-i}rx^i)(x^{-j}sx^j) = x^{-i+j}(\alpha^j(r)\alpha^i(s))x^{i+j}$$

for $r, s \in R$ and $i, j \geq 0$. Note that $\alpha : A(R, \alpha) \rightarrow A(R, \alpha)$, given by $\alpha(x^{-i}rx^i) = x^{-i}\alpha(r)x^i$, is actually an automorphism of $A(R, \alpha)$; this is because $x^{-i}rx^i = \alpha(x^{-i-1}rx^{i+1})$, for each $i \geq 0$ and each $r \in R$. We have $R[x, x^{-1}; \alpha] \simeq A[x, x^{-1}; \alpha]$. 

Note that [22, Example 2.1] shows that the $\alpha$-weakly rigid condition in Theorem 3.9 is not superfluous.

COROLLARY 3.10. If $R$ is a left p.q.-Baer ring with a derivation $\delta$, then $R[x; \delta]$ is a left p.q.-Baer ring.
Thus by way of an isomorphism which maps $x^{-i}rx^j$ to $\alpha^{-i}(r)x^{j-i}$, for each $i, j$, (See [16], for more details).

**Proposition 3.13.** Let $R$ be an $\alpha$-weakly rigid ring. Then $R$ is quasi-Baer if and only if $A(R, \alpha)$ is quasi-Baer.

**Proof.** Let $R$ be a quasi-Baer ring and $I$ an ideal of $A = A(R, \alpha)$. Put $B = \{a \in R \mid x^{-i}ax^j \in I \text{ for some } i \geq 0 \}$. Let $J = RBR$; then $\ell_R(J) = Re$ for some right semicentral idempotent $e$ of $R$. Now we show that $\ell_A(I) = Ae$. Since $eR(1-e) = 0$ and $R$ is $\alpha$-weakly rigid, $a(e)R(1-e) = 0$ so $a(e) = a(e)e$. So for each $i > 0$, $a_i(e) = a_i(e)a_{i-1}(e)\ldots a_0(e)$. Hence for each $x^{-i}ax^j \in I$, we have $ex^{-i}ax^j = x^{-i}a_i(e)ax^j = x^{-i}a_i(e)a_{i-1}(e)\ldots ea_0x^j = 0$, which implies that $Ae \subseteq \ell_A(I)$. Now if $x^{-i}ax^j \in \ell_A(I)$, then for each $x^{-i}bx^j \in I$ and $r \in R$, $x^{-i}ax^jrx^{-i}bx^j = 0$. So $x^{-i}a_i(\alpha)\alpha^{-i}(rb)x^{j-i} = 0$. Hence $\alpha^{-i}(\alpha)a_i\alpha^{-i}(rb) = 0$. Since $R$ is $\alpha$-weakly rigid, $\alpha^{-i}(\alpha)rb = 0$, and hence $arb = 0$ by Lemma 3.1. So $a \in \ell_R(I)$, and hence $a = ae$; thus $x^{-i}ax^j = x^{-i}ax^j$. On the other hand since $eR(1-e) = 0$ and $R$ is $\alpha$-weakly rigid, $e\alpha^{-i}(\alpha)(R(1-e)) = 0$, so $e = e\alpha^{-i}(\alpha)$. Thus $x^{-i}ax^j = x^{-i}a\alpha^{-i}(\alpha)x^j = x^{-i}a\alpha^{-i}(\alpha)e = x^{-i}ax^j$. Therefore $\ell_A(I) \subseteq Ae$, and the result follows. Conversely suppose that $A(R, \alpha)$ is a quasi-Baer ring and $I$ an ideal of $R$. So $r_A(I\alpha) = eA$ for some idempotent $e \in A$. Let $x = x^{-i}ax^j$, where $a = a_i^2 \in R$. We now show that $r_R(I) = aR$. Since $IAe = 0$, for each $r \in R$ and $b \in I$, $bx^{-i}rx^{-i}ax^j = 0$. So $x^{-i}a_i(\alpha)bx^j = 0$, and hence $\alpha^{-i}(\alpha)rb = 0$. Since $R$ is $\alpha$-weakly rigid, $\alpha^{-i}(\alpha)rb = 0$, and hence $arb = 0$. Now if $b \in r_R(I)$, then for each $c \in I$ and $x^{-i}rx^j \in A$, $cx^{-i}rx^{-i}bx^j = x^{-i}cx^j\alpha^{-i}(\alpha)c\alpha^{-i}(\alpha)\alpha^{-i}(rb)x^{j-i}$. On the other hand $cRb = 0$ so $\alpha^{-i}(\alpha)cRb(\alpha) = 0$. So $\alpha^{-i}(\alpha)c\alpha^{-i}(\alpha)b = 0$, and hence $x^{-i}bx^j \in r_A(I\alpha) = eA$. So $x^{-i}bx^j = x^{-i}ax^jx^{-i}bx^j$, and hence $b = ab$, and the result follows.

**Theorem 3.14.** Let $R$ be an $\alpha$-weakly rigid ring and $\alpha$ an automorphism of $R$. If $R$ is quasi-Baer, then $[x, x^{-1}; \alpha]$ is quasi-Baer.

**Proof.** Since $\alpha$ is an automorphism of $R$, each element of $[R, x^{-1}; \alpha]$ is of the form $\sum_{i=m}^{n} r_i x^j$, where $r_i \in R$ and $m, n \in \mathbb{Z}$, so the proof is similar to that of Theorem 3.4.

**Theorem 3.15.** Let $R$ be an $\alpha$-weakly rigid ring. If $R$ is a quasi-Baer ring, then $[x, x^{-1}; \alpha]$ is a quasi-Baer ring.

**Proof.** Since $R$ is $\alpha$-weakly rigid quasi-Baer, $A$ is quasi-Baer. Since $\alpha$ is an automorphism of $A$ and $[R, x^{-1}; \alpha] = A[x, x^{-1}; \alpha]$, so the result follows by Theorem 3.14.

**Lemma 3.16.** Let $R$ be an $\alpha$-weakly rigid ring and $\alpha$ an automorphism of $R$. Let $L = \{\ell_R(\alpha) \mid U \subseteq R \}$, $M = \{\ell_S(\alpha) \mid U \subseteq S = [R, x^{-1}; \alpha]\}$ and $\Phi: L \to M$, given by $\Phi(I) = I[x, x^{-1}; \alpha]$ and $\Psi: M \to L$, given by $\Psi(J) = J \cap R$; then $\Psi \circ \Phi = id_L$.

**Proof.** The proof is similar to that of Lemma 3.5.

**Theorem 3.17.** Let $R$ be an $\alpha$-weakly rigid ring and $\alpha$ an automorphism of $R$. If $[R, x^{-1}; \alpha]$ is quasi-Baer, then $R$ is quasi-Baer.

**Proof.** The proof is similar to that of Theorem 3.6.

**Proposition 3.18.** If $R$ is an $\alpha$-weakly rigid ring, then $A(R, \alpha)$ is an $\alpha$-weakly rigid ring.
**Theorem 3.19.** Let $R$ be an $\alpha$-weakly rigid ring. If $R[x, x^{-1}; \alpha]$ is quasi-Baer, then $R$ is quasi-Baer.

*Proof.* Since $R$ is $\alpha$-weakly rigid, $A$ is $\alpha$-weakly rigid by proposition 3.18. Since $\alpha$ is an automorphism of $A$ and $R[x, x^{-1}; \alpha] \simeq A[x, x^{-1}; \alpha]$, by Theorem 3.18, $A$ is quasi-Baer, and the result follows by Proposition 3.13. \hfill \Box

**Corollary 3.20.** Let $R$ be an $\alpha$-weakly rigid ring. Then $R$ is quasi-Baer if and only if $R[x, x^{-1}; \alpha]$ is quasi-Baer.

**Theorem 3.21.** Let $R$ be an $\alpha$-weakly rigid ring and $\alpha$ an automorphism of $R$. If $R$ is a left p.q.-Baer ring, then $R[x, x^{-1}; \alpha]$ is a left p.q.-Baer ring.

*Proof.* Let $f \in S = R[x, x^{-1}; \alpha]$. Since $\alpha$ is an automorphism, $f = x^{-m}a_m + x^{-m+1}a_{m+1} + \cdots + a_0 + a_1x + \cdots + a_nx^n$. Set $J = Ra_m + Ra_{m+1} + \cdots + Ra_0 + \cdots + Ra_n$, which is a left ideal of $R$, so $\ell_R(J) = Re$, by [6, Proposition 1.7]. By a similar method as in the proof of Theorem 3.4, we can show that $\ell_S(Sf) = Se$, and the result follows. \hfill \Box

**Proposition 3.22.** Let $R$ be an $\alpha$-weakly rigid ring. Then $R$ is a left p.q.-Baer ring if and only if $A(R, \alpha)$ is a left p.q.-Baer ring.

*Proof.* The proof is similar to that of Proposition 3.13. \hfill \Box

**Theorem 3.23.** Let $R$ be an $\alpha$-weakly rigid ring with an automorphism $\alpha$. If $R[x, x^{-1}; \alpha]$ is a left p.q.-Baer ring, then $R$ is left p.q.-Baer.

*Proof.* Using Lemma 3.16, the proof is similar to that of Theorem 3.6. \hfill \Box

**Theorem 3.24.** Let $R$ be an $\alpha$-weakly rigid ring. Then $R$ is a left p.q.-Baer ring if and only if $R[x, x^{-1}; \alpha]$ is a left p.q.-Baer ring.

*Proof.* Since $R[x, x^{-1}; \alpha] \simeq A[x, x^{-1}; \alpha]$ and $\alpha$ is an automorphism of $A$, the result follows using Theorems 3.21 and 3.23 and Proposition 3.22. \hfill \Box

The following (see [12, Example 3.6]) is an example of a ring $R$ such that $R[x, x^{-1}; \alpha]$ is quasi-Baer, but $R$ is not p.q.-Baer, so the condition $\alpha$-weakly rigid in Corollary 3.20 and Theorem 3.24 is not superfluous.

**Example 3.25.** Let $R$ and $\alpha$ be those given in Example 3.8. Then by [12, Example 3.6], $R[x, x^{-1}; \alpha]$ is quasi-Baer, but $R$ is not p.q.-Baer.

**Corollary 3.26.** Let $R$ be an $(\alpha, \delta)$-weakly rigid ring, $\alpha$ an endomorphism and $\delta$ an $\alpha$-derivation of $R$. Then the following are equivalent:

(i) $R$ is a left p.q.-Baer ring;
(ii) $A(R, \alpha)$ is a left p.q.-Baer ring;
(iii) $R[x; \alpha, \delta]$ is a left p.q.-Baer ring;
(iv) $R[x, x^{-1}; \alpha]$ is a left p.q.-Baer ring.
**Corollary 3.27** [7, Theorem 3.1]. The following are equivalent:

1. $R$ is a left p.q.-Baer ring;
2. $R[x]$ is a left p.q.-Baer ring;
3. $R[x, x^{-1}]$ is a left p.q.-Baer ring.

**Theorem 3.28.** Let $R$ be a $\alpha$-weakly rigid ring. If $R$ is quasi-Baer, then $R[[x; \alpha]]$ is quasi-Baer.

**Proof.** Let $I$ be an ideal of $S = R[[x; \alpha]]$. Let $J = \{a \in R \mid$ there exists $ax^m + a_{m+1}x^{m+1} + \cdots \in I$, for some non-negative integer $m$ and $a_i \in R\}$. Then $J$ is a left ideal of $R$. So $\ell_R(J) = Re$ for some right semi-central idempotent $e$ of $R$. We show that $\ell_S(I) = Se$. If $f = \sum_{i=m}^{\infty} a_i x^i \in I$, then $a_m \in J$, so $ea_m = 0$. Hence $ef = ea_{m+1}x^{m+1} + ea_{m+2}x^{m+2} + \cdots$. Since $ea_{m+1} \in J$, we have $ea_{m+1} = eea_{m+1} = 0$. By this way we get $ef = 0$ so $Se \subseteq \ell_S(I)$. Now assume $g = \sum_{i=m}^{\infty} b_i x^i \in \ell_S(I)$, so for each $f = \sum_{i=m}^{\infty} a_i x^i \in I$ and $e \in R$, $grf = 0$. So $b_i x^i e = 0$ for each $r \in R$, and hence $b_i x^i e = 0$. Since $e$ is right semi-central, $eR = eRe$ so $eR(1-e) = 0$. Since $R$ is $\alpha$-weakly rigid, $e\alpha^n(R(1-e)) = 0$ for each positive integer $n$. So $e = e\alpha^n(e)$, and hence $b_n x^n e = b_n x^n e = b_n x^n e = b_n x^n e$. But we have $ef = 0$, so $b_n x^n e = 0$, and hence $\sum_{j=m}^{\infty} b_j x^j e = 0$. By the same way we can see that $b_{n+1} = b_{n+1} e$ and by induction for each $i$ that $b_i = b_i e$, so $g = \sum_{i=m}^{\infty} b_i x^i e$. On the other hand, for each $j$, $e = e\alpha^j(e)$, so $g = \sum_{i=m}^{\infty} b_i x^i e = e = e$. So the result follows.

**Lemma 3.29.** Let $R$ be an $\alpha$-weakly rigid ring. Let $L = \{\ell_R(RU) \mid U \subseteq R\}$, $M = \{\ell_S(SU) \mid U \subseteq S = R[[x; \alpha]]\}$ and $\Phi : L \to M$, given by $\Phi(I) = I[[x; \alpha]]$, and $\Psi : M \to L$, given by $\Psi(J) = J \cap R$; then $\Psi \circ \Phi = id_L$.

**Proof.** The proof is similar to that of Lemma 3.5.

**Theorem 3.30.** Let $R$ be an $\alpha$-weakly rigid ring. If $R[[x; \alpha]]$ is quasi-Baer, then $R$ is quasi-Baer.

**Proof.** The proof is similar to that of Theorem 3.6.

**Corollary 3.31.** Let $R$ be an $(\alpha, \delta)$-weakly rigid ring, $\alpha$ an endomorphism and $\delta$ an $\alpha$-derivation of $R$. Then the following are equivalent:

1. $R$ is a quasi-Baer ring;
2. $A(R, \alpha)$ is a quasi-Baer ring;
3. $R[x; \alpha, \delta]$ is a quasi-Baer ring;
4. $R[x, x^{-1}; \alpha]$ is a quasi-Baer ring;
5. $R[[x; \alpha]]$ is a quasi-Baer ring.

**Theorem 3.32.** Let $R$ be an $\alpha$-weakly rigid ring, with $\alpha$ an automorphism of $R$. If $R$ is quasi-Baer, then $R[[x, x^{-1}; \alpha]]$ is quasi-Baer.

**Proof.** Since $\alpha$ is an automorphism of $R$, the elements of $R[[x, x^{-1}; \alpha]]$ can be written in the form $a_m x^m + \cdots + a_1 x + \cdots$, where $m$ is a positive integer and $a_i \in R$ for each $i$. So the proof is similar to that of 3.28.

**Lemma 3.33.** Let $R$ be an $\alpha$-weakly rigid ring and $\alpha$ an automorphism of $R$. Let $L = \{\ell_R(RU) \mid U \subseteq R\}$, $M = \{\ell_S(SU) \mid U \subseteq S = R[[x, x^{-1}; \alpha]]\}$ and $\Phi : L \to M$, given by $\Phi(I) = I[[x, x^{-1}; \alpha]]$, and $\Psi : M \to L$, given by $\Psi(J) = J \cap R$; then $\Psi \circ \Phi = id_L$.

**Proof.** The proof is similar to that of Lemma 3.5.
THEOREM 3.34. Let $R$ be an $\alpha$-weakly rigid ring and $\alpha$ an automorphism of $R$. If $R[[x, x^{-1}; \alpha]]$ is quasi-Baer, then $R$ is quasi-Baer.

Proof. The proof is similar to that of Theorem 3.6.

COROLLARY 3.35. Let $R$ be an $\alpha$-weakly rigid ring and $\alpha$ an automorphism of $R$. Then $R$ is quasi-Baer if and only if $R[[x, x^{-1}; \alpha]]$ is quasi-Baer.

COROLLARY 3.36. [5, Theorem 1.8] The following are equivalent.

(i) $R$ is a quasi-Baer ring;

(ii) $R[x]$ is a quasi-Baer ring;

(iii) $R[[x]]$ is a quasi-Baer ring;

(iv) $R[x, x^{-1}]$ is a quasi-Baer ring;

(v) $R[[x, x^{-1}]]$ is a quasi-Baer ring.

Notice that, Birkenmeier et al.’s proof of [5, Lemma 1.7] to show that either $R[x; x^{-1}]$ or $R[[x; x^{-1}]]$ is quasi-Baer implies $R$ is quasi-Baer involves a long and quite technical calculation.

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