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Nagoya Math. J.
Vol. 167 (2002), 157–216

ON CLASSIFICATION OF $\mathbb{Q}$-FANO 3-FOLDS OF GORENSTEIN INDEX 2. II

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Abstract. In the previous paper, we obtained a list of numerical possibilities of $\mathbb{Q}$-Fano 3-folds X with $\operatorname{Pic} X = \mathbb{Z}(-2K_X)$ and $h^0(-K_X) \geq 4$ containing index 2 points P such that $(X, P) \simeq (\{xy + z^2 + u^a = 0\}/\mathbb{Z}_2(1,1,1,0), o)$ for some $a \in \mathbb{N}$. Moreover we showed that such an X is birational to a simpler Mori fiber space. In this paper, we prove their existence except for a few cases by constructing a Mori fiber space with desired properties and reconstructing X from it.

Notation and Conventions

$\mathbb{N}$: The set of positive integers.
$\sim$: Linear equivalence.
$\equiv$: Numerical equivalence.
$\mathbb{F}_n$: Segre-del Pezzo scroll of degree n.
$\mathbb{F}_{n,0}$: Surface obtained by contracting the negative section of $\mathbb{F}_n$.
Q_3: Smooth quadric 3-fold.
ODP: Ordinary double point, i.e., singularity analytically isomorphic to $\{xy + z^2 + u^2 = 0 \subset \mathbb{C}^4\}$.
QODP: Singularity analytically isomorphic to $\{xy + z^2 + u^2 = 0 \subset \mathbb{C}^4/\mathbb{Z}_2(1,1,1,0)\}$.
B_i $(1 \leq i \leq 5)$: Factorial Gorenstein terminal Fano 3-fold of Fano index 2, and with Picard number 1 and $(-K)^3 = 8i$, where K is the canonical divisor.
A_{2g-2} $(1 \leq g \leq 12$ and $g \neq 11)$: Factorial Gorenstein terminal Fano 3-fold of Fano index 1, and with Picard number 1 and genus g.
Abuse of notation: We use the same notation for transforms of curves by birational maps as original ones.

Received November 30, 2000.
2000 Mathematics Subject Classification: Primary 14J45; Secondary 14N05, 14M20.

https://doi.org/10.1017/S0027763000025472 Published online by Cambridge University Press

§0. Introduction

In this paper we work over $\mathbb{C}$, the complex number field.

Definition 0.0. ($\mathbb{Q}$-Fano variety) Let X be a normal projective variety. X is said to be a *terminal* (resp. *canonical*, *klt*, etc.) *$\mathbb{Q}$-Fano variety* if X has only terminal (resp. canonical, Kawamata log terminal, etc.) singularities and $-K_X$ is ample. By replacing 'ample' with 'nef and big', *terminal* (resp. *canonical*, *klt*, etc.) *weak $\mathbb{Q}$-Fano varieties* are similarly defined. If X has only terminal singularities, then we say that X is a *$\mathbb{Q}$-Fano variety* for short and if X has only Gorenstein terminal (resp. canonical, klt, etc.) singularities, we say that X is a *Gorenstein terminal* (resp. *canonical*, *klt*, etc.) *Fano variety*.

Let $I(X) := \min\{I \mid IK_X \text{ is a Cartier divisor}\}$ and we call $I(X)$ the *Gorenstein index* of X.

Write $I(X)(-K_X) \equiv r(X)H(X)$, where $H(X)$ is a primitive Cartier divisor and $r(X) \in \mathbb{N}$. (Note that $H(X)$ is unique since $\operatorname{Pic} X$ is torsion free.) Then we call $r(X)/I(X)$ the *Fano index* of X and denote it by $F(X)$.

In the previous paper [Taka02], we formulate a generalization of Takeuchi's method [Take89] for the classification of smooth Fano 3-folds and use it for a partial classification of $\mathbb{Q}$-Fano 3-folds X with the following properties.

Main Assumption 0.1. (1) The Picard number of X is 1,

(2) the Gorenstein index of X is 2,
(3) the Fano index of X is $1/2$,
(4) $h^0(-K_X) \geq 4$, and
(5) there exists an index 2 point P such that

$$(X, P) \simeq (\{xy + z^2 + u^a = 0\}/\mathbb{Z}_2(1,1,1,0), o)$$

for some $a \in \mathbb{N}$.

Let $f\colon Y \to X$ be the weighted blow-up at P with weight $\frac{1}{2}(1,1,1,2)$. In the previous paper [Taka02], we proved that Y is a weak $\mathbb{Q}$-Fano 3-fold and obtained the following diagram.

$$\begin{array}{ccccc} & & Y \dashrightarrow Y' & & \\ & \swarrow^{f} & & \searrow^{f'} & \\ X & & & & X', \end{array}$$

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where

(1) $Y \dashrightarrow Y'$ is an isomorphism and f' is a crepant divisorial contraction or

(2) $Y \dashrightarrow Y'$ is a flop or a composite of a flop and a flip, and f' is an extremal contraction which is not isomorphic in codimension 1.

We use the following notation.

Notation 0.2.
- $\widetilde{E} :=$ the strict transform of E on Y',
- $n := 2((-K_Y)^3 - (-K_{Y'})^3)$,
- $e := E^3 - \widetilde{E}^3 - 4n$,
- Rational numbers z and u are defined as follows. In case f' is birational, the f'-exceptional divisor E' satisfies $E' \equiv z(-K_{Y'}) - u\widetilde{E}$. Otherwise the pull-back L of an ample generator of $\operatorname{Pic} X'$ satisfies $L \equiv z(-K_{Y'}) - u\widetilde{E}$,
- $h := h^0(-K_X)$, and
- N is the number of $\frac{1}{2}(1,1,1)$-singularities obtained by deforming non-Gorenstein points of X locally.

The following is the main theorem of [Taka02]:

Theorem 0.3. *Let X be as in Main Assumption 0.1. Consider the diagram*

$$\begin{array}{ccccc} & & Y \dashrightarrow Y' & & \\ & \swarrow^{f} & & \searrow^{f'} & \\ X & & & & X', \end{array}$$

as above. Then the possibilities of X are classified as in Tables 1–5 and Tables 1′–5′ with the notation of 0.2. In particular we have $(-K_X)^3 \leq 15$ and $h^0(-K_X) \leq 10$.

Table 1. f' is of $(2,1)$-type. I

No.	h	$(-K_X)^3$	N	e	n	z	$\deg C$	$g(C)$	X'
1.1	6	7	2	7	0	4	7	8	[5]
1.2	6	15/2	3	7	0	2	3	0	$[2], I(X') = 2$
1.3	6	15/2	3	6	1	4	6	3	[5]
1.4	7	17/2	1	6	0	3	9	9	$\mathbb{P}^3$

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1.5	7	9	2	6	0	2	6	3	[3]
1.6	7	9	2	5	1	3	8	5	$\mathbb{P}^3$
1.7	7	19/2	3	5	1	2	5	0	[3]
1.8	7	19/2	3	4	2	3	7	1	$\mathbb{P}^3$
1.9	8	21/2	1	6	0	1	3	0	B_3
1.10	8	21/2	1	5	0	2	9	6	Q_3
1.11	8	11	2	4	1	2	8	3	Q_3
1.12	9	25/2	1	5	0	1	5	1	B_4
1.13	10	29/2	1	4	0	1	7	2	B_5
1.14	10	15	2	3	1	1	6	0	B_5

Table 1′. *f' is of* (2, 1)*-type. I*

h	$(-K_X)^3$	N	e	n	z	$\deg C$	$g(C)$	X'
8	23/2	3	3	2	2	7	0	Q_3

Notation and Remarks for Table 1 *and Table* 1′.

$C := f'(E')$,

$\deg C := (H(X') \cdot C)$ (*see Definition* 0.0 *for the definition of* $H(X')$),

$g(C) :=$ *the genus of* C *in case* X *has only* $\frac{1}{2}(1,1,1)$*-singularities*,

see [San96] *for the definition of* $[i]$,

$u = z + 1$.

Table 2. *f' is of* (2, 1)*-type. II*

No.	$(-K_X)^3$	N	e	$\deg C$	X'
2.1	7/2	3	10	1	A_6
2.2	4	4	8	2	A_8
2.3	9/2	5	6	3	A_{10}
2.4	5	6	4	4	A_{12}

Table 2′. *f' is of* (2, 1)*-type. II*

$(-K_X)^3$	N	e	$\deg C$	X'
11/2	7	2	5	A_{14}

Notation and Remarks for Table 2 and Table 2′.

$$C := f'(E'),$$
$$\deg C := (-K_{X'} \cdot C),$$
$$z = u = 1,$$
$$h = 4 \text{ and } n = 0.$$

Table 3. *f' is* $(2,0)$*-type or crepant divisorial.*

No.	h	$(-K_X)^3$	N	e	n	*type of* f'
3.1	4	5/2	1	15	0	$(2,0)_4$
3.1′	4	5/2	1	/	/	*crep. div.*
3.2	4	3	2	12	0	$(2,0)_8$
3.3	4	4	4	9	3	$(2,0)_1$
3.4	4	9/2	5	8	3	$(2,0)_5$

Remarks for Table 3.

$z = u = 1$,
(No. 3.1) X' *also belongs to this class,*
(No. 3.1′) X' *is a Fano* 3*-fold of* $(-K_{X'})^3 = 2$ *and with a canonical singularity along the image of* f'*-exceptional divisor,*
(No. 3.2) $X' \simeq A_4$ *with one Gorenstein terminal singularity,*
(No. 3.3) X' *is smooth, isomorphic to* A_{10},
(No. 3.4) X' *is smooth, isomorphic to* A_{16}.

Table 4. *f' is of* $(3,2)$*-type.*

No.	h	$(-K_X)^3$	N	e	n	$\deg \Delta$
4.1	5	11/2	3	8	0	8
4.2	5	6	4	7	1	6
4.3	6	13/2	1	7	0	7
4.4	6	7	2	6	1	6
4.5	6	15/2	3	5	2	5
4.6	6	8	4	4	3	4
4.7	6	17/2	5	3	4	3
4.8	10	29/2	1	6	0	0

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Table 4′. f' is of $(3,2)$-type.

h	$(-K_X)^3$	N	e	n	$\deg\Delta$
5	13/2	5	6	2	4
5	7	6	5	3	2
5	15/2	7	4	4	0
6	9	6	2	5	2
6	19/2	7	1	6	1

Notation and Remarks for Table 4 and Table 4′.

$\Delta :=$ *the discriminant divisor of* f',

$\deg\Delta$ *is measured by the ample generator of* $\operatorname{Pic} X'$,

in case $h = 5$, $z = u = 2$ *and* $X' \simeq \mathbb{F}_{2,0}$,

in case $h = 6$, $z = u = 1$ *and* $X' \simeq \mathbb{P}^2$,

in case $h = 10$, $z = 1$, $u = 2$ *and* $X' \simeq \mathbb{P}^2$.

Table 5. f' is of $(3,1)$-type.

No.	h	$(-K_X)^3$	N	e	n	$\deg F$
5.1	4	9/2	5	9	0	6
5.2	5	9/2	1	9	0	3
5.3	5	5	2	8	1	4
5.4	5	11/2	3	7	2	5
5.5	5	6	4	6	3	6

Table 5′. f' is of $(3,1)$-type.

h	$(-K_X)^3$	N	e	n	$\deg F$
4	5	6	8	1	8

Notation and Remarks for Table 5 and Table 5′.

$F :=$ *a general fiber of* f',

in case $h = 4$, $z = u = 2$,

in case $h = 5$, $z = u = 1$.

Based on these lists, we derive some geometric properties of such a $\mathbb{Q}$-Fano 3-fold X in Sections 1–3.

Miles Reid conjectured that every $\mathbb{Q}$-Fano 3-fold has an effective anti-canonical divisor with only canonical singularities. The conjecture is affirmative in case of Gorenstein canonical Fano 3-folds [Sho79b] and [Reid83]. In §1, we prove the following:

THEOREM 0.4. (See Corollary 1.2) *Assume that for any index* 2 *point* P, *there is an isomorphism*

$$(X, P) \simeq (\{xy + z^2 + u^a = 0\}/\mathbb{Z}_2(1,1,1,0), o)$$

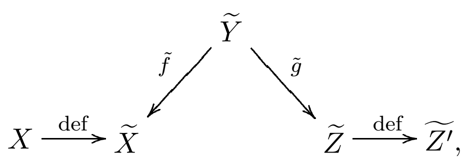
for some $a \in \mathbb{N}$. *Then* $|-K_X|$ *has a member with only canonical singularities.*

In §2, we study deformation theoretic properties of X and obtain the following:

THEOREM 0.5. (See Corollaries 2.2 and 2.3) *Let* X *be a* $\mathbb{Q}$*-factorial* $\mathbb{Q}$*-Fano* 3*-fold with* (1)–(4) *of Main Assumption* 0.1. *Let* $N := \mathrm{aw}(X)$ (*see* [Taka02, Definition 1.1]). *Then the following hold.*

(1) X *can be deformed to a* $\mathbb{Q}$*-factorial* $\mathbb{Q}$*-Fano* 3*-fold* X' *with* (1)–(4) *in Main Assumption 0.1 and with only QODP's or* $\frac{1}{2}(1,1,1)$*-singularities as its singularities.*

(2) *If* $N > 1$ (*resp.* $N = 1$), X *can be transformed to a* $\mathbb{Q}$*-factorial* $\mathbb{Q}$*-Fano* 3*-fold* $\widetilde{Z'}$ *with* (1)–(4) *of Main Assumption* 0.1 *and with only QODP's or* $\frac{1}{2}(1,1,1)$*-singularities as its singularities and* $h^0(-K_{\widetilde{Z'}}) = h$ *and* $\mathrm{aw}(\widetilde{Z'}) = N-1$ (*resp. a smooth Fano* 3*-fold* $\widetilde{Z'}$ *with* $\rho(\widetilde{Z'}) = 1$, $F(\widetilde{Z'}) = 1$ *and* $h^0(-K_{\widetilde{Z'}}) = h$) *as follows.*



where $* \xrightarrow{\text{def}} **$ *means that* $**$ *is a small deformation of* $*$,

$\widetilde{X}$ *is a* $\mathbb{Q}$*-Fano* 3*-fold with the properties* (1)–(4) *in Main Assumption* 0.1 *and with only ODP's, QODP's or* $\frac{1}{2}(1,1,1)$*-singularities as its singularities,*

$\tilde{f}\colon \widetilde{Y} \to \widetilde{X}$ *is similarly chosen to* f *in Theorem* 0.3, *and*

$\tilde{g}\colon \widetilde{Y} \to \widetilde{Z}$ *be the anti-canonical model.*

(2) is an analogue to Reid's fantasy about Calabi-Yau 3-folds [Reid87a].

In §3, we prove the following:

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Theorem 0.6. (See Corollary 3.1) *If any index* 2 *point is a* $\frac{1}{2}(1,1,1)$*-singularity,* X *can be embedded into a weighted projective space* $\mathbb{P}(1^h, 2^N)$*, where* $h := h^0(-K_X)$ *and* N *is the number of* $\frac{1}{2}(1,1,1)$*-singularities on* X*.*

We hope to determine the defining equation of X explicitely in some weighted projective space containing $\mathbb{P}(1^h, 2^N)$ as S. Mukai did in case of Fano 3-folds (see [Muk89], [Muk92] and [Muk95]).

In §4, as announced in [Taka02], we prove the existence of $\mathbb{Q}$-Fano 3-folds with Main Assumption 0.1. The main results are Thereoms 0.10–0.21. Proposition 0.8 gives a sufficient condition for the reconstruction of X.

Assumption 0.7. In Theorems 0.10–0.21 (A), we assume that X has only $\frac{1}{2}(1,1,1)$-singularities and fix f as in Theorem 0.3 (and then X is classified as in Tables 1–5). We use the notation of Tables 1–5 freely.

Proposition 0.8. *Let* Y' *be a projective* 3*-fold with only* $\frac{1}{2}(1,1,1)$*-singularities and* n *is a non-negative integer. Assume the following conditions.*

(1) $\rho(Y') = 2$.
(2) *In case* $n > 0$*, there are smooth rational curves* l_i $(0 \leq i \leq n-1)$ *such that*
   (2-1) l_i *are numerically equivalent.*
   (2-2) l_i *are mutually disjoint and are contained in* $\operatorname{Reg} Y'$.
   (2-3) $\operatorname{Bs}|-K_{Y'}|$ *is the union of* l_i *and* $\frac{1}{2}(1,1,1)$*-singularities.*
   (2-4) $-K_{Y'} \cdot l_i = -1$*, or*

   in case $n = 0$*,* $-K_{Y'}$ *is nef and big.*
(3) $(-K_{Y'})^3 + \frac{n}{2} > 0$.
(4) *There is an irreducible divisor* $\widetilde{E}$ *such that* $\widetilde{E} \cdot l_i = -1$ *in case* $n > 0$*,* $(-K_{Y'})^2 \widetilde{E} = 1 - n$ *and* $(-K_{Y'}) \widetilde{E}^2 = -2 - 2n$.
(5) *In case* $n = 0$*, there exists an extremal ray* R *of* Y' *such that* $\widetilde{E} \cdot R < 0$.

Then the following hold.

(i) *There is a birational map* $Y' \dashrightarrow Y$ *which is one flop, or a composite of one anti-flip and one flop.*
(ii) *There is an extremal contraction* $f\colon Y \to X$ *of* $(2,0)_4$*-type or* $(2,0)_{10}$*-type whose exceptional divisor is the strict transform of* $\widetilde{E}$*.*
(iii) X *is a* $\mathbb{Q}$*-Fano* 3*-fold with only* $\frac{1}{2}(1,1,1)$*-singularities or QODP's.*

Remark 0.9. In Theorems 0.10–0.21 (A), we assume that X has only $\frac{1}{2}(1,1,1)$-singularities. However, X reconstructed in Theorems 0.10–0.21 (B) by using Proposition 0.8 has possibly one singularity worse than $\frac{1}{2}(1,1,1)$-singularities.

THEOREM 0.10. (Table 1) (A) *Let X be a $\mathbb{Q}$-Fano 3-fold as in Table 1. Then*

(1) *images of n flipped curves l_i $(0 \le i \le n-1)$ are $(z+2)$-secant lines of C with respect to $\frac{1}{z+1}(-K_{X'})$,*
(2) *$l_i \subset \operatorname{Reg} Y'$ and l_i are mutually disjoint, and*
(3) *$\operatorname{Bs}|-K_{X'} - C|$ is the union of C, l_i and $\frac{1}{2}(1,1,1)$-singularities.*

(B) *Conversely let X' be a $\mathbb{Q}$-Fano 3-fold as in Table 1 and $C \subset X'$ a smooth curve of degree and genus given in the same row. Let n and z be integers given in the same row. Assume that*

(1) *C has $(z+2)$-secant lines l_i $(0 \le i \le n-1)$ with respect to $\frac{1}{z+1}(-K_{X'})$ such that $l_i \subset \operatorname{Reg} X'$ and l_i are mutually disjoint,*
(2) *$\operatorname{Bs}|-K_{X'} - C|$ is the union of C, l_i and $\frac{1}{2}(1,1,1)$-singularities, and*
(3) *There exists a surface $S \equiv \frac{z}{z+1}(-K_{X'})$ containing C.*

Let $f\colon Y' \to X'$ be the blow-up of X' along C and E' f'-exceptional divisor. Then the following hold.

(i) *S is irreducible and $C \not\subset \operatorname{Sing} S$.*
(ii) *Y', l_i and the strict transform $\widetilde{E}$ of S on Y' satisfy the conditions of Proposition 0.8. Let X be a $\mathbb{Q}$-Fano 3-fold obtained as in Proposition 0.8. Then X belong to the same row in Table 1 as X'.*

(C) *In any case of Table 1, there exists an example of (X', C, l_i) as in (B) and hence that of a $\mathbb{Q}$-Fano 3-fold X.*

THEOREM 0.11. (Table 2) (A) *Let X be a $\mathbb{Q}$-Fano 3-fold of No. 2.1 (resp. 2.2). Then*

(1) *$X' \simeq A_6$ (resp. A_8) and has 2 (resp. 3) singularities P_i on C such that (X', P_i) are isomorphic to $(\{xy+zw=0\}, o)$ in $\mathbb{C}^4$ or $(\{xy+z^2+w^3=0\}, o)$ in $\mathbb{C}^4$, and*
(2) *C is a smooth rational curve such that $(-K_{X'} \cdot C) = 1$ (resp. 2).*

(B) *Conversely let (X', C, P_i) be as in (A). Then the following hold.*

(i) *There exists a divisorial contraction* $f'\colon Y' \to X'$ *of* $(2,1)$*-type whose center is* C *(note that by* [Taka02, Proposition 2.2 (4c)], Y *has only* $\frac{1}{2}(1,1,1)$*-singularities).*

(ii) *There is a unique member* $\widetilde{E}$ *of* $|-K_{Y'} - E'|$, *where* E' *is* f'*-exceptional divisor.*

(iii) Y' *and* $\widetilde{E}$ *satisfy the conditions of Proposition* 0.8. *Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold obtained as in Proposition* 0.8. *Then* X *is of No.* 2.1 *(resp. No.* 2.2*).*

(C) *There exists an example of* (X', C) *as in* (A) *for No.* 2.1 *(resp. No.* 2.2*) and hence there exists a* $\mathbb{Q}$*-Fano* 3*-fold of No.* 2.1 *(resp. No.* 2.2*).*

Remark 0.12. Examples are not known for No. 2.3 or 2.4.

THEOREM 0.13. (Table 3) (No. 3.1 or 3.1′) *Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold of No.* 3.1 *or* 3.1′. *Then* $X \simeq ((5) \subset \mathbb{P}(1^4, 2))$. X' *is also of No.* 3.1 *if so is* X.

(No. 3.2) *Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold of No.* 3.2. *Then* $X \simeq ((3,4) \subset \mathbb{P}(1^4, 2^2))$ *and* $X' \simeq A_4$ *with one Gorenstein terminal singularity.*

(No. 3.3) (A) *Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold of No.* 3.3. *Then*

(1) X' *is smooth and isomorphic to* A_{10}, *and*

(2) *there exist exactly three lines through* Q, *which is the image of the* f'*-exceptional divisor* E'.

(B) *Conversely let* X' *be a smooth* 3*-fold isomorphic to* A_{10} *such that there exists a point* Q *where exactly three lines* l_i $(i = 0,1,2)$ *pass through. Let* $f'\colon Y' \to X'$ *be the blow-up at* Q *and* E' *the exceptional divisor. Then the following hold.*

(i) $\mathrm{Bs}|-K_{Y'}| = l_0 \cup l_1 \cup l_2$.

(ii) *There is a unique member* $\widetilde{E}$ *of* $|-K_{Y'} - E'|$.

(iii) Y', l_i *and* $\widetilde{E}$ *satisfy the conditions of Proposition* 0.8. *Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold obtained as in Proposition* 0.8. *Then* X *is of No.* 3.3.

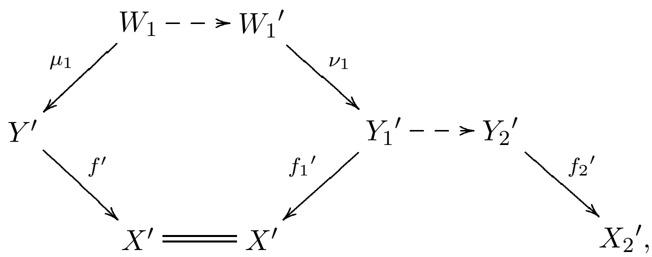
(C) *There exists an example of* (X', Q) *as in* (B) *and hence there exists a* $\mathbb{Q}$*-Fano* 3*-fold of No.* 3.3.

Remark 0.14. Examples are not known for No. 3.4.

THEOREM 0.15. (Table 4, I.) (A) *Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold such that* $n \geq 1$ *and* $X' \simeq \mathbb{F}_{2,0}$. *Let* $\mu_1\colon W_1 \to Y'$ *be the blow-up along a flipped* *curve l_0. Then there is a flop $W_1 \dashrightarrow W_1{}'$ over X' and an extremal contraction of $(2,1)$-type $\nu_1\colon W_1{}' \to Y_1{}'$ over X'. Let m_1 be the image of ν_1-exceptional divisor. Then the following hold.*

(1) $n = 1$.
(2) *m_1 is a smooth rational curve with $(-K_{Y_1{}'} \cdot m_1) = 8$ such that $m_1 \subset \operatorname{Reg} Y_1{}'$, $f_1{}'|_{m_1}$ is an isomorphism and $\operatorname{Bs}|-K_{Y_1{}'} - m_1|$ is the union of m_1 and $\frac{1}{2}(1,1,1)$-singularities.*
(3) *$Y_1{}'$ is a $\mathbb{Q}$-Fano 3-fold with $(-K_{Y_1{}'})^3 = 17$ and a unique flipping ray.*
(4) *Let $Y_1{}' \dashrightarrow Y_2{}'$ be the flip. Then $Y_2{}'$ is a smooth divisor in $\mathbb{P} := \mathbb{P}(\mathscr{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2})$ over $X_2{}' \simeq \mathbb{P}^1$ and linearly equivalent to $2H+M$, where H is the tautological divisor of $\mathbb{P}$ and M is a fiber of the natural projection $\mathbb{P} \to X_2{}'$. $Y_2{}'$ has two disjoint sections which are connected components of the intersection of $Y_2{}'$ and the subvariety V of $\mathbb{P}$ associated to the surjection $\mathscr{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2} \to \mathscr{O}_{\mathbb{P}^1}^{\oplus 2}$.*

Consequently we obtain the following diagram.



where $f_i{}'$ are the natural projections.

(B) *Conversely let $Y_2{}'$ be as in* (A) (4)*. Then there is an anti-flip $Y_2{}' \dashrightarrow Y_1{}'$. Moreover let $m_1 \subset Y_1{}'$ be as in* (A) (2) *and $\nu_1\colon W_1{}' \to Y_1{}'$ the blow-up along m_1. Then there is a flop $W_1{}' \dashrightarrow W_1$ over X' and an extremal contraction of $(2,1)$-type $\mu_1\colon W_1 \to Y'$ over X'. Let l_0 be the image of μ_1-exceptional divisor and L the pull-back of a line of X' on Y'. Then the following hold.*

(i) *There is a unique member $\widetilde{E}$ of $|-K_{Y'} - D|$ and $\widetilde{E}$ is irreducible, where D is a Weil divisor such that $2D \sim L$.*
(ii) *Y', l_0 and $\widetilde{E}$ satisfy the conditions of Proposition* 0.8*. Let X be a $\mathbb{Q}$-Fano 3-fold obtained as in Proposition* 0.8*. Then X is of No.* 4.2.

(C) *There exists an example of* $({Y_2}', m_1)$ *as in* (B) *and hence there exists a* $\mathbb{Q}$*-Fano* 3*-fold of No.* 4.2*.*

Remark 0.16. Examples are not known for No. 4.1.

THEOREM 0.17. (Table 4, II) (A) *Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold of No.* 4.3*. Then the following hold.* Y' *is a weak Fano* 3*-fold with* $\rho(Y') = 2$, $(-K_{Y'})^3 = 6$ *and a conic bundle structure over* $\mathbb{P}^2$*.* $\widetilde{E}$ *is an irreducible divisor which is generically a* 2*-section such that* $(-K_{Y'})^2\widetilde{E} = 1$ *and* $(-K_{Y'})\widetilde{E}^2 = -2$*.*

(B) *Conversely let* $(Y', \widetilde{E})$ *be as in* (A)*. Then they satisfy the conditions of Proposition* 0.8*. Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold obtained as in Proposition* 0.8*. Then* X *is of No.* 4.3*.*

(C) *There exists an example of* $(Y', \widetilde{E})$ *as in* (A) *and hence there exists a* $\mathbb{Q}$*-Fano* 3*-fold of No.* 4.3*.*

THEOREM 0.18. (Table 4, III) (A) *Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold of No.* 4.4*–*4.7*. Let* $\mu_1 \colon W_1 \to Y'$ *be the blow-up along a flipped curve* l_0*. Then there is a flop* $W_1 \dashrightarrow {W_1}'$ *over* X' *and an extremal contraction of* $(2,1)$*-type* $\nu_1 \colon {W_1}' \to {Y_1}'$ *over* X'*. Let* m_1 *be the image of* ν_1*-exceptional divisor.*

(A-1) *Assume that* X *is a* $\mathbb{Q}$*-Fano* 3*-fold of No.* 4.4*. Then* ${Y_1}' \simeq ((2,2) \subset \mathbb{P}^2 \times \mathbb{P}^2)$ *or a double cover of* $((1,1) \subset \mathbb{P}^2 \times \mathbb{P}^2)$ *ramified along a smooth anti-canonical divisor. Let* p_i $(i = 1, 2)$ *be the two structure morphism of conic bundles. Then* m_1 *is a smooth rational curve such that* $p_i|_{m_1}$ *are isomorphisms and* $p_i(m_1)$ *are lines.*

(A-2) *Assume that* X *is a* $\mathbb{Q}$*-Fano* 3*-fold of No.* 4.5*–No.* 4.7*. Then* ${Y_1}'$ *is a weak Fano* 3*-fold. Let* l_1 *be the transform of a flipped curve other than* l_0*. Let* $\mu_2 \colon W_2 \to {Y_1}'$ *be the blow-up along* l_1*. Then there is a flop* $W_2 \dashrightarrow {W_2}'$ *over* X' *and an extremal contraction of* $(2,1)$*-type* $\nu_2 : {W_2}' \to {Y_2}'$ *over* X'*.* ${Y_2}'$ *is the blow-up of* ${X_2}' \simeq B_{n+1}$ *along a curve* γ*. Moreover* γ*,* m_1 *and the image of* ν_2*-exceptional divisor* m_2 *are normal rational curves of degree* $n-1$ *intersecting the common* $n-2$ *points simply.*

Consequently we obtain the following diagram.

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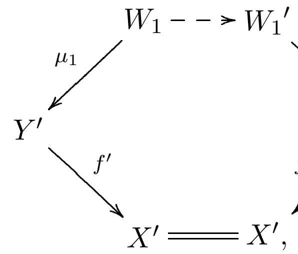
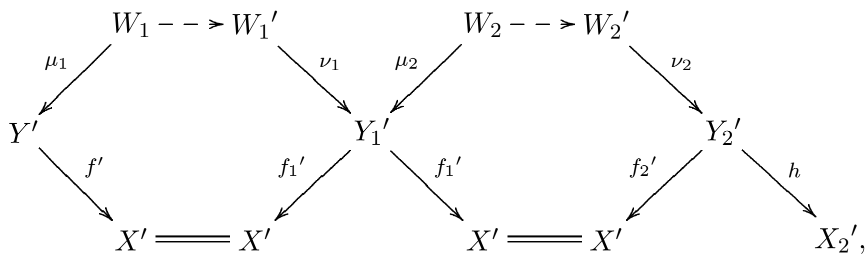
(*No.* 4.4)



where ${f_1}'$ *is the natural projection.*

(*No.* 4.5–*No.* 4.7)



where ${f_i}'$ *are the natural projections and* h *is the blow-up along* γ*.*

(B)(B-1) *Conversely let* $({Y_1}', m_1)$ *be as in* (A1). *Let* $\nu_1 \colon {W_1}' \to {Y_1}'$ *the blow-up along* m_1*. Then there is a flop* ${W_1}' \dashrightarrow W_1$ *over* X' *and an extremal contraction of* $(2,1)$*-type* $\mu_1 \colon W_1 \to Y'$ *over* X'*. Let* l_0 *be the image of* μ_1*-exceptional divisor and* L *the pull-back of a line of* X' *on* Y'*.*

(B-2) *Conversely let* $({X_2}', m_1, m_2, \gamma)$ *be as in* (A2). *Let* $h \colon {Y_2}' \to {X_2}'$ *be the blow-up of* ${X_2}'$ *along* γ *and* $\nu_2 \colon {W_2}' \to {Y_2}'$ *the blow-up along* m_2*. Then there is a flop* ${W_2}' \dashrightarrow W_2$ *over* X' *and an extremal contraction of* $(2,1)$*-type* $\mu_2 \colon W_2 \to {Y_1}'$ *over* X'*. Let* l_1 *be the image of* μ_2*-exceptional divisor. Let* $\nu_1 \colon {W_1}' \to {Y_1}'$ *the blow-up along* m_1*. Then there is a flop* ${W_1}' \dashrightarrow W_1$ *over* X' *and an extremal contraction of* $(2,1)$*-type* $\mu_1 \colon W_1 \to Y'$ *over* X'*. Let* l_0 *be the image of* μ_1*-exceptional divisor and* L *the pull-back of a line of* X' *on* Y'*.*

Then the following hold.

(i) *There is a unique member* $\widetilde{E}$ *of* $|-K_{Y'} - L|$ *and* $\widetilde{E}$ *is irreducible,*

(ii) Y', l_i *and* $\widetilde{E}$ *satisfy the conditions of Proposition* 0.8. *Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold obtained as in Proposition* 0.8. *Then* X *is of No.* 4.4 *in the case* (B1), *or No.* 4.5*–No.* 4.7 *in the case* (B2).

(C) *There exists an example of* $(Y_2{}', m_1)$ *as in* (A1) (*resp.* $(X_2{}', m_1, m_2, \gamma)$ *as in* (A2)) *and hence there exists a* $\mathbb{Q}$*-Fano* 3*-fold of No.* 4.4 (*resp. No.* 4.5–4.7).

THEOREM 0.19. (Table 4, IV) (A) *Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold of No.* 4.8*. Then the following hold.* Y' *is a weak Fano* 3*-fold with* $(-K_{Y'})^3 = 14$ *and a* $\mathbb{P}^1$*-bundle structure over* $\mathbb{P}^2$*.* $\widetilde{E}$ *is an irreducible divisor which is generically a* 2*-section such that* $(-K_{Y'})^2\widetilde{E} = 1$ *and* $(-K_{Y'})\widetilde{E}^2 = -2$.

(B) *Conversely let* $(Y', \widetilde{E})$ *be as in* (A)*. Then they satisfy the conditions of Proposition* 0.8*. Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold obtained as in Proposition* 0.8*. Then* X *is of No.* 4.8.

(C) *There exists an example of* $(Y', \widetilde{E})$ *as in* (A) *and hence there exists a* $\mathbb{Q}$*-Fano* 3*-fold of No.* 4.8.

THEOREM 0.20. (Table 5, I) (*No.* 5.2) *Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold of No.* 5.2*. Then* $X \simeq ((3,3) \subset \mathbb{P}(1^5, 2))$*. Moreover by* [Take99]*,* Y' *is embedded in* $\mathbb{P}(\mathscr{O}_{\mathbb{P}_1}^{\oplus 3} \oplus \mathscr{O}_{\mathbb{P}_1}(1))$ *as a divisor linearly equivalent to* $3H + F$*, where* H *is the tautological divisor and* F *is a fiber.*

(*No.* 5.3) (A) *Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold of No.* 5.3*. Then the following hold.* Y' *is a smooth complete intersection of two members of* $|2H - F|$ *of* $\mathbb{P}(\mathcal{E})$*, where* $\mathcal{E} := \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 3} \oplus \mathscr{O}_{\mathbb{P}^1}(2)$*,* H *is the tautological divisor of* $\mathbb{P}(\mathcal{E})$*,* F *is a fiber of the natural projection* $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$.

(B) *Conversely* Y' *is given as in* (A)*. Then the following hold.*

(i) $\rho(Y') = 2$.

(ii) $\mathrm{Bs}|-K_{X'}| = l_0$*, where* l_0 *is the section associated to the surjection* $\mathcal{E} \to \mathscr{O}_{\mathbb{P}^1}$.

(iii) *There exists a unique member* $\widetilde{E}$ *of* $|(H - 2F)|_{Y'}|$ *and* $\widetilde{E}$ *is irreducible.*

(iv) Y'*,* l_0 *and* $\widetilde{E}$ *satisfy the conditions of Proposition* 0.8*. Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold obtained as in Proposition* 0.8*. Then* X *is of No.* 5.3.

(C) *There exists an example of* Y' *as in* (B) *and hence there exists a* $\mathbb{Q}$*-Fano* 3*-fold of No.* 5.3.

THEOREM 0.21. (Table 5, II) (A) *Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold of No.* 5.4 *or No.* 5.5*. Let* $\mu_1 \colon W_1 \to Y'$ *be the blow-up along a flipped curve* l_0*. Then there is a flop* $W_1 \dashrightarrow W_1{}'$ *over* X' *and an extremal contraction*

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of $(2,1)$*-type* $\nu_1 \colon {W_1}' \to {Y_1}'$ *over* X'*. Let* ${F_1}''$ *be the strict transform of* μ_1*-exceptional divisor on* ${Y_1}'$ *and* m_1 *the image of* ν_1*-exceptional divisor.*

(A-1) *Assume that* X *is a* $\mathbb{Q}$*-Fano* 3*-fold of No.* 5.4*. Then*

(1) ${Y_1}' \simeq \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2})$,

(2) ${F_1}''$ *is a surface linearly equivalent to* $2H + L_1$*, where* H *is the tautological divisor of* ${Y_1}'$ *and* L_1 *is a fiber of the natural projection* ${f_1}' \colon {Y_1}' \to X' \simeq \mathbb{P}^1$*, and*

(3) m_1 *is a curve on* ${F_1}''$ *with* $g(m_1) = 9$ *and* $(-K_{{Y_1}'} \cdot m_1) = 33$ *such that* $\mathrm{Bs}|-K_{{Y_1}'} - m_1| = m_1 \cup l_1$ *and* m_1 *and* l_1 *intersect at one point simply, where* l_1 *is the section of* ${f_1}'$ *associated to the surjection* $\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2} \to \mathscr{O}_{\mathbb{P}^1}$.

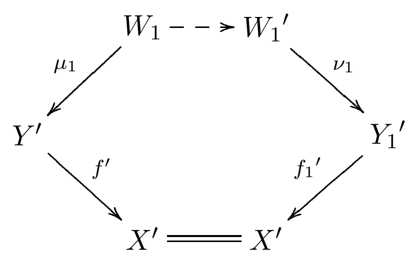
(A-2) *Assume that* X *is a* $\mathbb{Q}$*-Fano* 3*-fold of No.* 5.5*. Then the following hold.*

(1) ${Y_1}'$ *is a smooth divisor in* $\mathbb{P} := \mathbb{P}(\mathscr{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2})$ *linear equivalent to* $2H$*, where* H *is the tautological divisor. Let* V *be the subvariety of* $\mathbb{P}$ *associated to the surjection* $\mathscr{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2} \to \mathscr{O}_{\mathbb{P}^1}^{\oplus 2}$*. Then* $V \cap {Y_1}'$ *is a disjoint union of two sections* l_i *of* ${f_1}'$*.*

(2) ${F_1}''$ *is a surface linearly equivalent to* $2H|_{{Y_1}'} + L_1$*, where* L_1 *is a fiber of the natural morphism* ${f_1}' \colon {Y_1}' \to X'$*.*

(3) m_1 *is a curve on* ${F_1}''$ *with* $g(m_1) = 3$ *and* $(-K_{{Y_1}'} \cdot m_1) = 16$ *such that* $\mathrm{Bs}|-K_{{Y_1}'} - m_1| = m_1 \cup l_1 \cup l_2$ *and* m_1 *and* l_i $(i = 1, 2)$ *intersect at one point simply.*

Consequently we obtain the following diagram.



(B) *Conversely let* $({Y_1}', {F_1}'', m_1, l_i\,(i \geq 1))$ *be as in* (A1) *for No.* 5.4 (*resp.* (A2) *for No.* 5.5)*. Let* $\nu_1 \colon {W_1}' \to {Y_1}'$ *be the blow-up along* m_1*. Then there is a flop* ${W_1}' \dashrightarrow W_1$ *over* X' *and an extremal contraction of* $(2,1)$*-type* $\mu_1 \colon W_1 \to Y'$ *over* X'*. Let* l_0 *be the image of* μ_1*-exceptional* *divisor and L a fiber of the natural projection $f'\colon Y' \to X'$. Then the following hold.*

(i) *There is a unique member $\widetilde{E}$ of $|-K_{Y'} - L|$ and $\widetilde{E}$ is irreducible.*
(ii) *Y', l_i and $\widetilde{E}$ satisfy the conditions of Proposition* 0.8. *Let X be a $\mathbb{Q}$-Fano 3-fold obtained as in Proposition* 0.8. *Then X is of No.* 5.4 (*resp. No.* 5.5).

(C) *There exists an example of $({Y_1}', {F_1}'', m_1, l_i)$ be as in* (A1) *for No.* 5.4 (*resp.* (A2) *for No.* 5.5) *and hence there exists a $\mathbb{Q}$-Fano 3-fold of No.* 5.4 (*resp. No.* 5.5).

Remark 0.22. Examples are not known for No. 5.1.

Moreover in Section 5, we deny the possibilities of $\mathbb{Q}$-Fano 3-folds in Tables $1'$–$5'$ in Theorem 0.3.

Acknowledgements. I express my sincere thanks to Professor Shigefumi Mori for stimulating discussion about the construction of examples of $\mathbb{Q}$-Fano 3-folds. This paper is part of my doctoral thesis. I express my gratitude to my thesis advisor Professor Yujiro Kawamata for giving me useful comments. I am grateful to Dr. Tatsuhiro Minagawa for stimulating discussion about the deformation of $\mathbb{Q}$-Fano 3-folds, and also to Professor Miles Reid for his interest on my paper and for showing me a different point of view. I am grateful to Dr. Natsuo Saito for giving me TEX advice. I was partially supported by the Japan Society for the Promotion of Science.

§1. On existence of an anti-canonical divisor with only canonical singularities for a $\mathbb{Q}$-Fano 3-fold as in Theorem 0.3

Theorem 1.0. *Let X be as in Theorem* 0.3 *and P an index* 2 *point satisfying* (5) *of Main Assumption* 0.1. *Let $f\colon Y \to X$ be the weighted blow-up with weights $\frac{1}{2}(1,1,1,2)$ and E the exceptional divisor. If $n \geq 2$ for P, then let Q be the unique index* 2 *point on E. Then the following hold.*

(1) $H^0(\mathscr{O}_Y(-K_Y)) \to H^0(\mathscr{O}_E(-K_Y|_E))$ *is surjective. Moreover if $n \geq 2$, then* $\mathrm{Bs}|-K_Y| = \{Q\}$ *near E or if $n = 1$, then* $\mathrm{Bs}|-K_Y| = \emptyset$ *near E.*
(2) $\mathrm{Bs}|-K_X| = \{P\}$ *near P.*

Proof. Consider the exact sequence

$$0 \longrightarrow \mathscr{O}_Y(-K_Y - E) \longrightarrow \mathscr{O}_Y(-K_Y) \longrightarrow \mathscr{O}_E(-K_Y|_E) \longrightarrow 0.$$

To see that the map $H^0(\mathscr{O}_Y(-K_Y)) \to H^0(\mathscr{O}_E(-K_Y|_E))$ is surjective, it suffices to prove $h^0(\mathscr{O}_Y(-K_Y - E)) = h - 3$ by [Taka02, Proposition 2.3]. Note that this is equivalent to $h^0(\mathscr{O}_{Y'}(-K_{Y'} - \widetilde{E})) = h - 3$ (we use the notation of Theorem 0.3). We can prove this using the data of Tables 1–5 and 1′–5′ in Theorem 0.3 as follows.

Tables 1 and 1′: We have $-K_{Y'} - \widetilde{E} \sim f'^* D$, where D is a primitive ample Weil divisor (we can easily see that the linear equivalent class of D is unique). Hence $h^0(-K_{Y'} - \widetilde{E}) = h^0(D)$. $h^0(D) = h - 3$ is easy to see.

Tables 2, 2′ and 3: We have $-K_{Y'} - \widetilde{E} \sim E'$ whence $h^0(-K_{Y'} - \widetilde{E}) = 1 = h - 3$.

Table 4 and 4′: Since $-K_{Y'} - \widetilde{E} - K_{Y'}$ is nef and big, we can compute $h^0(-K_{Y'} - \widetilde{E})$ by Riemann-Roch theorem and we are done. But if $h = 5, 6$, then we can prove this more directly as follows. If $h = 5$, then $h^0(-K_{Y'} - \widetilde{E}) = h^0(f'_* \mathscr{O}_{Y'}(-K_{Y'} - \widetilde{E})) = h^0(\mathscr{O}_{X'}(l)) = 2 = h - 3$, where l is a ruling of X'. Similarly if $h = 6$, then $h^0(-K_{Y'} - \widetilde{E}) = h^0(L) = 3 = h - 3$.

Table 5 and 5′: Since $-K_{Y'} - \widetilde{E} - K_{Y'}$ is nef and big, we can compute $h^0(-K_{Y'} - \widetilde{E})$ by Riemann-Roch theorem and we are done.

Note that

$$(Y, Q) \simeq (\{xy + z^2 + u^{a-1} = 0\}/\mathbb{Z}_2(1,1,1,0), o).$$

Hence inductively we can construct the sequence of the weighted blow-ups with weights $\frac{1}{2}(1,1,1,2)$ at index 2 points on exceptional divisors and denote it by

$$Z_a \xrightarrow{f_a} Z_{a-1} \xrightarrow{f_{a-1}} \cdots \xrightarrow{f_2} Z_1 := Y$$

and set $f_1 := f$. Let F_i be the f_i-exceptional divisor and $F_i{}'$ its strict transform on Z_{i+1}. We prove that $H^0(\mathscr{O}_{Z_i}(-K_{Z_i})) \to H^0(\mathscr{O}_{F_i}(-K_{Z_i}|_{F_i}))$ is surjective. For $i = 1$, we proved the claim as above. Assume that the assertion holds for $i - 1$. Then by $H^0(\mathscr{O}_{Z_i}(-K_{Z_i})) \simeq H^0(\mathscr{O}_{Z_{i-1}}(-K_{Z_{i-1}}))$ and $H^0(\mathscr{O}_{F_{i-1}}(-K_{Z_{i-1}}|_{F_{i-1}})) \simeq H^0(\mathscr{O}_{F_{i-1}'}(-K_{Z_i}|_{F_{i-1}'}))$, $H^0(\mathscr{O}_{Z_i}(-K_{Z_i})) \to H^0(\mathscr{O}_{F_{i-1}'}(-K_{Z_i}|_{F_{i-1}'}))$ is also surjective. Note that

$$H^0(\mathscr{O}_{F_{i-1}'}(-K_{Z_i}|_{F_{i-1}'})) \simeq H^0(\mathscr{O}_{F_{i-1}' \cap F_i}(-K_{Z_i}|_{F_{i-1}' \cap F_i}))$$

and

$$H^0(\mathscr{O}_{F_i}(-K_{Z_i}|_{F_i})) \simeq H^0(\mathscr{O}_{F_{i-1}' \cap F_i}(-K_{Z_i}|_{F_{i-1}' \cap F_i})).$$

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Hence $H^0(\mathscr{O}_{Z_i}(-K_{Z_i})) \to H^0(\mathscr{O}_{F_i}(-K_{Z_i}|_{F_i}))$ is surjective. Hence we know that for the exceptional set F for $Z_a \to X$, $\mathrm{Bs}|-K_{Z_a}| \cap F = \emptyset$. Since $H^0(\mathscr{O}_X(-K_X)) \simeq H^0(\mathscr{O}_Y(-K_Y)) \simeq H^0(\mathscr{O}_{Z_a}(-K_{Z_a}))$, we finish the proof of Theorem 1.0. ∎

Proposition 1.1. *Let X be a klt weak $\mathbb{Q}$-Fano 3-fold satisfying the following conditions.*

(1) $|-K_X| \neq \emptyset$.
(2) *There are a finite number of non-Gorenstein points on X.*
(3) *There is a member of $|-K_X|$ which is normal near non-Gorenstein points.*

Then $|-K_X|$ has a member which is normal and has only canonical singularities outside non-Gorenstein points of X.

Proof. The proof is almost the same as one of [Amb99, Main Theorem] or [Mel99, Theorem 1]. So we only give an outline of the proof. Let $U := \{x \mid x \text{ is a Gorenstein point of } X\}$. Let S be a general member of $|-K_X|$. Let $\gamma := \max\{t \mid K_X + tS|_U \text{ is log canonical}\}$. It suffices to prove that if there is an element of $\mathrm{CLC}(K_X + \gamma S|_U)$ contained in $\mathrm{Bs}|-K_X|$, it is $S|_U$. Assume the contrary and let Z be a minimal element of $\mathrm{CLC}(K_X + \gamma S|_U)$ contained in $\mathrm{Bs}|-K_X|$. By the assumption (3), Z is a complete variety. Hence by using [Taka02, Theorem 1.0] (KKV vanishing theorem), we know that it suffices to prove $H^0(\mathscr{O}_Z(-K_X|_Z)) \neq 0$. It is done by Adjunction Theorem and a non-vanishing argument. ∎

Corollary 1.2. *Let X be a $\mathbb{Q}$-Fano 3-fold with Main Assumption 0.1 and assume moreover that any index 2 point satisfies (5). Then $|-K_X|$ has a member with only canonical singularities.*

Proof. Fix an index 2 point P and the weighted blow-up f as in Theorem 1.0. By Theorem 1.0 and Proposition 1.1, we can find a member $S \in |-K_Y|$ such that S is normal and has only canonical singularities outside index 2 points of Y. But by Theorem 1.0 again, S has only canonical singularities near E. So since $f|_S$ is crepant, $f(S)$ has only canonical singularity outside index 2 points of X except P. Since P is any index 2, we can find a member of $|-K_X|$ with only canonical singularities. ∎

§2. On deformations of ℚ-Fano 3-folds as in Theorem 0.3

Our starting point in this section is the following theorem proved by T. Minagawa:

THEOREM 2.0. (T. Minagawa) *Let X be a $\mathbb{Q}$-Fano 3-fold (resp. weak $\mathbb{Q}$-Fano 3-fold) with $I(X) = 2$. Assume that there exists a smooth member of $|-2K_X|$. Then there exists a flat family $\mathfrak{f}\colon \mathfrak{X} \to (\Delta, 0)$ over a 1-dimensional disc $(\Delta, 0)$ such that $X \simeq \mathfrak{f}^{-1}(0)$ and $\mathfrak{f}^{-1}(t)$ is a $\mathbb{Q}$-Fano 3-fold (resp. a weak $\mathbb{Q}$-Fano 3-fold) with only ODP's, QODP's or $\frac{1}{2}(1,1,1)$-singularities as its singularities for $t \in \Delta\backslash\{0\}$.*

Proof. See [Min01, Theorem 2.4]. □

THEOREM 2.1. *Let X be a (not necessarily $\mathbb{Q}$-factorial) weak $\mathbb{Q}$-Fano 3-fold with $I(X) = 2$. Assume that*

(1_X) *$h^0(-K_X) \geq 4$,*
(2_X) *near an index 2 point P, $\mathrm{Bs}|-K_X| = \{P\}$, and*
(3_X) *there is no divisor contracted to a point by the morphism defined by $|-mK_X|$ for $m \gg 0$.*

Then X can be deformed to a weak $\mathbb{Q}$-Fano 3-fold with only $\frac{1}{2}(1,1,1)$-singularities as its singularities.

Proof. By (1_X) and [Taka02, Theorem 4.1], $|-2K_X|$ is free. So by Theorem 2.0, we may assume that X has ODP's, $\frac{1}{2}(1,1,1)$-singularities or QODP's as its singularities. Let $f\colon Y \to X$ be the composite of the weighted blow-ups at all QODP's of X (as in Theorem 0.3), $g\colon Z \to Y$ the composite of the blow-ups at all $\frac{1}{2}(1,1,1)$-singularities of Y and $h := g \circ f$. Then by the choice of h and (2_X), $-K_Z$ is nef. Moreover by (1_X), we have $h^0(-K_Z) = h^0(-K_X) \geq 4$. Hence by Riemann-Roch theorem, $(-K_Z)^3 > 0$. So Z is a Gorenstein weak Fano 3-fold. We verify that the assumption (3_Z) holds. Assume that there is a divisor S on Z which is contracted to a point by the morphism defined by $|-mK_Z|$. By the choice of h, S is not h-exceptional since an h-exceptional divisor contains a curve negative for K_Z. If $E \cap S \neq \emptyset$ for a prime h-exceptional divisor E, then $E \cap S$ is a curve since E is a Cartier divisor. By the nature of S, $E \cap S$ is numerically trivial for $-K_Z$. So since E contains a curve which is numerically trivial for $-K_Z$, E must be the strict transform of an f-exceptional divisor E' and moreover S must intersect also the g-exceptional

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divisor F of the blow-up at the $\frac{1}{2}(1,1,1)$-singularity on E'. Then $S \cap F$ is numerically positive for $-K_Z$, a contradiction. Hence S is disjoint from h-exceptional divisors. However, $h(S)$ is contracted to a point by the morphism defined by $|-mK_X|$, a contradiction. Hence Z satisfies (3_Z).

Step 1. smoothing ODP's. We prove that Z is smoothable by the same method as [Nam97]. Note that the following claim (which is [Nam97, Proposition 2]) holds for a weak Fano 3-fold Z satisfying (3_Z) without any change in his proof.

Claim. *Let D be a member of $|-K_Z|$ with only canonical singularities. Then* $\operatorname{Pic} Z \to \operatorname{Pic} D$ *is an injection.*

Let $\mathfrak{Z} \to \Delta$ be a 1-parameter smoothing of Z. Then by [KM92, Proposition 11.4], we obtain the deformation $\mathfrak{Y} \to \Delta$ of Y which satisfies the commutative diagram

$$\begin{array}{ccc} \mathfrak{Z} & \longrightarrow & \mathfrak{Y} \\ & \searrow \quad \swarrow & \\ & \Delta. & \end{array}$$

Then $\mathfrak{Z}_t \to \mathfrak{Y}_t$ is a composite of $(2,0)_4$ type contractions for $t \in \Delta$ since a contraction of type $(2,0)_4$ is stable under a deformation by [Kod63]. Hence Y_t has only $\frac{1}{2}(1,1,1)$-singularities as its singularities. Similarly we can prove that this can be blown down to a smoothing of ODP's of X using [KM92, the proof of Theorem 12.3.1].

Step 2. deforming QODP's to $\frac{1}{2}(1,1,1)$-singularities. By induction, we only have to deform one QODP to two $\frac{1}{2}(1,1,1)$-singularities. Let P be a QODP of X and E (resp. F) the strict transform of f-exceptional divisor over P (resp. g-exceptional divisor over P). Then $E \simeq \mathbb{F}_4$ and there exists a primitive crepant birational morphism $p\colon Y \to W$ which contracts E to a smooth rational curve. Then by [Min99, Proposition 3.5 (i)], there is a small deformation

$$\begin{array}{ccc} \mathfrak{Y} & \longrightarrow & \mathfrak{W} \\ & \searrow \quad \swarrow & \\ & \Delta & \end{array}$$

of p such that $\mathfrak{Y}_t \to \mathfrak{W}_t$ is an isomorphism for $t \neq 0$. By the same argument as Step 1, we have a birational morphism $\mathfrak{Y}_t \to \mathfrak{X}_t$, where $\mathfrak{X}_t$ is a deformation of X. If there exists a QODP which specializes to P, then $\mathfrak{Y}_t \to \mathfrak{W}_t$

must be a contraction of the same type as p, a contradiction. Hence P is deformed to two $\frac{1}{2}(1,1,1)$-singularities. $\square$

Corollary 2.2. *Let X be a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-fold with (1)–(4) in Main Assumption 0.1. Then X can be deformed to a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-fold X' with (1)–(4) in Main Assumption 0.1 and with only $\frac{1}{2}(1,1,1)$-singularities as its singularities.*

Proof. The proof is similar to the next corollary. $\square$

Corollary 2.3. *Let X be a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-fold with (1)–(4) in Main Assumption 0.1. Let $N := \mathrm{aw}(X)$. Then if $N > 1$ (resp. $N = 1$), X can be transformed to a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-fold $\widetilde{Z'}$ with (1)–(4) in Main Assumption 0.1 and with only $\frac{1}{2}(1,1,1)$-singularities as its singularities and $h^0(-K_{\widetilde{Z'}}) = h$ and $\mathrm{aw}(\widetilde{Z'}) = N-1$ (resp. a smooth Fano 3-fold $\widetilde{Z'}$ with $\rho(\widetilde{Z'}) = 1$, $F(\widetilde{Z'}) = 1$ and $h^0(-K_{\widetilde{Z'}}) = h$) as follows.*

$$\begin{array}{ccccccc} & & & \widetilde{Y} & & & \\ & & \tilde{f}\swarrow & & \searrow\tilde{g} & & \\ X & \xrightarrow{\text{def}} & \widetilde{X} & & & \widetilde{Z} \xrightarrow{\text{def}} & \widetilde{Z'}, \end{array}$$

where $* \xrightarrow{\text{def}} **$ means that $**$ is a small deformation of $*$,

$\widetilde{X}$ is a $\mathbb{Q}$-Fano 3-fold with Main Assumption 0.1 and with only ODP's, QODP's or $\frac{1}{2}(1,1,1)$-singularities as its singularities,

$\tilde{f}\colon \widetilde{Y} \to \widetilde{X}$ is similarly chosen to f in Theorem 0.3, and $\tilde{g}\colon \widetilde{Y} \to \widetilde{Z}$ be the anti-canonical model.

Proof. By Theorem 2.0, there is a deformation $X \xrightarrow{\text{def}} \widetilde{X}$ as stated above. Note that we may assume that the $\mathbb{Q}$-factoriality is preserved by [KM92, Theorem 12.1.10]. Moreover by Tables 1–5 and 1′–5′ of Theorem 0.3, and the result of [San95], [San96], $\widetilde{X}$ satisfies Main Assumption 0.1. By Theorem 0.3, we obtain $\widetilde{Y}$ and $\widetilde{Z}$ as above and $\rho(\widetilde{Z}) = 1$. If $N = 1$ and $h = 4$, then $\widetilde{Z}$ may have canonical singularities but in this case $\Phi_{|-K_X|}$ is a double cover of $\mathbb{P}^3$ by [Muk95, Theorem 6.5 and Proposition 7.8] and so $\widetilde{Z}$ has a smoothing $\widetilde{Z'}$. Except this case, we apply Theorem 2.1 for $\widetilde{Z}$. We only have to check that $\widetilde{Z}$ satisfies the assumption $(2_{\widetilde{Z}})$. By Theorem 1.0, $(2_{\widetilde{Y}})$ holds and so does $(2_{\widetilde{Z}})$. Hence $\widetilde{Z}$ can be deformed to a $\mathbb{Q}$-Fano 3-fold

https://doi.org/10.1017/S0027763000025472 Published online by Cambridge University Press

$\widetilde{Z'}$ with only QODP's or $\frac{1}{2}(1,1,1)$-singularities as its singularities. Next we show $\widetilde{Z'}$ has the properties as stated above. By [KM92, the proof of Corollary 12.3.4], we have $\rho(\widetilde{Z'})=1$. If $N>1$, $F(\widetilde{Z'})=1/2$ by Tables 1–5 and $1'$–$5'$ of Theorem 0.3 and [San95], [San96]. If $N=1$, we have clearly $F(\widetilde{Z'})=1$. Hence we are done. □

The following is similar to Shokurov's theorem [Sho79a]:

Corollary 2.4. *Let X be a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-fold with* (1)–(4) *of Main Assumption* 0.1. *Then for any index* 2 *point P, there exists a smooth rational curve l through P such that $-K_X \cdot l = 1/2$.*

Proof. First we treat the case that any index 2 point is of type as in Main Assumption 0.1 (5). By Tables 1–5 and $1'$–$5'$ in Theorem 0.3, e is positive or f' is a crepant divisorial contraction for any choice of an index 2 point P. Let $g\colon Y \to Z$ be the anti-canonical model. Let l' be a flopping curve if g is a flopping contraction or a general fiber of E' if g is a crepant divisorial contraction. Then in the former case, by [Taka02, Lemma 4.3], (resp. in the latter case, by the proof of Theorem 0.3 (see [Taka02]) in Case 5), we have $g(E) \simeq E$ whence $E.l'=1$. Hence $l := f(l')$ is what we want.

Next we treat the general case. Let $\mathfrak{f}\colon \mathfrak{X} \to \Delta$ be a flat family as in Theorem 2.0. By [KM92, Corollary 12.3.4], $\rho(\mathfrak{X}_t)=1$ and moreover by Tables 1–5 and $1'$–$5'$, [San95] and [San96], $\mathfrak{X}_t$ $(t \neq 0)$ satisfies Main Assumption 0.1. Let P be an index 2 point on X and P_t an index 2 point on $\mathfrak{X}_t$ which specializes to P. By the first part of this proof, there is a curve l_t on $\mathfrak{X}_t$ $(t \neq 0)$ such that $l_t \simeq \mathbb{P}^1$, $P_t \in l_t$ and $-K_{\mathfrak{X}_t} \cdot l_t = 1/2$. Since there are only countably many components of relative Hilbert scheme $\mathrm{Hilb}(\mathfrak{X}/\Delta)$, we may assume that they form a flat family over Δ. Moreover by the properness of a component of relative Hilbert scheme, this family extends over 0. Let l be its fiber over 0. Then l is what we want. □

§3. Embedding $\mathbb{Q}$-Fano 3-folds as in Theorem 0.3 into weighted projective spaces

The next result is a first step for the classification of Mukai's type [Muk95, Theorem 1.10].

Theorem 3.0. *Let X be a* (*not necessarily $\mathbb{Q}$-factorial*) *canonical $\mathbb{Q}$-Fano* 3-*fold of $I(X)=2$. Assume that X has the following properties.*

(1) $|-K_X|$ *is indecomposable, i.e.,* $|-K_X|$ *contains no member which is a sum of two movable Weil divisors.*
(2) $|-K_X|$ *has no base curve containing an index* 2 *point.*
(3) $|-K_X|$ *has a member with only canonical singularities.*
(4) $h^0(X, \mathscr{O}(-K_X)) \geq 4$.
(5) *Any index* 2 *point of* X *is* $\frac{1}{2}(1,1,1)$*-singularity.*

Then except the following two cases (*a*) *and* (*b*), X *is embedded into* $\mathbb{P}(1^h, 2^N)$ *and* $-K_X$ *is the restriction of* $\mathscr{O}(1)$, *where* $h := h^0(-K_X)$ *and* N *is the number of* $\frac{1}{2}(1,1,1)$*-singularities.*

(a) $\Phi_{|-K_X|}$ *is a double cover of* $\mathbb{P}^3$ *branched along a sextic.*
(b) $\Phi_{|-K_X|}$ *is a double cover of a quadric hypersurface branched along the intersection with a quartic.*

(*Note that in case* (*a*),

$$X \simeq ((6) \subset \mathbb{P}(1^4, 3)).$$

Note also that in case (*b*),

$$X \simeq ((2,4) \subset \mathbb{P}(1^5, 2)).$$

The number of weight 2 *is not equal to the number of index* 2 *point.*)

Moreover $\bigoplus_{m=0}^{\infty} H^0(X, -mK_X)$ *is generated by elements of degree* ≤ 2 *and related by elements of degree* ≤ 6.

If $h = 4$ *and* $N = 1$, *then* $X \simeq ((5) \subset \mathbb{P}(1^4, 2))$.
If $h = 4$ *and* $N = 2$, *then* $X \simeq ((3,4) \subset \mathbb{P}(1^4, 2^2))$.
If $h = 5$ *and* $N = 1$, *then* $X \simeq ((3,3) \subset \mathbb{P}(1^5, 2))$.

Proof. We prove this by induction on N.

In case $N = 0$, the assertion follows from [Muk95, Theorem 6.5 and Proposition 7.8]. Next we prove that if the assertion holds in case X has $N-1$ $\frac{1}{2}(1,1,1)$-singularities, then so does it in case X has N $\frac{1}{2}(1,1,1)$-singularities. Let X be a $\mathbb{Q}$-Fano 3-fold satisfying the assumptions of this theorem and with N $\frac{1}{2}(1,1,1)$-singularities. Let $f\colon Y \to X$ be the blow-up at a $\frac{1}{2}(1,1,1)$-singularity. Let E be the exceptional divisor of f. Then Y is a weak $\mathbb{Q}$-Fano 3-fold by [Taka02, Proposition 4.2]. By the assumption (4), Y is not a $\mathbb{Q}$-Fano 3-fold. Let $g\colon Y \to Z$ be the anti-canonical model and $\overline{E} := g(E)$.

Claim 1. *Z satisfies the assumption of this theorem and has $N-1$ $\frac{1}{2}(1,1,1)$-singularities.*

Proof. By $-K_Y = g^*(-K_Z)$, if $|-K_Z|$ is decomposable, $|-K_X|$ must be decomposable, a contradiction. Hence (1) is satisfied. By (2) for X, neither $|-K_Y|$ has a base curve containing an index 2 point. Hence any g-exceptional curve does not contain an index 2 point. So by $-K_Y = g^*(-K_Z)$, (2) is satisfied and (5) is also satisfied. Let D be a member of $|-K_X|$ with only canonical singularities. Then the strict transform D' of D on Y has the same property since $D' \to D$ is crepant. Since $D' \to g(D')$ is crepant, $g(D')$ has also the same property. Hence (3) is satisfied. By $-K_Y = g^*(-K_Z)$ and $h^0(-K_Y) = h^0(-K_X)$, we know that (4) is satisfied. □

Hence by the assumption of the induction, one of the following three cases occurs.
Case α. $Z \subset \mathbb{P}(1^h, 2^{N-1})$ and $-K_Z = \mathscr{O}_Z(1)$.
Case β. Z is of type (a).
Case γ. Z is of type (b).

Claim 2. $\mathrm{Bs}|-K_X|$ *coincides with $\frac{1}{2}(1,1,1)$-singularities as a set.*

Proof. If $N = 0$, the assertion follows from [Muk95, Theorem 6.5 and Proposition 7.8]. Hence by Claim 1, the assertion follows by induction with respect to the number of $\frac{1}{2}(1,1,1)$-singularities. □

Case α. We first show that $\overline{E} \simeq E$. By the proof of Claim 2, the assertion similar to Claim 2 holds for $\mathrm{Bs}|-K_Y|$. Hence $H^0(\mathscr{O}_Y(-K_Y)) \to H^0(\mathscr{O}_E(-K_Y)) \simeq H^0(\mathscr{O}_{\mathbb{P}^2}(1))$ is surjective. Hence $H^0(\mathscr{O}_Y(-mK_Y)) \to H^0(\mathscr{O}_E(-mK_Y))$ is also surjective for all $m \geq 0$ since $\bigoplus_{m\geq 0} H^0(\mathscr{O}_{\mathbb{P}^2}(m))$ is simply generated. So $\overline{E} \simeq E$ since g is defined by $|-mK_Y|$ for some $m > 0$.

We note here that there is an elementary transformation $\mathbb{P}(1^h, 2^N) \dashrightarrow \mathbb{P}(1^h, 2^{N-1})$ which is decomposed as follows. Let $\mathbb{P}$ be the projective bundle over $\mathbb{P}(1^h, 2^{N-1})$ whose associated vector bundle is $\mathscr{O} \oplus \mathscr{O}(-2)$ and T the effective tautological divisor (which is unique). Let a be the contraction morphism of T. Then the image of $\mathbb{P}$ by a is isomorphic to $\mathbb{P}(1^h, 2^N)$. Let $b\colon \mathbb{P} \to \mathbb{P}(1^h, 2^{N-1})$ be the natural projection. Then our elementary transformation is $b \circ a^{-1}$.

We seek a natural morphism $Y \to \mathbb{P}$. For this, we prove that there is a natural surjection $g^*(\mathscr{O}_Z \oplus \mathscr{O}_Z(-2)) \to \mathscr{O}_Y(E)$.

There is a natural injection $\mathscr{O}_Y(-E) \to \mathscr{O}_Y$ which represents $\mathscr{O}_Y(-E)$ as the ideal sheaf of E. By [Taka02, Theorem 4.1], there is a member $S \in |-2K_X|$ such that $f^*S \cap E = \emptyset$. Associated to S, there is an injection $\mathscr{O}_Y(-f^*S) \to \mathscr{O}_Y$. This gives an injection $\mathscr{O}_Y(-E) \to g^*\mathscr{O}_Z(2)$ since $g^*\mathscr{O}_Z(2) \simeq \mathscr{O}_Y(-2K_Y)$ and $-f^*(-2K_X) \sim -(-2K_Y) - E$. Hence we can define an injection $\mathscr{O}_Y(-E) \to g^*(\mathscr{O}_Z \oplus \mathscr{O}_Z(2))$. Since $f^*S \cap E = \emptyset$, the cokernel of this map is locally free and hence the dual of this map is a surjection. Let $\iota \colon Y \to \mathbb{P}$ be the morphism over $\mathbb{P}(1^h, 2^{N-1})$ associated to the surjection $d : g^*(\mathscr{O}_Z \oplus \mathscr{O}_Z(-2)) \to \mathscr{O}_Y(E)$ and $p' := p|_{\iota(Y)}$. Note that ι is finite since E is g-ample.

Claim 3. *$\iota(Y)$ is normal.*

Proof. First we see that $\iota(Y)$ is smooth near $\iota(E)$. Let y' be a point of $\iota(E)$ and $z = p'(y')$. Let $m := g^{-1}(z)_{\mathrm{red}}$. If m is 0-dimensional, then g and p' are isomorphisms over z since Z is normal whence ι is an isomorphism over y'. In particular $\iota(Y)$ is smooth at y' since so is Y at $\iota^{-1}(y')$. So we may assume that m is 1-dimensional. By the surjectivity of d, its restriction to m is also surjective. By $E \simeq \overline{E}$, $m \simeq \mathbb{P}^1$. Hence $\iota|_m$ is isomorphism whence ι is injective on m. Let $y := \iota^{-1}(y')$, $A := \mathscr{O}_{Y,y}$ and $B := \mathscr{O}_{Z,z}$. We will prove that the natural morphism $B[t] \to A$ is surjective, where t is a local parameter of ${p'}^{-1}(z)$ at y'. By this map $B[t] \to A$, t is sent to a local parameter of m and two local parameters of $\iota(E)$ at y' are sent to that of E at y. Hence $B[t] \to A$ is surjective. So ι is an isomorphism over y'.

Next we complete the proof of the claim. It suffices to prove that $\iota_*\mathscr{O}_Y = \mathscr{O}_{\iota(Y)}$. The natural morphism $\mathscr{O}_{\iota(Y)} \to \iota_*\mathscr{O}_Y$ is injective since the kernel is at most a torsion sheaf. Let $\mathcal{C}$ be its cokernel. We prove that ${p'}_*\mathcal{C} = 0$. By the exact sequence

$$0 \longrightarrow \mathscr{O}_{\iota(Y)} \longrightarrow \iota_*\mathscr{O}_Y \longrightarrow \mathcal{C} \longrightarrow 0,$$

we have

$$0 \longrightarrow {p'}_*\mathscr{O}_{\iota(Y)} \longrightarrow {p'}_*\iota_*\mathscr{O}_Y \longrightarrow {p'}_*\mathcal{C} \longrightarrow R^1{p'}_*\mathscr{O}_{\iota(Y)}.$$

Since ${p'}_*\mathscr{O}_{\iota(Y)} \to p_*\iota_*\mathscr{O}_Y$ is an isomorphism by the normality of Z, it suffices to prove that $R^1p_*\mathscr{O}_{\iota(Y)} = 0$. Consider the exact sequence

$$0 \longrightarrow \mathcal{I}_{\iota(Y)} \longrightarrow \mathscr{O}_{\mathbb{P}} \longrightarrow \mathscr{O}_{\iota(Y)} \longrightarrow 0.$$

Since the dimension of a fiber of $p \leq 1$, we have $R^2 p_* \mathcal{I}_{\iota(Y)} = 0$. Since $\mathbb{P}$ is a $\mathbb{P}^1$-bundle, we have $R^1 p_* \mathscr{O}_{\mathbb{P}} = 0$. Thus we obtain $R^1 {p'}_* \mathscr{O}_{\iota(Y)} = 0$ whence ${p'}_* \mathcal{C} = 0$.

Since every fiber of $g\colon Y \to Z$ intersects $\iota(E)$ and $\iota(Y)$ is smooth at points of $\iota(E)$, any fiber is not contained in the singular locus of $\iota(Y)$. Let l be any 1-dimensional fiber of g. By the theorem on formal functions, we have $\mathcal{C} \otimes \mathscr{O}_l = 0$ because $\dim \operatorname{Supp} \mathcal{C} \otimes \mathscr{O}_l = 0$ (note that l is not contained in the singular locus of $\iota(Y)$) and ${p'}_* \mathcal{C} = 0$. Hence by Nakayama's lemma, $\mathcal{C} = 0$. □

Hence $\iota\colon Y \to \iota(Y)$ is finite and birational and $\iota(Y)$ is normal, it is an isomorphism by the Zariski's Main Theorem. Hence $X \simeq a(\iota(Y))$ is naturally embedded into $\mathbb{P}(1^h, 2^N)$ and $-K_X = \mathscr{O}(1)$.

For the next two cases, we directly prove that if $h = 4$ and $N = 1$, then $X \simeq ((5) \subset \mathbb{P}(1^4, 2))$ and if $h = 5$ and $N = 1$, then $X \simeq ((3,3) \subset \mathbb{P}(1^5, 2))$ below, which complete the induction.

Recall that $|-2K_X|$ is free by [Taka02, Theorem 4.1]. So we can take a smooth curve C which is the intersection of general members of $|-K_X|$ and $|-2K_X|$. Let $L := -K_X|_C$. Note that L is a Cartier divisor such that $K_C = 2L$. We describe $R(C, L) := \bigoplus_{m \geq 0} H^0(\mathscr{O}_C(mL))$ by using [Reid90, Theorem 3.4]. By a composite of blow-ups of $\frac{1}{2}(1,1,1)$-singularities and crepant contractions, we can reach a Gorenstein Fano 3-fold W. Let $W' \subset \mathbb{P}(1^h)$ be the image of $\Phi_{|-K_W|}$. Let $\pi\colon C \to C'$ be the restriction of the rational map $X \dashrightarrow W'$ to C.

Assume that W does not satisfy (a) or (b). Then $X \dashrightarrow W'$ is birational whence by choosing C generally, we may assume that π is a birational map. Assume that W satisfies (a) (resp. (b)). Then $W \to W'$ is a double cover of $\mathbb{P}^3$ (resp. a (possibly singular) quadric 3-fold). Since X does not satisfy (a) (resp. (b)), $(-K_X)^3 \geq 5/2$ (resp. $(-K_X)^3 \geq 9/2$) whence $L \cdot C \geq 5$ (resp. $L \cdot C \geq 9$). So by choosing C generally, we may assume that π is a birational map or a double cover of a plane curve of degree ≥ 3 (resp. a space curve of degree ≥ 5).

In any case, C' is not a normal rational curve in $\mathbb{P}(1^{h-1})$. Note that $C' = \Phi_{|L|}(C)$. Hence $H^0(\mathscr{O}_C(L)) \otimes H^0(\mathscr{O}_C(2L)) \to H^0(\mathscr{O}_C(3L))$ is surjective. (Note that $K_C = 2L$.) So by [ibid.], $R(C, L)$ is generated by elements of degree ≤ 2 and related by elements of degree ≤ 6, which in turn show that the same things hold for $\bigoplus_{m \geq 0} H^0(\mathscr{O}_X(-mK_X))$.

Assume that $h = 4$ and $N = 1$. Since $\deg L = 5$, π is birational. By the genus formula of a plane curve, we have $p_a(C') = 6$. On the other hand,

$g(C) = 6$. So π is an isomorphism, which in turn show that $X \simeq ((5) \subset \mathbb{P}(1^4, 2))$.

Assume that $h = 4$ and $N = 2$. If there is a relation of degree 2 in $R(C, L)$, C' is a conic in $\mathbb{P}^2$, a contradiction. Hence there is no relation of degree 2 in $R(C, L)$. Then we can find easily the relation of $R(C, L)$ by [ibid.] and conclude $C \simeq ((3, 4) \subset \mathbb{P}(1^3, 2))$, which in turn shows that $X \simeq ((3, 4) \subset \mathbb{P}(1^4, 2^2))$.

Assume that $h = 5$ and $N = 1$. Since $\deg L = 9$, π is birational. By easy computations, we have $h^0(\mathscr{O}(L)) = 4$, $h^0(\mathscr{O}(2L)) = 10$ and $h^0(\mathscr{O}(3L)) = 18$. Hence there are at least two relations of degree 3 among elements of degree 1. This means that C' is contained in two cubics in $\mathbb{P}^3$ whence $C' \simeq ((3, 3) \subset \mathbb{P}^3)$. We have $g(C) = p_a(C') = 10$. Hence $C \simeq C'$, which in turn shows that $X \simeq ((3, 3) \subset \mathbb{P}(1^5, 2))$.

Now we complete the proof of this theorem. □

Remark 3.1. The assumption that $h^0(-K_X) \geq 4$ is necessary for Theorem 3.0 by the existence of the following.

$$X \simeq ((12) \subset \mathbb{P}(1^3, 4, 6))$$

which satisfies $h^0(-K_X) = 3$.

COROLLARY 3.2. *Let X be a $\mathbb{Q}$-Fano 3-fold with Main Assumption 0.1. Assume that any index 2 point of X is $\frac{1}{2}(1, 1, 1)$-singularity. Then X is embedded into $\mathbb{P}(1^h, 2^N)$ and $-K_X$ is the restriction of $\mathscr{O}(1)$, where $h := h^0(-K_X)$ and N is the number of $\frac{1}{2}(1, 1, 1)$-singularities. Moreover X is an intersection of weighted hypersurfaces of degree ≤ 6.*

If $h = 4$ and $N = 1$, then $X \simeq ((5) \subset \mathbb{P}(1^4, 2))$.

If $h = 4$ and $N = 2$, then $X \simeq ((3, 4) \subset \mathbb{P}(1^4, 2^2))$.

If $h = 5$ and $N = 1$, then $X \simeq ((3, 3) \subset \mathbb{P}(1^5, 2))$.

Proof. By Corollary 2.4, Theorem 1.0 and Corollary 1.2, we can see that the assumptions of Theorem 3.0 are satisfied for X. Hence we are done.

Remark 3.3. There are two possibilities of $f' \colon Y' \to X'$ for $X \simeq ((5) \subset \mathbb{P}(1^4, 2))$. Both of them actually occurs. Indeed, let $f_5(x_0, x_1, x_2, x_3, y) = 0$ be a quintic in $\mathbb{P}(1^4, 2)$, where $\operatorname{wt} x_i = 1$ and $\operatorname{wt} y = 2$. If we take f_5 generally, then $\{f_5 = 0\}$ is an example for No. 3.1. If we take f_5 specially, for example, $f_5 \equiv x_0 y^2 + \sum_{i=0}^{3} x_i{}^5$, then $\{f_5 = 0\}$ is an example for No. 3.1′.

By Corollary 3.2, we can improve [Taka02, Theorem 4.1] and Theorem 1.0 for X as in Corollary 3.2 as follows:

COROLLARY 3.4. *Let X be a $\mathbb{Q}$-Fano 3-fold with Main Assumption* 0.1. *Assume that any non-Gorenstein point is $\frac{1}{2}(1,1,1)$-singularity. Then*

(1) $-2K_X$ *is very ample,*
(2) $\mathrm{Bs}|-K_X|$ *is the union of $\frac{1}{2}(1,1,1)$-singularities and a general member of $|-K_X|$ has only ordinary double points, and*
(3) *Fix $f, Y, Y', \ldots$ etc. as in Theorem* 0.3. *Then $\mathrm{Bs}|-K_{Y'}|$ is the union of flipped curves and $\frac{1}{2}(1,1,1)$-singularities.*

Proof. The proof of (1) and (2) are clear from Corollary 3.2. (3) follows from (2) and [Taka02, Proposition 2.1 (4)]. ∎

§4. Construction of examples of $\mathbb{Q}$-Fano 3-folds as in Theorem 0.3

Note that the assertions for No. 3.1, No. 3.1′, No. 3.2 and No. 5.2 are proved in Corollary 3.2 and Remark 3.3.

Proof of Proposition 0.8. Assume that $n > 0$. Let R' be the ray of $\overline{\mathrm{NE}}(Y')$ generated by the numerical class of l_i. By (2-3) and (2-4), R' is extremal. By (2-2) and (2-3), a general member D of $|-K_{Y'}|$ is smooth along l_i and has only canonical singularities (see [MM85, Proposition 6.8]). By (2-4), $D \cdot l_i = -1$. Hence we have $\mathcal{N}_{l_i/Y'} \simeq \mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}(-2)$. Moreover R' is contractible since for another general member D' of $|-K_{Y'}|$, $(Y', D + \epsilon D')$ is klt for $0 < \epsilon \ll 1$ and $(-(K_{Y'} + D + \epsilon D') \cdot R') > 0$. Moreover the contraction associated to R' can be regarded as a log flipping contraction for $(Y', D + \epsilon D')$. Let $Y' \dashrightarrow {Y_0}'$ be the log flip. The log flip coincides the anti-flip for $K_{Y'}$ and by $\mathcal{N}_{l_i/Y'} \simeq \mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}(-2)$, ${Y_0}'$ has only $\frac{1}{2}(1,1,1)$-singularities. By (2-3) and [Taka02, Proposition 2.1 (4)], $\mathrm{Bs}|-K_{{Y_0}'}|$ is the union of $\frac{1}{2}(1,1,1)$-singularities. In particular $-K_{{Y_0}'}$ is nef.

In case $n = 0$, set ${Y_0}' := Y'$.

In any case $-K_{{Y_0}'}$ is nef and moreover by (3) and [Taka02, Proposition 2.1 (5)], we have $(-K_{{Y_0}'})^3 = (-K_{Y'})^3 + \frac{n}{2} > 0$. Hence $-K_{{Y_0}'}$ is big.

Let $\widetilde{E}_0$ be the strict transform of $\widetilde{E}$ on ${Y_0}'$. By (4), we can show that $(-K_{{Y_0}'})^2 \widetilde{E}_0 = 1$ and $(-K_{{Y_0}'}) \widetilde{E}_0^2 = -2$ by [Taka02, Lemma 3.2 (3)].

Since $-K_{{Y_0}'}$ is nef and big, we can construct a diagram

$${Y_0}' \dashrightarrow \cdots \dashrightarrow {Y_i}' \dashrightarrow {Y_{i+1}}' \dashrightarrow \cdots \dashrightarrow Y := {Y_k}' \xrightarrow{f} X$$

as in [Taka02, Set up 3.3] starting from ${Y_0}'$ by setting $D = \widetilde{E}_0$, where ${Y_i}' \dashrightarrow {Y_{i+1}}'$ is a flop or a flip for $i = 0$ and a flip for $i \geq 1$. Let $\widetilde{E}_i$ (resp. E) be the strict transform of $\widetilde{E}_0$ on ${Y_i}'$ (resp. Y). Let $R_i := R$ if $n = 0$ and $i = 0$ and the $K_{{Y_i}'}$-negative extremal ray otherwise. By [Taka02, Lemma 3.1], we have

$$(-K_Y)^2 E = 1 - \sum a_i d_i, \tag{4.1}$$

$$(-K_Y)E^2 = -2 - \sum {a_i}^2 d_i, \quad \textit{and} \tag{4.2}$$

$$E^3 = \widetilde{E}_0{}^3 - \sum {a_i}^3 d_i - e, \tag{4.3}$$

Claim 4.1. *$\widetilde{E}_i \cdot R_i < 0$. In particular a_i are non-positive. Moreover a_i are integers.*

Proof. We can prove the assertion by induction. For $i = 0$, $\widetilde{E}_0 \cdot R_0 < 0$ can be easily checked. Assume that the assertion holds for the numbers less than i. Then we have

$$a_j \leq 0 \quad (j < i) \tag{4.4}$$

and moreover the other extremal ray than R_i is positive for $\widetilde{E}_i$. Note that the linear system of a sufficient multiple of $-K_{{Y_i}'}$ is free outside a finite number of curves because the linear system of a sufficient multiple of $-K_{{Y_1}'}$ is free. So $-K_{{Y_i}'}|_{\widetilde{E}_i}$ is numerically equivalent to an effective 1-cycle. Note that $-K_{{Y_i}'}\widetilde{E}_i^2 \leq -K_{Y'}\widetilde{E}^2 = -2$ by (4.2) and (4.4). Hence we have $\widetilde{E}_i \cdot R_i < 0$.

Since ${Y_0}'$ has only at worst index 2 singularities, so is ${Y_i}'$. Hence $a_i = 2(\widetilde{E}_i \cdot \gamma_i) \in \mathbb{Z}$ if ${Y_i}' \dashrightarrow {Y_{i+1}}'$ is a flip. ∎

By this claim, we know that f is a divisorial contraction whose exceptional divisor is E. If f is a crepant divisorial contraction, then $l = 0$. But $(-K_{Y'})^2\widetilde{E} = 1$, a contradiction. Hence f is a K_Y-negative contraction. Assume that f is of $(2,1)$-type which contracts E to a curve C'. Then $(-K_X \cdot C') = (-K_Y + E)(-K_Y)E = -1 - \sum d_i a_i(a_i + 1) < 0$, a contradiction since X is a $\mathbb{Q}$-Fano 3-fold. So f is of $(2,0)$-type. Then we have $-K_Y E^2 \geq -2$ by [Taka02, Proposition 2.3]. On the other hand $-K_Y E^2 \leq -K_{Y'}\widetilde{E}^2 = -2$. Hence there is no flip. So $(-K_Y)^2 E = (-K_{Y'})^2 \widetilde{E} = 1$ and hence again by [Taka02, Proposition 2.3], f is the blow-up at a $\frac{1}{2}(1,1,1)$-singularity or the weighted blow-up at a QODP with weights $\frac{1}{2}(1,1,1,2)$ (we use the coordinate as stated in the definition of QODP). ∎

TABLE 1.

Proof of Theorem 0.10 (A). (1) is easily checked. The former half of (2) follows from [Taka02, Proposition 2.1 (4)]. (3) follows from Corollary 3.4. We prove the latter half of (2). Assume the contrary. Then there is a non-trivial fiber l of f' intersecting two l_i's. l must be a flopping curve containing two $\frac{1}{2}(1,1,1)$-singularities on Y_1, where $Y' \dashrightarrow Y_1$ is the anti-flip, a contradiction to Corollary 3.4. □

Proof of Theorem 0.10 (B).

(i) Assume that S is reducible. The following argument is similar to [Muk93, Section 4]. The possibilities of an irreducible component T are classified in [Reid94]. We have genus formulae and degree formulae of curves on T and by virtue of these formulae, we obtain a contradiction except No. 1.6, 1.10 and 1.11. We treat only No. 1.6 here. In this case, there is a possibility that C is contained in a smooth quadric T such that C is a divisor of $(2,6)$-type. Note that $-K_{X'}|_T$ is of $(4,4)$-type. Hence $-K_{X'}|_T - C$ is not effective, a contradiction to the assumption (2).
Note that if S is irreducible and nonnormal, then $C' := \operatorname{Sing} S \simeq \mathbb{P}^1$ and $-K_S \cdot C' = 1$ by [Reid94]. Hence $C \not\subset \operatorname{Sing} S$.

(ii) We show that conditions of Proposition 0.8 are satisfied. By (i), $\widetilde{E} = f'^*S - E'$ and $\widetilde{E}$ is irreducible. On the other hand l_i are $(z+2)$-secant lines with respect to $\frac{1}{z+1}(-K_{X'})$. Thus by easy case by case calculations, we see that (2-4), (3) and (4) hold. Since $\operatorname{Bs}|-K_{X'} - C|$ is the union of C, l_i and $\frac{1}{2}(1,1,1)$-singularities, $\operatorname{Bs}|-K_{Y'}|$ is the union of l_i and $\frac{1}{2}(1,1,1)$-singularities. For (5) in case $n = 0$, we have only to take R as the extremal ray different from one associated to f'. Other conditions are clearly satisfied.

□

Proof of Theorem 0.10 (C). We construct an example of the data given in (B).

Let S be a smooth Cartier divisor in X' such that $S \equiv \frac{z}{z+1}(-K_{X'})$. We can take such an S (it is well-known if $I(X') = 1$. If $I(X') > 1$, see [San96, Remark 4.1]). S is a del Pezzo surface. We represent S as blow-up at r points of $\mathbb{P}^2$ in a general position, where $r := e + n$. Let l_i $(0 \leq i \leq r-1)$ be its exceptional curves and l the total transform of a line in $\mathbb{P}^2$. Let $D := l + l_0 + \cdots + l_{n-1}$ and $C := -K_{X'}|_S - D$. Then we show that $|C|$ is free. By computing intersection numbers with (-1)-curves, we can check that C is nef in any case in the table. Let $M := C - K_S$. Check that $M^2 > 4$. Hence if $|C|$ is not free, there is an effective divisor l such that $M \cdot l = 1$ and $l^2 = 0$ whence $-K_S \cdot l = 1$ by Reider's theorem [RI]. But $l \cdot (K_S + l) = -1$ is a contradiction. So $|C|$ is free.

Hence we can take a smooth member from $|C|$. We denote it by C. l_i are $(z+2)$-secant lines of C which are mutually disjoint. Moreover since $h^1(\mathscr{O}_{X'}(-K_{X'} - S) = 0$, $\mathrm{Bs}|-K_{X'} - S|$ is the union of $\frac{1}{2}(1,1,1)$-singularities and $-K_{X'}|_S - C = D$, $\mathrm{Bs}|-K_{X'} - C|$ is the union of C, l_i and $\frac{1}{2}(1,1,1)$-singularities. □

TABLE 2.

Proof of Theorem 0.11 (A). The proof is almost clear. □

Proof of Theorem 0.11 (B). For simplicity, we assume that P_i are ODP's.

(i) f' is constructed as follows (see also [Taka02, Proposition 2.2]). Let $\nu' \colon \widetilde{X'} \to X'$ be the composite of the blow-ups at $P_0 \sim P_{N-2}$ and F_i' the exceptional divisor over P_i. Let $\mu' \colon \widehat{X'} \to \widetilde{X'}$ be the blow-up along the strict transform $\widetilde{C}$ of C and F' the μ'-exceptional divisor. We denote the strict transforms of the two fibers of $F_i' \simeq \mathbb{P}^1 \times \mathbb{P}^1$ through $F_i' \cap \widetilde{C}$ by l_{ij} $(j = 1, 2)$. Note that $-K_{\widehat{X'}} \cdot l_{ij} = 0$. We can easily see that $|-K_{\widehat{X'}}|$ is free by $P \cap X' = C$, where P is the linear subspace spanned by C and $-K_{\widehat{X'}}$ is big. Hence l_{ij}'s are flopping curves on $\widehat{X'}$ and we can see that the classes of l_{i1} and l_{i2} belong to the same ray. Let $\widehat{X'} \dashrightarrow \widehat{X'}^+$ be the flop. Then the strict transforms of F_i''s on $\widehat{X'}^+$ are $\mathbb{P}^2$'s and we can contract them to $\frac{1}{2}(1,1,1)$-singularities. Let $g' \colon \widehat{X'}^+ \to Y'$ be the contraction morphism. Then the natural morphism $f' \colon Y' \to X'$ is what we want.

(ii) We use the notation in the proof of (i) below. Since $|-K_{\widehat{X'}^+}|$ is free, the assertion follows.

(iii) Let E' (resp. F'^+) be the strict transform of F' on Y' (resp. $\widehat{X'}^+$). Then $-K_{\widehat{X'}^+} - F'^+ = g'^*(-K_{Y'} - E')$. Moreover $h^0(-K_{\widehat{X'}^+} - F'^+) = h^0(-K_{\widehat{X'}} - F')$. Note that there is a smooth member of $|-K_{\widetilde{X'}}|$ containing $\widetilde{C}$ because $\mathrm{Bs}|-K_{\widetilde{X'}} - \widetilde{C}| = \widetilde{C}$. Hence we have $\mathcal{N}_{\widetilde{C}/\widetilde{X'}} \simeq \mathscr{O}(-1) \oplus \mathscr{O}(-2)$. Hence $F' \simeq \mathbb{F}_1$ and $-K_{\widehat{X'}}|_{F'} \sim C_0 + l$, where

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C_0 is the minimal section of F' and l is a fiber of F'. So we have $h^0(-K_{\widehat{X'}}|_{F'}) = 3$ and $H^0(-K_{\widehat{X'}}) \to H^0(-K_{\widehat{X'}}|_{F'})$ is surjective since $|-K_{\widehat{X'}}|$ is free. Thus $h^0(\mathscr{O}_{Y'}(-K_{Y'} - E')) = 1$ since $h^0(-K_{\widehat{X'}}) = 4$.

(iv) For (5), we have only to take R as the extremal ray different from one associated to f'. The assertion of (3) and (4) follows from (i) and (ii) except the irreducibility of $\widetilde{E}$. Note that in the proof of Proposition 0.8, we need the irreducibility of $\widetilde{E}$ only after the proof of Claim 4.1. Hence we can proceed as in the proof of Proposition 0.8 to the proof of Proposition 4.1 and know that f is a divisorial contraction. Let F be f-exceptional divisor and $\overline{E}'$ the strict transform of E' on Y. By setting $D = \overline{E}'$ and $D' = F$, we can apply the construction in [Taka02, Set up 3.2]. Moreover by taking f' as f in [Taka02, Set up 3.3] and applying [Taka02, Lemma 3.5], we can write $\overline{E}' = z(-K_Y) - uF$. By [Taka02, Lemma 3.6], $z, u \in \mathbb{N}$. Moreover we have $z \leq u$ by (ii) and [Taka02, Claim 3.8]. Note that $(-K_{Y'})^3 = \frac{N+3}{2}$, $(-K_{Y'})^2 E' = \frac{N+1}{2}$, $(-K_{Y'})E'^2 = \frac{N-5}{2}$ and $E'^3 = \frac{N-23}{2}$. Thus we can figure out the solutions of the equations in [Taka02, §3]. We see that $z = u = 1$, $Y' \dashrightarrow Y$ is a flop and f is an extremal contraction of $(2,0)_4$-type or $(2,0)_{10}$-type. As a consequence, F is the strict transform of $\widetilde{E}$.

□

In the proof below, the hardest part is the proof of $\mathbb{Q}$-factoriality of X'.

Proof of Theorem 0.11 (C) *for No.* 2.1. Let C be a line in $\mathbb{P}^5$ and P_1, $P_2 \in C$ two points. Let $\nu\colon B \to \mathbb{P}^5$ be the composite of the blow-ups at P_1 and P_2 and $\mu\colon A \to B$ the blow-up along the transform C' of C. Let $E_i{}'$ be the strict transform of the exceptional divisor over P_i, F the exceptional divisor over C' and H the total transform of a hyperplane of $\mathbb{P}^5$. Let $G := 3H - 2E_1{}' - 2E_2{}' - F$.

Claim 1. (1) *$|G|$ is free.*

(2) *For an irreducible curve on A, $G \cdot l = 0$ if and only if l is a fiber of $F \simeq \mathbb{P}^3 \times \mathbb{P}^1 \to \mathbb{P}^3$.*

Proof.

(1) Since $G = (H - E_1{}' - E_2{}' - F) + (H - E_1{}') + (H - E_2{}')$, and $|H - E_1{}' - E_2{}' - F|$, $|H - E_1{}'|$ and $|H - E_2{}'|$ are free, $|G|$ is free.

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(2) First we show that $F \simeq \mathbb{P}^3 \times \mathbb{P}^1$. Note that $A \to \mathbb{P}^5$ is the composite of the blow-up along C and the blow-ups along the fiber over P_i's. Hence $F \simeq \mathbb{P}^3 \times \mathbb{P}^1$.
By the decomposition as in (1), if $G \cdot l = 0$, then $({E_i}' + F) \cdot l = 0$ for $i = 1, 2$. From this, it is easy to see that $l \subset F$. By easy calculations, we can see that $-F|_F \sim H_1 + H_2$, where H_1 (resp. H_2) is the pull-back of a hyperplane of $\mathbb{P}^3$ (resp. $\mathbb{P}^1$). Hence $G|_F \sim H_1$. So (2) is now clear.

□

In particular $|G|$ is not composed of a pencil and hence its general member G' is smooth.

Set $Q := 2H - {E_1}' - {E_2}' - F$. Similarly we see that $|Q|$ is free and general member Q' of $|Q|$ is smooth. Set $X'' := G' \cap Q'$. We can assume that X'' is smooth. Let $g\colon A \to \overline{A}$ be the Stein factorization of the morphism defined by $|G|$. We denote the image on $\overline{A}$ of an object $*$ on A by $\overline{*}$. Then by Claim 1, we can see that $\overline{Q'}$ has only terminal hypersurface singularities (along $\overline{Q'} \cap \overline{F}$) and $\overline{X''}$ is an ample divisor of $\overline{Q'}$. Hence by the Grothandieck-Lefschetz theorem [Gro68, p. 135, 3.18], we have $\rho(\overline{X''}) = \rho(\overline{Q'})$. Since $g|_{X''}$ and $g|_{Q'}$ is primitive, we have $\rho(X'') = \rho(Q') = 4$. Denote the image of X'' on $\mathbb{P}^5$ by X'. Then we can see that X' is factorial and P_i's are ODP's of X'. □

Proof of Theorem 0.11 (C) *for No.* 2.2. We construct X' with only ODP's.

Claim 1. *Let V (resp. X') be a $(2,2)$-complete intersection in $\mathbb{P}^6$ (resp. a quadric section of V) with the following properties.*

(1) *V (resp. X') contains a smooth conic C, and*
(2) *V (resp. X') has three ODP's $P_0 \sim P_2$ on C and outside P_i's, V (resp. X') is smooth.*

Then X' is factorial.

Proof. We claim that V contains the plane P spanned by C. Let σ be the pencil which consists of quadrics in $\mathbb{P}^6$ containing V. Since P_i is an ODP on V, there is a quadric in σ which is singular at P_i. If there is a quadric in σ which is singular at all P_i's, then it is singular on P and hence V is singular along C, a contradiction. So σ is generated by two quadrics

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which are singular at some P_i. But such quadrics contains P and hence V contains P.

Let $\nu\colon \widetilde{V} \to V$ be the composite of the blow-ups at $P_0 \sim P_2$ and F_i the exceptional divisor over P_i. Let $\widetilde{X'}$ be the strict transform of X' on $\widetilde{V}$ and H the total transform of a hyperplane section of V. Then $\widetilde{X'} \sim 2H - F_0 - F_1 - F_2$. Note that $|H - F_i - F_j|$ is free outside the strict transform l_{ij} of the line through P_i and P_j and $|H - F_k|$ is free (note that l_{ij} is contained in V since $l_{ij} \subset P$). By this, we can easily see that $|\widetilde{X'}|$ is free and $\widetilde{X'}$ is numerically trivial only for l_{ij}'s $((i,j) = (0,1),(1,2),(2,0))$.

Let ϕ be the morphism defined by $|\widetilde{X'}|$. Then ϕ-exceptional curves are l_{ij}'s. We prove that $\mathrm{Leff}(\widetilde{V}, \widetilde{X'})$ holds and $\widetilde{X'}$ meets every effective divisor on $\widetilde{V}$. By [H, p. 165, Proposition 1.1] and the argument of [H, p. 172, the proof of Theorem 1.5], it suffices to prove that $\mathrm{cd}(\widetilde{V} - \widetilde{X'}) < 3$, i.e., for any coherent sheaf F on $\widetilde{V} - \widetilde{X'}$, $H^i(\widetilde{V} - \widetilde{X'}, F) = 0$ for all $i \geq 3$. Let $\overline{V} := \phi(\widetilde{V})$ and $\overline{X'} := \phi(\widetilde{X'})$. Consider the Leray spectral sequence

$$E_2^{pq} = H^p(\overline{V} - \overline{X'}, R^q{\phi'}_* F) \Longrightarrow E^{p+q} = H^{p+q}(\widetilde{V} - \widetilde{X'}, F),$$

where $\phi' := \phi|_{\widetilde{V} - \widetilde{X'}}$. Since $\overline{V} - \overline{X'}$ is affine and the dimension of every fiber of $\phi \leq 1$, we have $E_2^{pq} = 0$ for $p \geq 1$ or $q \geq 2$ whence $E^{p+q} = 0$ for $p+q \geq 2$. So the assertion follows.

Moreover since $\widetilde{X'}$ is nef and big, $H^i(\widetilde{V}, \mathscr{O}(-n\widetilde{X'})) = 0$ for $n \geq 1$ and $i = 1,2$ by KKV vanishing theorem. Hence by the Grothandieck-Lefschetz theorem [G, p. 135, 3.18] (or [H, p. 178, Theorem 3.1]), we have $\mathrm{Pic}\,\widetilde{X'} \simeq \mathrm{Pic}\,\widetilde{V} \simeq \mathbb{Z}^4$. So $\rho(\widetilde{X'}/X') = 3$ which imply that X' is factorial. ∎

We give a pair (V, X') satisfying the condition of Claim 1. Let C be a smooth conic in $\mathbb{P}^6$ and $P_0 \sim P_2$ three points on C. We can choose a coordinate of $\mathbb{P}^6$ such that $C = \{x_0x_1 + x_1x_2 + x_2x_0 = x_3 = x_4 = x_5 = x_6 = 0\}$ and $P_i = \{x_j = 0 \text{ for } j \neq i\}$.

Claim 2. *Let X' be a $(2,2,2)$-complete intersection in $\mathbb{P}^6$ satisfying the following conditions.*

(1) *X' is factorial,*
(2) *X' contains a smooth conic C, and*
(3) *X' has three ODP's $P_0 \sim P_2$ on C and outside P_i's, X' is smooth.*

Then X' is the intersection of three quadrics $Q_1 \sim Q_3$ of the following forms by permuting P_i's if necessary.

$$Q_1 := \{m_0x_0 + m_1x_1 + q_1 = 0\},$$
$$Q_2 := \{pm_1x_1 + m_2x_2 + q_2 = 0\}, \quad \textit{and}$$
$$Q_3 := \left\{x_0x_1 + x_1x_2 + x_2x_0 + \sum_{i=3}^{6} l_ix_i = 0\right\},$$

where $p \in \mathbb{C}$, m_i (resp. q_i) is a linear form (resp. a quadratic form) of $x_3 \sim x_6$ and l_i is a linear form of $x_0 \sim x_6$.

Conversely if $X' = Q_1 \cap Q_2 \cap Q_3$, where Q_i is of the form as above and m_i, q_i and l_i are suitably general, then X' satisfies $(1) \sim (3)$.

Proof. Let γ be the net which consists of quadrics containing X'. γ contains a member Q_1 which is singular at P_2. Then Q_1 is of the form as above. If $m_1 = m_2 = 0$, then Q_1 is singular on the plane P spanned by C and hence X' is singular along C, a contradiction. Hence $m_1 \neq 0$ or $m_2 \neq 0$. By permuting P_1 and P_2 if necessary, we may assume that $m_1 \neq 0$. γ contains a member Q_2 which is singular at P_0. Q_2 is of the form as

$$\{{m_1}'x_1 + m_2x_2 + q_2 = 0\},$$

where ${m_1}'$ and m_2 (resp. q_2) are linear forms (resp. is a quadratic form) of $x_3 \sim x_6$. γ also contains a member Q' which is singular at P_1. If Q_1, Q_2 and Q' generate γ, then X' contains the plane P, a contradiction to the factoriality and $F(X') = 1$. Hence Q' is contained in the pencil generated by Q_1 and Q_2. So ${m_1}' = pm_1$ for some $p \in \mathbb{C}$ and

$$Q = \{-pm_0x_0 + m_2x_2 + (q_2 - pq_1) = 0\}.$$

Since X' does not contain P as noted above, γ contains a member Q_3 of the form as in the statement. Q_3 is not contained in the pencil generated by Q_1 and Q_2 and hence Q_i's generate γ.

Conversely let $X' := Q_1 \cap Q_2 \cap Q_3$, where Q_i is of the form as above and m_i, q_i and l_i are suitably general. We can easily check that X' satisfies (2) and (3). Set $V := Q_1 \cap Q_2$. We may assume that V satisfies the condition of Claim 1. Hence by Claim 1, X' is factorial. □

□

TABLE 3.

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Proof of Theorem 0.13 (A). This is almost clear. ∎

Proof of Theorem 0.13 (B). (1) The assertion follows because X' is an intersection of quadrics.

(2) By (i), the rank of the natural map $H^0(-K_{Y'}) \to H^0(\mathscr{O}(-K_{Y'}|_{E'}))$ is 3. Hence there is a unique member $\widetilde{E}$ of $|-K_{Y'}-E'|$ since $h^0(-K_{Y'}) = 4$.

(3) The conditions of Proposition 0.8 are easily checked except irreducibility of $\widetilde{E}$. The proof of irreducibility of $\widetilde{E}$ is similar to one of No. 2.2 so we omit it.

∎

Proof of Theorem 0.13 (C). We construct an example of the data given in (B). The Grassmannian $G(2,5)$ (parameterizing 2-dimensional subspaces of 5-dimensional vector space) can be embedded into $\mathbb{P}^9$ by the Plücker embedding. Its defining equations are $x_{ij}x_{kl} - x_{ik}x_{jl} + x_{jk}x_{il} = 0$ for all $1 \le i < j < k < l \le 5$, where x_{pq} $(1 \le p < q \le 5)$ is a Plücker coordinate. Let Q be the point defined by $x_{pq} = 0$ for any $(p,q) \neq (1,2)$. Let m_1 (resp. m_2) be the line $\subset G(2,5)$ defined by $x_{pq} = 0$ for any $(p,q) \neq (1,2),(1,3)$ (resp. $(p,q) \neq (1,2),(2,4)$). Let m_3 be the line $\subset G(2,5)$ defined by the equations $x_{pq} = r_{pq}x_{12}$ for $(p,q) \neq (1,2)$ such that $r_{34} = r_{35} = r_{45} = 0$, $r_{13}r_{24} - r_{23}r_{14} = 0$, $r_{13}r_{25} - r_{23}r_{15} = 0$, $r_{14}r_{25} - r_{24}r_{15} = 0$ and $r_{15}r_{25} \neq 0$. Let H be the 3-plane spanned by m_1, m_2 and m_3. Then $G(2,5) \cap H = m_1 \cup m_2 \cup m_3$. Hence by [MM85, Proposition 6.8], there are two hyperplane H_1, H_2 and a quadric Q such that $X' := G(2,5) \cap H_1 \cap H_2 \cap Q$ is smooth and X' contains m_1, m_2 and m_3. Since the tangent space of X' at Q also contains all the lines on X' through Q, it is equal to H. Hence there are only three lines on X' through Q. ∎

The proof of the following lemma is easy so we omit it.

Lemma 4.2. *Let $f\colon X \to Y$ be the blow-up of a smooth 3-fold Y along a smooth curve C on Y and E the exceptional divisor. Then*

(1) $E^3 = -\deg \mathscr{N}_{C/Y}$,
(2) $(-K_X)E^2 = 2g(C) - 2$,
(3) $(-K_X)^2E = (-K_Y \cdot C) + 2 - 2g(C)$,
(4) $(-K_X)^3 = (-K_Y)^3 - 2\{(-K_Y \cdot C) - g(C) + 1\}$.

Claim 4.3. *Consider the situation as in* [Taka02, Set up 3.2]. *Assume that*

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(1) *f is of $(3,2)$-type.*
(2) *$X \simeq \mathbb{P}^2$.*
(3) *There exists a degenerate fiber $l \subset \operatorname{Reg} Y$ of f.*

Then $z, u \in \mathbb{N}$.

Proof. $z \in \mathbb{N}$ and the positivity of u are proved in [Taka02, Claim 3.6]. Note that there exists a smooth rational curve $m \subset \operatorname{Reg} Y$ such that $D \cdot m = 1$. Hence $u = z(-K_{Y'}) \cdot m - \overline{D'} \cdot m \in \mathbb{N}$. □

TABLE 4, I.

Proof of Theorem 0.15 (A).

CLAIM 4.4. *$f'|_{l_0}$ is an isomorphism and $f'(l_0)$ does not pass the vertex v of X'.*

Proof. Let $L' \in |-K_{Y'} - \widetilde{E}|$ be a general member. Note that $f'(L')$ is a ruling of X'. Then $L' \cdot l_0 = 1$ and l_0 does not pass through $\frac{1}{2}(1,1,1)$-singularities. Hence $f'|_{l_0}$ is birational. By $L' \cdot l_0 = 1$, $f'(l_0) \sim 2r$, where r is a ruling of X'. Thus $f'(l_0)$ is smooth and so $f'|_{l_0}$ is an isomorphism. If $v \in f'(l_0)$, then $f'(l_0)$ must be reducible, a contradiction. □

Let F_1 be the μ_1-exceptional divisor and γ_i irreducible components intersecting l_0 of degenerate fibers of f'. Then $-K_{W_1}$ is relatively nef over X'. Let R_1 be the other extremal ray of W_1 over X' than that associated to μ_1. Then $\operatorname{supp} R_1 = \bigcup \gamma_i$ and R_1 is a flopping ray. Let $W_1 \dashrightarrow {W_1}'$ be the flop and ${R_1}'$ the other extremal ray of ${W_1}'$ over X' than the flopped ray. Let γ be the transform of a non-degenerate fiber intersecting l_0 and ${\gamma_i}^+$ the flopped curves. Since $G_1 \cdot {\gamma_i}^+ > 0$ and $G_1 \cdot \gamma < 0$, we have $G_1 \cdot {R_1}' < 0$, where G_1 is the strict transform of $f'^{-1} f'(l_0)$ on ${W_1}'$. So the contraction $\nu_1 \colon {W_1}' \to {Y_1}'$ is an extremal contraction of $(2,1)$-type whose exceptional divisor is G_1. Let l_i $(i \geq 1)$ be flipped curves different from l_0. Since $\operatorname{Bs}|-K_{Y'}|$ is the union of l_i $(i \geq 0)$ and $\frac{1}{2}(1,1,1)$-singularities, $\operatorname{Bs}|-K_{{W_1}'}|$ is the union of l_i $(i \geq 1)$ and $\frac{1}{2}(1,1,1)$-singularities. Then by easy calculations based on Lemma 4.2, we have $(-K_{{Y_1}'} \cdot m_1) = 2n+6$ and $(-K_{{Y_1}'})^3 = 4n+13$.

CLAIM 4.5. *l_i $(i \geq 1)$ does not intersect γ_j on ${W_1}'$.*

Proof. If l_i $(i \geq 1)$ intersects γ_j on ${W_1}'$, then γ_j is a flopped curve on Y_1, where $Y' \dashrightarrow Y_1$ is the anti-flip. But γ_j passes through two $\frac{1}{2}(1,1,1)$-singularities, a contradiction to Corollary 3.3. □

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Hence we know that $-K_{Y_1'} \cdot l_i = 1$ $(i \geq 1)$ and Y_1' is a weak Fano 3-fold. We can apply the construction in [Taka02, Set up 3.3] starting from $f_1' \colon Y_1' \to X'$ by setting $D = L_1$. By Claim 4.3, $z, u \in \mathbb{N}$.

Claim 4.6. $|-K_{Y_1'} - L_1| \neq \emptyset$.

Proof. By $h^0(-K_{W_1}) = 5$ and $h^0(-K_{W_1}|_{F_1}) = 3$, we have $|-K_{W_1} - F_1| \neq \emptyset$. Hence we have the assertion. □

Then we can figure out the solutions of the equations as in [Taka02, §3] and we know that

(i) the case $n = 2$ does not occur, and
(ii) in case $n = 1$, Y_1' is a $\mathbb{Q}$-Fano 3-fold with the properties as stated in Theorem 0.15 (1), (2) and Y_2' is a quadric bundle over $X_2' \simeq \mathbb{P}^1$, where Y_2' is obtained from Y_1' by the flip.

From now on assume that $n = 1$. Y_2' can be embedded in a $\mathbb{P}^3$-bundle $\mathbb{P}(\mathscr{E})$ over $\mathbb{P}^1$, where $\mathscr{E} := \bigoplus_{i=0}^3 \mathscr{O}(a_i)$ is a vector bundle of rank 4. We may assume that

$$a_0 = 0 \leq a_1 \leq a_2 \leq a_3. \tag{4.5}$$

Let H be the tautological divisor and M a fiber. In $\mathbb{P}(\mathscr{E})$, Y_2' is linearly equivalent to $2H - aM$ for some $a \in \mathbb{Z}$. Since $-K_{Y_2'} = 2H|_{Y_2'} + (2 + a - \sum_{i=0}^3 a_i)L_2'$ and $(-K_{Y_2'}) = 16$, we have $(-K_{Y_2'})^3 = 16a - 8\sum_{i=0}^3 a_i + 48 = 16$. So we obtain

$$\sum_{i=0}^{3} a_i = 2a + 4. \tag{4.6}$$

Note that $\widetilde{L_1} \sim -K_{Y_2'} - L_2 = 2H|_{Y_1'} - (a+3)L_2$ by (4.6), where $\widetilde{L_1}$ is the strict transform of L_1 on Y_2'. Let α^+ be a flipped curve for $Y_1' \dashrightarrow Y_2'$. Then since $\widetilde{L_1} \cdot \alpha^+ = -2$ and $-K_{Y_2'} \cdot \alpha^+ = -1$, we have

$$2H \cdot \alpha^+ = a + 1 \geq 0. \tag{4.7}$$

Moreover since $-3M'$ is not effective, we have

$$h^0(2H - (a+3)M) \leq h^0((2H|_{Y_1'} - (a+3)L_2) = 4. \tag{4.8}$$

Thus we have $a = -1$ and $a_1 = 0$ and $a_2 = a_3 = 1$ by (4.5), (4.6), (4.7) and (4.8). □

Proof of Theorem 0.15 (B). Similarly to the proof of Proposition 0.8, we see that there is an anti-flip ${Y_2}' \dashrightarrow {Y_1}'$ whose flipped curves are connected components of $V \cap {Y_2}'$ and ${Y_1}'$ has only $\frac{1}{2}(1,1,1)$-singularities. Moreover we know that $L_1 := -K_{{Y_1}'} - \widetilde{L_2}$ is nef, where $\widetilde{L_2}$ is the strict transform of L_2. By the base point free theorem (see [KMM87]), L_1 is semi-ample. Since $(L_1)^3 = 0$ and $(L_1)^2 \not\equiv 0$, a sufficient multiple of L_1 defines a conic bundle structure ${f'}_1\colon {Y_1}' \to X'$. In particular ${Y_1}'$ is a $\mathbb{Q}$-Fano 3-fold since both of its extremal rays are $K_{{Y_1}'}$-negative. Since $-K_{{Y_2}'} - L_2 = 2(H|_{{Y_2}'} - L_2)$ and the transform of $H|_{{Y_2}'} - L_2$ on ${Y_1}'$ is not Cartier, we know that $X' \simeq \mathbb{F}_{2,0}$ whence we know that $|L_1|$ is actually free.

Let G_1 be ν_1-exceptional divisor and ${F_1}'$ the strict transform of ${f_1'}^{-1} f_1'(m_1)$. By a similar argument to (A), we know that there is a flop ${W_1}' \dashrightarrow W_1$ over X' and an extremal contraction $\mu_1\colon W_1 \to Y'$ over X' of $(2,1)$-type, whose exceptional divisor F_1 is the strict transform of ${F_1}'$ on W_1. Let $l_0 := \mu(F_1)$. Since $\mathrm{Bs}|-K_{{Y_1}'} - m_1|$ is the union of m_1 and $\frac{1}{2}(1,1,1)$-singularities, $\mathrm{Bs}|-K_{Y'} - l_0|$ is the union of l_0 and $\frac{1}{2}(1,1,1)$-singularities. It is easy to see that $-K_{Y'} \cdot l_0 = -1$ by (A)(1) and Lemma 4.2. Hence $\mathrm{Bs}|-K_{Y'}|$ is the union of l_0 and $\frac{1}{2}(1,1,1)$-singularities.

(1) $D_2 \in |H|_{{Y_2}'} - L_2|$ be a general smooth member and D_1 (resp. D) its strict transform on ${Y_1}'$ (resp. Y'). Since $-K_{D_2} = H|_{{Y_2}'}$, $-K_{D_2}$ is nef and big and numerically trivial only for two flipped curves on D_2. Thus D_1 is a del Pezzo surface with two ODP's at $\frac{1}{2}(1,1,1)$-singularities of ${Y_1}'$. Since $D_1 \dashrightarrow D$ is a composite of a blow-up at a smooth point and a blow-down at a smooth point, D is a weak del Pezzo surface with two ODP's. Consider the exact sequence

$$0 \longrightarrow \mathscr{O}_{Y'}(-K_{Y'} - D) \longrightarrow \mathscr{O}_{Y'}(-K_{Y'}) \longrightarrow \mathscr{O}_D(-K_{Y'}) \longrightarrow 0.$$

By singular Riemann-Roch theorem on surfaces with only canonical singularities [Reid87b, (9.1) Theorem] and KKV vanishing theorem, we have $h^0(\mathscr{O}_D(-K_{Y'})) = 5$. Note that $\mathrm{Bs}|-K_{Y'}| \cap D$ consists of one point and $|-K_{Y'}|_D|$ is free outside two ODP's. Thus $\dim \mathrm{Im}(H^0(\mathscr{O}_{Y'}(-K_{Y'})) \to H^0(\mathscr{O}_D(-K_{Y'}))) = 4$. On the other hand, we have $h^0(\mathscr{O}_{Y'}(-K_{Y'})) = 5$ and so $|-K_{Y'} - D|$ has a unique member $\widetilde{E}$. We prove that $\widetilde{E}$ is irreducible. If $\widetilde{E}$ is reducible, it is a union of an irreducible surface $\widetilde{E}'$ which is a generically a 2-section for f' and surfaces which are mapped to curves on X'. Hence $h^0(-K_{Y'} - L) \neq 0$. Since $|D|$ is movable, $\widetilde{E}|_D$ is effective for a general D. Hence by the

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exact sequence

$$0 \longrightarrow \mathscr{O}_{Y'}(-K_{Y'}-L) \longrightarrow \mathscr{O}_{Y'}(-K_{Y'}-D) \longrightarrow \mathscr{O}_D(-K_D) \longrightarrow 0,$$

we have $h^0(-K_{Y'}-L)=0$, a contradiction.

(2) (2-3) is checked before the proof of (i). Other conditions are easily checked.

□

Proof of Theorem 0.15 (C). Let $Y_2{}'$ be a smooth divisor in $\mathbb{P}(\mathscr{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2})$ linearly equivalent to $2H+M$. Let $L_2 := M|_{Y_2{}'}$. Since $Y_2{}'$ is an ample divisor, $\rho(Y_2{}')=2$ by [Gro68, p. 135, 3.18]. By looking on the local charts, we can easily see that if we take $Y_2{}'$ generally, $\mathrm{Bs}|-K_{Y_2{}'}| = \mathrm{Bs}\,|H|_{Y_2{}'} - L_2| = V \cap Y_2{}'$, which is a disjoint union of two sections. We denote by L_1 a general member of $|L_1|$.

Claim 4.7. *L_1 is a del Pezzo surface of degree* 2.

Proof. Denote by $\widetilde{L_1}$ the strict transform of L_1 on $Y_2{}'$. Since $-K_{\widetilde{L_1}} = L_2|_{\widetilde{L_1}}$, $|-K_{\widetilde{L_1}}|$ is free and $(-K_{\widetilde{L_1}})^2=0$. Since $\pi\colon \widetilde{L_1} \to L_1$ is the blow-up at two points, L_1 is a weak del Pezzo surface of degree 2. Assume that there is a (-2)-curve δ on L_1. Then on $\widetilde{L_1}$, δ is a component of a degenerate fiber of $\widetilde{L_1} \to X_2{}'$ and does not intersect any π-exceptional curve. Then another component δ' of the fiber containing δ intersects both of two π-exceptional curves. Then, however, δ' satisfies $-K_{Y_1{}'}\cdot\delta'=0$ on $Y_1{}'$. But this contradicts the fact that $Y_1{}'$ is a $\mathbb{Q}$-Fano 3-fold. □

We can regard L_1 as a surface obtained by blowing up $\mathbb{P}^2$ at 7 points and let e_i $(i=1,\dots,7)$ be the exceptional curves, where we may assume that e_1 is a section of $f_1{}'$ and e_i $(i \geq 2)$ are components of different 6 degenerate fibers. Let $m_1{}' := 4l - 3e_1 - \sum_{i=2}^7 e_i$, where l is the pull-back of a line in $\mathbb{P}^2$.

Claim 4.8. *$|m_1{}'|$ is free.*

Proof. Since $m_1{}' = (3l - 2e_1 - \sum_{i=2}^7 e_i) + (l - e_1)$, $|m_1{}'|$ is nef. Assume that $|m_1{}'|$ is not free. Since $m_1{}' - K_{\widetilde{L_1{}'}}$ is nef and $(m_1{}' - K_{\widetilde{L_1{}'}})^2 > 4$, by [Reide88], there is an effective divisor Z such that

(i) $(m_1{}' - K_{\widetilde{L_1{}'}})\cdot Z = 0$ and $Z^2=-1$ or

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(ii) $(m_1{}' - K_{\widetilde{L_1{}'}}) \cdot Z = 1$ and $Z^2 = 0$.

In case (i), we have $(-K_{\widetilde{L_1{}'}}) \cdot Z = 0$ and $Z^2 = -1$ but this contradicts the genus formula. In case (ii), we have $(-K_{\widetilde{L_1{}'}}) \cdot Z = 0$ and $Z^2 = 0$ by the genus formula and moreover $(-K_{\widetilde{L_1{}'}}) \equiv aZ$ for some $a \in \mathbb{Q}$. By $(-K_{\widetilde{L_1{}'}}) \cdot m_1{}' = 3$ and $Z \cdot m_1{}' = 1$, we have $a = 3$. On the other hand, by $(-K_{\widetilde{L_1{}'}}).e_1 = 1$, $Z.e_1 = 1/3$, a contradiction. Hence we have the assertion. ∎

Let $m_1 \in |m_1{}'|$ be a general smooth member.

Claim 4.9. $\mathrm{Bs}|-K_{Y_1{}'} - m_1| = m_1$.

Proof. Since $|-K_{Y_2{}'} - \widetilde{L_1}| = |L_2|$, $\mathrm{Bs}|-K'_{Y_1} - m_1|$ is at most the union of L_1 and two flipping curves. Consider the exact sequence

$$0 \longrightarrow \mathscr{O}_{Y_1{}'}(-K_{Y_1{}'} - L_1) \longrightarrow \mathscr{O}_{Y_1{}'}(-K_{Y_1{}'}) \longrightarrow \mathscr{O}_{L_1}(-K_{Y_1{}'}) \longrightarrow 0.$$

Since $-2K_{Y_1{}'} - L_1$ is nef and big, $h^1(\mathscr{O}(-2K_{Y_1{}'} - L_1)) = 0$ by KKV vanishing. Hence $H^0(\mathscr{O}_{Y_1{}'}(-K_{Y_1{}'})) \to H^0(\mathscr{O}_{L_1}(-K_{Y_1{}'}))$ is surjective. Moreover we have $|-K'_{Y_1}|_{L_1} - m_1| = |l|$. We may assume that m_1 is disjoint from two flipping curves on $Y_1{}'$. Thus $\mathrm{Bs}|-K_{Y_1{}'} - m_1| = m_1$. ∎

It is easy to see that m_1 has other properties as in (A)(2). ∎

Table 4, II.

Proof of Theorem 0.17 (A) *and* (B). These are almost clear. ∎

Proof of Theorem 0.17 (C). We know that there is an example for this case by Corollary 2.3 and the existence of an example for No. 1.1 ($h = 6$ and $N = 2$) since only No. 4.3 satisfies $h = 6$ and $N = 1$. ∎

Table 4, III.

Proof of Theorem 0.18 (A). Let F_1 be μ_1-exceptional divisor. By an argument similar to the proof of Theorem 0.15 (A), we know that there is a flop $W_1 \dashrightarrow W_1{}'$ over X' and an extremal contraction $\nu_1 \colon W_1{}' \to Y_1{}'$ of $(2,1)$-type over X' whose exceptional divisor G_1 is the strict transform of $f'^{-1}f'(l_0)$ on $W_1{}'$. Let l_i $(i \geq 1)$ be flipped curves different from l_0. Since

$\mathrm{Bs}|-K_{Y'}| = \bigcup_{i=0}^{n-1} l_i$, $\mathrm{Bs}|-K_{W_1{}'}| = \bigcup_{i=1}^{n-1} l_i$. Then by easy calculations, we have $(-K_{W_1{}'})^2 G_1 = n+3$ and $(-K_{W_1{}'})^3 = 6$. By Lemma 4.2, $(-K_{Y_1{}'} \cdot m_1) = n+1$ and $(-K_{Y_1{}'})^3 = 2n+10$. Similarly to the proof of Claim 4.5, we know that any l_i ($i \geq 1$) does not intersect flopping curves for $W_1 \dashrightarrow W_1{}'$. Hence we have $G_1 \cdot l_i = 1$ and $-K_{Y_1{}'} \cdot l_i = 0$ ($i \geq 1$). Thus $Y_1{}'$ is a weak Fano 3-fold.

Assume that X is a $\mathbb{Q}$-Fano 3-fold of No. 4.4. We can apply the construction in [Taka02, Set up 3.3] starting from $f_1{}' \colon Y_1{}' \to X'$ by setting $D = L_1$. By Claim 4.3, $z, u \in \mathbb{N}$ and similarly to the proof of Claim 4.6, we can see that $|-K_{Y_1{}'} - L_1| \neq \emptyset$. Then we can figure out the solutions of the equations as in [Taka02, §3] and we know that $Y_1{}'$ is a Fano 3-fold with $(-K_{Y_1{}'})^3 = 12$ and two conic bundle structures. By [MM81, Table 1], we know that $Y_1{}'$ and m_1 are as in the statement of Theorem 0.18 (A-1).

Assume that X is a $\mathbb{Q}$-Fano 3-fold of No. 4.5–No. 4.7. $Y_1{}'$ has a flopping ray since $n \geq 2$. We use the notation of Theorem 0.18 (A-2). Let F_2 be μ_2-exceptional divisor. By an argument similar to the proof of Theorem 0.15 (A), we know that there is a flop $W_2 \dashrightarrow W_2{}'$ over X' and an extremal contraction $\nu_2 \colon W_2{}' \to Y_2{}'$ of $(2,1)$-type over X' whose exceptional divisor G_2 is the strict transform of ${f_1{}'}^{-1} f_1{}'(l_1)$ on $W_2{}'$. Let $m_2 := \nu_2(G_2)$. Then by easy calculations based on Lemma 4.2, we have $(-K_{Y_2{}'} \cdot m_2) = n$ and $(-K_{Y_2{}'})^3 = 4n+10$. Note that $\mathrm{Bs}|-K_{W_2{}'}| = \bigcup_{i=2}^{n-2} l_i$ outside m_1.

Claim 4.10. *$l_1 \cap l_i = \emptyset$ ($i \geq 2$).*

Proof. If $l_1 \cap l_i \neq \emptyset$, then Y' has a non-degenerate fiber δ intersecting l_0, l_1 and l_2. Then $-K_{Y_1} \cdot \delta = 1/2$ and there are three $\frac{1}{2}(1,1,1)$-singularities on it. If there is a member of $|-K_{Y_1}|$ which does not contain δ, then $-K_{Y_1} \cdot \delta \geq 3/2$, a contradiction. So $\delta \subset \mathrm{Bs}|-K_{Y_1}|$. But this contradicts Corollary 3.3. ∎

Similarly to the proof of Claim 4.5, we know that any l_i ($i \geq 2$) does not intersect flopping curves for $W_2 \dashrightarrow W_2{}'$. Hence we have $G_2 \cdot l_i = 1$ and $-K_{Y_1{}'} \cdot l_i = 1$ ($i \geq 1$). Hence $Y_2{}'$ is a weak Fano 3-fold. We can apply the construction in [Taka02, Set up 3.3] starting from $f_1{}' \colon Y_1{}' \to X'$ by setting $D = L_1$. By Claim 4.3, $z, u \in \mathbb{N}$ and similarly to the proof of Claim 4.6, we can see that $|-K_{Y_1{}'} - L_1| \neq \emptyset$. Then we can figure out the solutions of the equations as in [Taka02, §3] and we know that $Y_2{}'$ is actually a Fano 3-fold with $(-K_{Y_2{}'})^3 = 4n+10$. Since $Y_2{}'$ has a conic bundle structure, we know that $Y_2{}'$ and γ are as in the statement of Theorem 0.18 (A-2) by [MM81, Table 1].

Claim 4.11. (i) *γ, m_1 and m_2 intersect the common $n-2$ points simply.*

(ii) *l_i $(i \geq 2)$ are fibers of h which intersect m_1 or m_2.*

(iii) *γ, m_1 and m_2 are smooth rational curves of degree $n-1$.*

Proof. By Claim 4.10, (i) follows. By [MM83, Theorem 5.1 (1)], we have $-K_{Y_2{}'} = L_2 + M_2$, where L_2 (resp. M_2) is the pull-back of a hyperplane section of X' (resp. $X_2{}'$). Since $-K_{Y_2{}'} \cdot l_i = 1$ and $L_2 \cdot l_i = 1$, we have $M_2 \cdot l_i = 0$. Hence l_i are fibers of h. Moreover by $-K_{Y_2{}'} \cdot m_j = n$ and $L_2 \cdot l_i = 1$, we have $M_2 \cdot m_j = n-1$. Hence by $-K_{Y_2{}'} = 2M_2 - G_3$, where G_3 is h-exceptional divisor, we have $G_3 \cdot m_j = n-2$. Thus (ii) holds. (iii) follows from (i) and (ii). □

□

Proof of Theorem 0.18 (B). First we consider the case X is a $\mathbb{Q}$-Fano 3-fold of No. 4.4. Let G_1 be ν_1-exceptional divisor and $F_1{}'$ the strict transform of $f_1'^{-1} f_1'(m_1)$, where $f_1' \colon Y_1{}' \to X' \simeq \mathbb{P}^2$ is one of two structure morphisms of conic bundles. By an argument similar to the proof of 0.15 (A), we know that there is a flop $W_1{}' \dashrightarrow W_1$ over X' and an extremal contraction $\mu_1 \colon W_1 \to Y'$ of $(2,1)$-type over X' whose exceptional divisor F_1 is the strict transform of $F_1{}'$ on W_1.

(i) Let $f_1{}'' \colon Y_1{}' \to X_1{}' \simeq \mathbb{P}^2$ be the morphism of the other conic bundle structure. By [MM83, Theorem 5.1 (1)], we have $-K_{Y_2{}'} = L_1 + M_1$, where L_1 (resp. M_1) is the pull-back of a hyperplane section of X' (resp. $X_1{}'$). Hence $f_1'^{-1} f_1'(m_1) + f_1''^{-1} f_1''(m_1) \in |-K_{Y_1{}'}|$. Let $\widetilde{E}''$ be the strict transform of $f_1''^{-1} f_1''(m_1)$ on $W_1{}'$. Then $F_1{}' + \widetilde{E}'' + G_1 \in |-K_{W_1{}'}|$. Hence the strict transform $\widetilde{E}$ of $\widetilde{E}''$ on Y' satisfies $\widetilde{E} \sim -K_{Y'} - L$. Conversely we can easily see that if $\widetilde{E} \sim -K_{Y'} - L$ is effective, it is the strict transform of $f_1''^{-1} f_1''(m_1)$. Hence the uniqueness of $\widetilde{E}$ follows. We prove that $\widetilde{E}$ is irreducible. If $\widetilde{E}$ is reducible, it is a union of an irreducible surfece $\widetilde{E}'$ which is a generically a 2-section for f' and surfaces which are mapped to curves on X'. Hence $h^0(-K_{Y'} - 2L) \neq 0$. Since $|L|$ is free, $\widetilde{E}|_L$ is effective for a general L. Hence by the exact sequence

$$0 \longrightarrow \mathcal{O}_{Y'}(-K_{Y'} - 2L) \longrightarrow \mathcal{O}_{Y'}(-K_{Y'} - L) \longrightarrow \mathcal{O}_L(-K_L) \longrightarrow 0,$$

we have $h^0(-K_{Y'} - 2L) = 0$, a contradiction.

(ii) Since $\mathrm{Bs}|-K_{Y_1{}'}-m_1|=m_1$, $\mathrm{Bs}|-K_{Y'}-l_0|=l_0$. We can easily see that other conditions of Proposition 0.8 are satisfied.

Next we consider the case X is a $\mathbb{Q}$-Fano 3-fold of No. 4.5–No. 4.7. By [MM85, (7.7)], $Y_2{}'$ is a Fano 3-fold with a conic bundle structure $f_2'\colon Y_2{}'\to X'\simeq\mathbb{P}^2$. By the assumption, $m_1\cap m_2=\emptyset$ on $Y_2{}'$. Let l_i $(2\le i\le n-1)$ be fibers of h over $m_1\cap m_2\subset X_2{}'$.

Claim 4.12. (a) *There is a unique hyperplane section of $X_2{}'$ containing m_1 and m_2.*

(b) $\mathrm{Bs}|-K_{Y_2{}'}-m_1-m_2|=m_1\cup m_2\cup l_2\cup\cdots\cup l_{n-1}$.

Proof.

(a) Let $h'\colon Y_2{}''\to X_2{}'$ be the blow-up of $X_2{}'$ along m_1 and $G_3{}'$ h'-exceptional divisor. Then by [MM85, (7.7)], $Y_2{}''$ is a Fano 3-fold with a conic bundle structure $f_2{}''\colon Y_2{}''\to X''$. By [MM83, Theorem 5.1 (1)], we have $-K_{Y_2{}''}=L_2{}'+M_2{}'$, where $L_2{}'$ (resp. $M_2{}'$) is the pull-back of a hyperplane section of X'' (resp. $X_2{}'$). Hence we have $L_2{}'=M_2{}'-G_3{}'$. So the image of $f_2{}''^{-1}f_2{}''(m_2)$ on $X_2{}'$ is a hyperplane section of $X_2{}'$ containing m_1 and m_2 and such a hyperplane section is obtained by this way.

(b) By [MM83, Theorem 5.1 (1)], we have $-K_{Y_2{}'}=L_2+M_2$, where L_2 (resp. M_2) is the pull-back of a hyperplane section of X' (resp. $X_2{}'$). Since $-K_{Y_2{}'}\cdot m_j=n$ and $M_2\cdot m_j=n-1$, we have $L_2\cdot m_j=1$. Let $L_{2,j}$ $(j=1,2)$ be the member of $|L_2|$ containing m_j. Let $\widetilde{E}''$ be the strict transform of a hyperplane section of $X_2{}'$ containing m_1 and m_2. Since $\mathrm{Bs}\,|M_2-m_j|=m_j\cup l_2\cup\cdots\cup l_{n-1}$, we have $\mathrm{Bs}|-K_{Y_2{}'}-m_1-m_2|\subset(L_{2,1}\cap L_{2,2}\cap\widetilde{E}'')\cup(m_1\cup m_2\cup l_2\cup\cdots\cup l_{n-1})$. Since $L_{2,1}\cap L_{2,2}\cap m_i\subset L_{2,1}\cap L_{2,2}\cap\widetilde{E}''$ and $L_{2,1}\cdot L_{2,2}\cdot\widetilde{E}''=2$, we have $L_{2,1}\cap L_{2,2}\cap\widetilde{E}''=L_{2,1}\cap L_{2,2}\cap(m_1\cup m_2)$. So we have the assertion. □

Let G_2 be ν_2-exceptional divisor and $F_2{}'$ the strict transform of $L_{2,2}$, where f_2' is the structure morphism of conic bundle. By similar argument to the proof of Theorem 0.15 (A), we know that there is a flop $W_2{}'\dashrightarrow W_2$ over X' and an extremal contraction $\mu_2\colon W_2\to Y_1{}'$ of $(2,1)$-type over X' whose exceptional divisor F_2 is the strict transform of $F_2{}'$ on W_2. By Claim 4.12,

$$\mathrm{Bs}|-K_{Y_1{}'}-m_1-l_1|=m_1\cup l_1\cup\cdots\cup l_{n-1}\qquad(4.9)$$

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Let G_1 be ν_1-exceptional divisor and ${F_1}'$ the strict transform of ${f_1'}^{-1} f_1'(m_1)$, where $f_1' \colon {Y_1}' \to X'$ is the natural morphism. By a similar argument to above, we know that there is a flop ${W_1}' \dashrightarrow W_1$ over X' and an extremal contraction $\mu_1 \colon W_1 \to Y'$ of $(2,1)$-type over X' whose exceptional divisor F_1 is the strict transform of ${F_1}'$ on W_1.

(i) Since $L_{2,1} + \widetilde{E}'' \in |-K_{{Y_2}'}|$ (see the proof of Claim 4.12 for the notation), the strict transform $\widetilde{E}$ of $\widetilde{E}''$ on Y' satisfies $\widetilde{E} \sim -K_{Y'} - L$, where L is the pull-back of a line of X'. Conversely we can easily see that if $\widetilde{E} \sim -K_{Y'} - L$ is effective, it is the strict transform of $\widetilde{E}''$. Hence the uniqueness of $\widetilde{E}$ follows from Claim 4.12. The irreducibility of $\widetilde{E}$ can be proved similarly to No. 4.4.

(ii) By 4.9, $\mathrm{Bs}|-K_{Y'} - l_0 - l_1| = l_0 \cup l_1 \cup \cdots \cup l_{n-1}$. But since $l_0 \cup l_1 \subset \mathrm{Bs}|-K_{Y'}|$, we have $\mathrm{Bs}|-K_{Y'}| = l_0 \cup l_1 \cup \cdots \cup l_{n-1}$.

CLAIM 4.13. *l_i are mutually disjoint.*

Proof. It suffices to prove that l_i $(i \geq 2)$ do not intersect flopping curves for ${W_2}' \dashrightarrow W_2$ or ${W_1}' \dashrightarrow W_1$. This follows from Claim 4.12 because otherwise there exists a member of $|-K_{{W_1}'}|$ (resp. $|-K_{{W_2}'}|$) which intersects a flopping curve for ${W_2}' \dashrightarrow W_2$ (resp. ${W_1}' \dashrightarrow W_1$) but does not contain it, a contradiction. □

We can easily see that other conditions of Proposition 0.8 are satisfied. □

Proof of Theorem 0.18 (C). For No. 4.4, the proof is almost clear. For No. 4.5–No. 4.7, we prove that on B_d $(d = 3, 4, 5)$, there exist three smooth rational curves γ, m_1 and m_2 of degree $d-2$ which intersect the common $d-3$ points simply. First note that there exists at least one such a curve on a smooth hyperplane section H of B_d. Call it γ. Since $\mathrm{Bs}\,|H - \gamma| = \gamma$ by [MM85, Proposition 6.8], There exists two other hyperplane sections H_1 and H_2 containing γ. It is easy to see that we can take m_i on H_i as desired. □

TABLE 4, IV.

Proof of Theorem 0.19 (A) *and* (B). These are almost clear. □

Proof of Theorem 0.19 (C). Let $Z_1 := \mathbb{P}(\mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(-1))$, $\pi\colon Z_1 \to X' \simeq \mathbb{P}^2$ the natural projection and $\overline{E} \simeq \mathbb{P}^2$ the unique member of the tautological linear system. Let $P_1 \sim P_6$ be six points in a general position on X'. We denote by P_i' the point on $\overline{E}$ corresponding to P_i. Let F_{ij} be the pull-back of a line of X' through P_i and P_j and l_{ij} the section of F_{ij} intersecting $\overline{E}$ at two points P_i' and P_j'. We may assume l_{12}, l_{34} and l_{56} do not intersect mutually. Let $Z_1 \dashrightarrow Z_2$ be the elementary transformation along l_{12}, $Z_2 \dashrightarrow Z_3$ the elementary transformation along the strict transform of l_{34}, $Z_3 \dashrightarrow Y'$ the elementary transformation along the strict transform of l_{56}. Let $\widetilde{E}$ be the strict transform of $\overline{E}$ and L the pull-back of a line on Y'. $\widetilde{E}$ is obtained by blow-up $\overline{E}$ at ${P_1}' \sim {P_6}'$. Let m_i be the exceptional curve over P_i'. Then we can prove that $-K_{Y'} = L + 2\widetilde{E}$ and $\widetilde{E}|_{\widetilde{E}} = 2l - \sum m_i$, where l is the total transform of a line of $\overline{E}$. Let $C_{i_1i_2i_3i_4i_5}$ be the (-1)-curve on $\widetilde{E}$ linearly equivalent to $2l - m_{i_1} - m_{i_2} - m_{i_3} - m_{i_4} - m_{i_5}$. Let R be the other extremal ray of $\overline{\mathrm{NE}}(Y')$ than one generated by the class of fiber of $Y' \to X'$. We check the following.

(1) $-K_{Y'}$ is nef and big,
(2) R is generated by the class of $C_{i_1i_2i_3i_4i_5}$ and $\operatorname{Supp} R = \bigcup C_{i_1i_2i_3i_4i_5}$, and
(3) $-K_{Y'} \cdot R = 0$.

Since $\widetilde{E} \cdot C_{i_1i_2i_3i_4i_5} = -1$ and $\rho(Y') = 2$, we have $\widetilde{E} \cdot R < 0$. Hence $R \subset \operatorname{Im}(\overline{NE}(\widetilde{E}) \to \overline{NE}(\widetilde{Y'}))$. On the other hand it is easy to check that $-K_{Y'}|_{\widetilde{E}}$ is nef and numerically trivial only for $C_{i_1i_2i_3i_4i_5}$'s. Hence we have $-K_{Y'} \cdot R = 0$ and (2) follows. (3) becomes also clear. For (1), the nefness is already checked. The bigness follows from a direct calculation. In fact we have $(-K_{Y'})^3 = 14$. □

Table 5, I.

Proof of Theorem 0.20 (A). The proof is similar to [Take96]. Since Y' is a smooth del Pezzo fibration whose fibers are del Pezzo surfaces of degree 4, Y' can be embedded in $\mathbb{P}^4$-bundle. Let $\mathcal{E} := \sum_{i=0}^{4} \oplus \mathscr{O}(a_i)$ be the associated vector bundle of rank 5, where we may choose $a_0 = 0$ and $a_0 \le a_1 \le a_2 \le a_3 \le a_4$. Let H be the tautological divisor and F a fiber. In $\mathbb{P}(\mathcal{E})$, Y' is a complete intersection of $V_1 \in |2H - aF|$ and $V_2 \in |2H - bF|$ for some a and b. We may assume that $a \ge b$. Since $-K_{Y'} = H|_{Y'} + (a+b+2-\sum a_i)L$, we have $(-K_{Y'})^3 = 10(a+b) - 8\sum a_i + 24 = 4$. So we obtain

$$5(a+b) = 4\sum a_i - 10. \tag{A1}$$

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Note that $\widetilde{E} \sim -K_{Y'} - L = H - (\frac{a+b+6}{4})L$ by (A1). Let $c := \frac{a+b+6}{4}$. Since $H - cF - V_1$ and $(H-cF)|_{V_1} - Y'$ is not effective, we have $h^0(H-cF) \leq h^0((H-cF)|_{V_1}) \leq h^0((H-cF)|_{Y'}) = 1$. So we obtain

$$\text{(A2)} \qquad a_3 < c \quad \text{and} \quad a_4 \leq c.$$

This gives $\sum a_i \leq 4c - 3 = a + b + 3$ and hence by (A1), we obtain $a + b \leq 2$. On the other hand, there is a flipped curve $m_1{}'$ on $\widetilde{E}$, which satisfies $\widetilde{E} \cdot m_1{}' = -2$ and $F \cdot m_1{}' = 1$. Hence $H \cdot m_1{}' = \frac{a+b-2}{4}$. This is non-negative so we have $a + b \geq 2$. So we obtain $a + b = 2$ and by (A1), $\sum a_i = 5$. Together with (A2), we have $\mathcal{E} := \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 3} \oplus \mathscr{O}_{\mathbb{P}^1}(2)$. If $a \geq 3$, then V_1 must be non-reduced or reducible by looking at a local coordinate. If $a = 2$, then $\widetilde{E}$ must be singular along $m_1{}'$. Hence we have $a = b = 1$ and we are done. ∎

Proof of Theorem 0.20 (B).

(1) Let $\mu\colon Q \to \mathbb{P}(\mathcal{E})$ be the blow-up along l_0 and G the exceptional divisor. Since $N_{l_0/\mathbb{P}(\mathcal{E})} = \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 3} \oplus \mathscr{O}_{\mathbb{P}^1}(2)$, G contains the subvariety W which corresponds to the surjection $\mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 3} \oplus \mathscr{O}_{\mathbb{P}^1}(2) \to \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 3}$. Note that $|V' := \mu^*(2H - F) - G|$ is free since $\mu^*(H-F) - G$ and μ^*H is free. We can easily prove that for an irreducible curve l, $V' \cdot l = 0$ if and only if l is a fiber of the natural projection $W \simeq \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^2$ and if $V' \cdot l = 0$, then $K_Q \cdot l = 0$. So the Stein factorization of $\Phi_{|V'|}$ (we call it $\nu\colon Q \to R$) is a crepant primitive birational contraction and R has only hypersurface singularities. Hence by the Grothandieck-Lefschetz theorem [Gro68, p. 135, 3.18], a complete intersection of two members of $|\nu(V')|$ has the same Picard group as R. So for the strict transform V_i' of V_i, $\rho(\nu(V_1{}') \cap \nu(V_2{}')) = 2$ and hence $\rho(Y') = 2$.

(2) Since $-K_{Y'} = H - F|_{Y'}$, $\mathrm{Bs}|-K_{Y'}| = l_0$.

(3) By $h^0(H - 2F) = 1$, it is easy to see that $h^0((H-2F)|_{Y'}) = 1$. The irreducibility of $\widetilde{E}$ can be proved similarly to No. 4.4.

(4) This is easily proved.

∎

Proof of Theorem 0.20 (C). It is easy to see that we can take Y' as in (A) by looking on the local charts. ∎

TABLE 5, II.

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Proof of Theorem 0.21 (A). Let F_1 be μ_1-exceptional divisor. Since $-K_{Y'}$ is relatively very ample over X', $|-K_{W_1}|$ is relatively free and big over X'. Let R_1 be the extremal ray of W_1 over X' other than that associated to μ_1.

CLAIM 4.14. (1) *R_1 is a flopping ray and an irreducible curve whose numerical class generates R_1 is a transform of a curve γ on X with $-K_X \cdot \gamma = 1/2$ passing the $\frac{1}{2}(1,1,1)$-singularity on ${l_0}^-$, where ${l_0}^- \subset X$ is the strict transform of flipping curve corresponding to l_0.*

(2) *Let l_1 be a flipped curve different from l_0. Then γ does not intersect l_1 on W_1.*

Proof.

(1) By Theorem 0.3, there is a curve γ on X with $-K_X \cdot \gamma = 1/2$ passing the $\frac{1}{2}(1,1,1)$-singularity on ${l_0}^-$. By [Taka02, Proposition 2.1 (4)], we have $-K_{Y'} \cdot \gamma = \frac{1}{2} + \alpha$ and $\widetilde{E} \cdot \gamma = 2\alpha + \beta$ for a positive rational number $\alpha \in \frac{\mathbb{Z}}{2}$ and a non-negative rational number β. α (resp. β) describes the effect of the flip $Y_1 \dashrightarrow Y'$ (resp. the flop $Y \dashrightarrow Y_1$) (see [Taka02, §3] for the notation). By $L \sim -K_{Y'} - \widetilde{E}$, we have $L \cdot \gamma = \frac{1}{2} - \alpha - \beta$. Since L is nef, we have $\alpha = 1/2$ and $\beta = 0$. Moreover by $-K_{W_1} = \mu_1^*(-K_{Y'}) - F_1$, we have $-K_{W_1} \cdot \gamma = 0$. So the numerical class of γ generates R_1. Conversely let γ' be an irreducible curve whose numerical class generates R_1. When the fiber of f' containing γ' is anti-canonically embedded in a projective space, γ' is a line and hence $F_1 \cdot \gamma' = 1$. Since $\widetilde{E}$ is smooth along l_0, we have $\mu_1^* L \sim -K_{W_1} - \widetilde{E}'$, where $\widetilde{E}'$ is the strict transform of $\widetilde{E}$ on W_1. Hence we have $\widetilde{E}' \cdot \gamma' = 0$ and $\widetilde{E} \cdot \gamma' = 1$. By reversing the argument above, we can easily see that γ' is a curve as in the statement of Claim 4.14 on X. The finiteness of the number of γ''s follows from Theorem 0.3. In particular R_1 is a flopping ray.

(2) If γ intersects l_1, then γ is a flopped curve on Y_1, where $Y \dashrightarrow Y_1$ is the flop. But γ passes two $\frac{1}{2}(1,1,1)$-singularities, a contradiction to Corollary 3.3.

□

Let R_1' be the extremal ray of ${W_1}'$ over X' other than that associated to the flop $W_1 \dashrightarrow {W_1}'$ and $\nu_1 \colon {W_1}' \to {Y_1}'$ is the associated contraction. Let L' be a general fiber of $W_1 \to X'$ and denote by L the image of L' on Y'. Then by Claim 4.14, we may assume that L' is a del Pezzo surface of degree 4 in case X is a $\mathbb{Q}$-Fano 3-fold of No. 5.4 (resp. degree 5 in case X is a $\mathbb{Q}$-Fano 3-fold of No. 5.5). We consider L' is anti-canonically embedded in a projective space.

Claim 4.15. *ν_1 is of $(2,1)$-type.*

Proof. It suffices to prove that ν_1 is not of $(3,2)$-type. Assume that ν_1 is of $(3,2)$-type. Then ${Y_1}'$ is a $\mathbb{P}^1$-bundle over X'. Let M be the pull-back of a general section of ${Y_1}' \to X'$ on ${W_1}'$ and M' (resp. M'') the transform of M on W_1 (resp. Y'). We may assume that $M'|_{L'}$ is a smooth conic. Note that $F_1|_{L'}$ is a line. Since L' is an intersection of quadrics, $M'|_{L'}$ intersects a line at most one point. Hence $(M''|_L)^2 \leq 1$, a contradiction to the fact that the image of $\operatorname{Pic} Y' \to \operatorname{Pic} L$ is generated by $-K_L$ and $(-K_L)^2 \geq 4$. ∎

Let G_1 be the ν_1-exceptional divisor and $m_1 := \nu_1(G_1)$. Let ${G_1}'$ (resp. ${G_1}''$) the strict transform of G_1 on W_1 (resp. Y'). Note that ${G_1}'|_{L'}$ is a union of lines intersecting $F_1|_{L'}$ at one point. Since the image of $\operatorname{Pic} Y' \to \operatorname{Pic} L$ is generated by $-K_L$, we know that in case X is a $\mathbb{Q}$-Fano 3-fold of No. 5.4 (resp. No. 5.5), $G'|_{L'}$ is a union of five (resp. three) lines and ${W_1}'$ is a $\mathbb{P}^2$-bundle (resp. a quadric bundle). Hence we can write ${G_1}'' = 2(-K_{Y'}) + aL$ (resp. ${G_1}'' = (-K_{Y'}) + aL$) for some $a \in \mathbb{Z}$. Note that ${G_1}' = \mu_1^* {G_1}'' - 5F_1$ (resp. ${G_1}' = \mu_1^* {G_1}'' - 3F_1$). Then by easy calculations, we have

$$(-K_{{W_1}'})^2 G = 4a + 5. \tag{4.10}$$

$$(-K_{{W_1}'}) G^2 = 10a - 14. \tag{4.11}$$

$$\text{(resp.}\quad (-K_{{W_1}'})^2 G = 5a + 2. \tag{4.12}$$

$$(-K_{{W_1}'}) G^2 = 6a - 8). \tag{4.13}$$

Assume that X is a $\mathbb{Q}$-Fano 3-fold of No. 5.4. Since ${W_1}'$ is a $\mathbb{P}^2$-bundle, $(-K_{{W_1}'})^3 = 54$. So we have $a = 3$, $g(m_1) = 9$ and $(-K_{{Y_1}'} \cdot m_1) = 33$ by (4.10), (4.11) and Lemma 4.2. Let l_1 be flipped curves different from l_0. Since $\operatorname{Bs}|-K_{Y'}| = l_0 \cup l_1$, $\operatorname{Bs}|-K_{{W_1}'}| = l_1$. By Claim 4.14 (2), we have $G_1 \cdot l_1 = (2(-K_{Y'}) + 3L \cdot l_1) = 1$ whence $(-K_{{Y_1}'} \cdot l_1) = 0$. Thus ${Y_1}'$ is a weak Fano 3-fold.

Claim 4.16. *${Y_1}'$ has no crepant divisorial contraction.*

Proof. Assume the contrary. Then by the method of [Take99], we can easily see that ${Y_1}' \simeq \mathbb{P}(\mathscr{O}^{\oplus 2} \oplus \mathscr{O}(2))$. Let H be the tautological divisor, L_1 a fiber and T the subvariety associated to the surjection $\mathscr{O}^{\oplus 2} \oplus \mathscr{O}(2) \to \mathscr{O}^{\oplus 2}$. Then $T \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and a horizontal fiber l satisfies $-K_{{Y_1}'} \cdot l = 0$. Since there exists only a finite number of curves intersecting $-K_{{W_1}'}$ negatively, $m_1 \not\subset T$. If there exists an exceptional curve m for ${W_1}' \dashrightarrow W_1$ contained in the strict transform T' of T on ${W_1}'$, it must be the transform of a fiber of $T \to X'$ intersecting m_1. Note that $T \cdot m_1 = (H - 2L_1) \cdot m_1 = 1$. Hence m and m_1 intersect at one point simply. Then, however, $-K_{{W_1}'} \cdot m = 2$, a contradiction. Hence if we take l generally, l does not intersect an exceptional curve for ${W_1}' \dashrightarrow W_1$. Thus $F_1 \cdot l = {F_1}'' \cdot l = (2H + L_1) \cdot l = 1$ and so $-K_{Y'} \cdot l = 1$. If l intersects l_1, then l must be a flopping curve on Y_1 containing two $\frac{1}{2}(1,1,1)$-singularities, where $Y' \dashrightarrow Y_1$ is the anti-flip. This contradicts Corollary 3.4. If l intersects F_1 at a point of the negative section of F_1, then l must be a flopping curve on Y_1. Hence by the finiteness of the number of flopping curves, we may assume that l intersects F_1 outside the negative section of F_1 by taking l generally. Then $-K_{Y_1} \cdot l = 1/2$. Since $\widetilde{E} \cdot l = (-K_{Y'} - L) \cdot l = 0$, we have $E \cdot l = 0$. Since the strict transform of $\widetilde{E}$ on ${Y_1}'$ is linearly equivalent to $-K_{{Y_1}'} - L_1$, it is not equal to T. Hence we may assume that $l \cap E = \emptyset$. Thus we have $-K_Y \cdot l = 1/2$. However this contradicts the finiteness of the number of such curves. $\square$

Hence by the list of [Take99], ${Y_1}' \simeq \mathbb{P}(\mathscr{O} \oplus \mathscr{O}(1)^{\oplus 2})$. By ${G_1}' = \mu_1^* {G_1}'' - 5F_1$ and ${G_1}'' = 2(-K_{Y'}) + 3L$, we have $3{F_1}'' = 2(-K_{{Y_1}'}) + 3L_1$. So we obtain the description of ${F_1}''$.

Assume that X is a $\mathbb{Q}$-Fano 3-fold of No. 5.5. By (4.12), (4.13) and Lemma 4.2, we have $g(m_1) = 3a - 3$, $(-K_{{Y_1}'} \cdot m_1) = 11a - 6$ and $(-K_{{Y_1}'})^3 = 16a$. Since $g(m_1) \geq 0$, we have $a \geq 1$. Let l_i $(i \geq 1)$ be transforms of flipped curves different from l_0. By Claim 4.14 (2), we have $G_1 \cdot l_i = ((-K_{Y'}) + aL \cdot l_i) = a - 1$ whence $(-K_{{Y_1}'} \cdot l_i) = a - 2$.

Claim 4.17. $a \neq 1$.

Proof. Assume that $a = 1$. ${Y_2}'$ can be embedded in a $\mathbb{P}^3$-bundle $\mathbb{P}(\mathscr{E})$ over $\mathbb{P}^1$, where $\mathscr{E} := \bigoplus_{i=0}^{3} \mathscr{O}(a_i)$ is a vector bundle of rank 4. We may assume that

$$a_0 = 0 \leq a_1 \leq a_2 \leq a_3. \tag{4.14}$$

Let H be the tautological divisor and M a fiber. In $\mathbb{P}(\mathscr{E})$, ${Y_2}'$ is linearly equivalent to $2H - aM$ for some $a \in \mathbb{Z}$. Since $-K_{{Y_1}'} = 2H|_{{Y_1}'} + (2 + a -$

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$\sum_{i=0}^{3} a_i)L_1$ and $(-K_{Y_1{}'}) = 16$, we have $(-K_{Y_1{}'})^3 = 16a - 8\sum_{i=0}^{3} a_i + 48 = 16$. So we obtain

$$\sum_{i=0}^{3} a_i = 2a + 4. \tag{4.15}$$

Let $\widetilde{E}'$ be the strict transform of $\widetilde{E}$ on $Y_1{}'$. Note that $\widetilde{E} \sim -K_{Y_1{}'} - L_1 = 2H|_{Y_1{}'} - (a+3)L_1$ by (4.15). Since $L_1 \cdot l_1 = 1$ and $-K_{Y_1{}'} \cdot l_1 = -1$, we have

$$2H \cdot l_1 = a + 1 \geq 0. \tag{4.16}$$

Moreover since $h^j(-3M) = 0$ $(j = 0, 1)$, we have $h^0(2H - (a+3)M) = h^0((2H|_{Y_1{}'} - (a+3)L_1) = 1$. Hence

$$2a_3 = a + 3,\ \ a_2 < a_3. \tag{4.17}$$

By (4.17), we have $2(\sum_{i=0}^{3} a_i) \leq 2(a+1) + a + 3$ and then by (4.15), $a \leq -3$. But this contradicts (4.16). ∎

Assume that $a \geq 2$. Since $\mathrm{Bs}|-K_{Y'}| = \bigcup_{i=0}^{n-1} l_i$, $\mathrm{Bs}|-K_{W_1{}'}| = \bigcup_{i=1}^{n-1} l_i$. Moreover we know that $(-K_{Y_1{}'} \cdot l_i) = a - 2 \geq 0$ by the assumption. So $Y_1{}'$ is a weak Fano 3-fold. If $Y_1{}'$ has a crepant divisorial contraction, then by the method of [Take99], we can easily see that $(-K_{Y_1{}'})^3 = 8$, a contradiction. Hence by the list of [Take99], we have $a = 2$. Moreover since $2F_1{}'' = -K_{Y_1{}'} + 2L_1$, $-K_{Y_1{}'}$ is divisible by 2. Thus $Y_1{}'$ is a smooth divisor in $\mathbb{P}(\mathscr{O}^{\oplus 2} \oplus \mathscr{O}(1)^{\oplus 2})$ linearly equivalent to $2H$, where H is the tautological divisor. By $G_1{}' = \mu_1^* G_1{}'' - 3F_1$ and $G_1{}'' = (-K_{Y'}) + 2L$, we have $2F_1{}'' = (-K_{Y_1{}'}) + 2L_1$. So we obtain the description of $F_1{}''$. ∎

Proof of Theorem 0.21 (B). Let G_1 be ν_1-exceptional divisor and $F_1{}'$ the strict transform of $F_1{}''$. Since $\mathrm{Bs}|-K_{Y_1{}'} - m_1| = m_1 \cup l_1 \cup \cdots \cup l_{n-1}$, where l_i are all the flopping curves on $Y_1{}'$, $|-K_{W_1{}'}|$ is free outside l_i. By an argument similar to the proof of (A), we know that there is a flop $W_1{}' \dashrightarrow W_1$ over X' and an extremal contraction $\mu_1 \colon W_1 \to Y'$ of $(2,1)$-type over X' whose exceptional divisor F_1 is the strict transform of $F_1{}'$ on W_1. Since $\mathrm{Bs}|-K_{Y_1{}'} - m_1| = m_1 \cup l_1 \cup \cdots \cup l_{n-1}$, $\mathrm{Bs}|-K_{Y'} - l_0| = l_0 \cup \cdots \cup l_{n-1}$. It is easy to see that $-K_{Y'} \cdot l_i = -1$ by (A1)(3) or (A2)(3). Hence $\mathrm{Bs}|-K_{Y'}| = l_0 \cup \cdots \cup l_{n-1}$.

(i) Consider the exact sequence

$$0 \longrightarrow \mathscr{O}_{Y'}(-K_{Y'} - L) \longrightarrow \mathscr{O}_{Y'}(-K_{Y'}) \longrightarrow \mathscr{O}_L(-K_L) \longrightarrow 0,$$

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where L is a general fiber of f'. First we treat No. 5.4. Since $\mathrm{Bs}|-K_{Y'}| \cap L = (L \cap l_0) \cup (L \cap l_1)$ consists of two points and $-K_L$ is very ample, the dimension of $\mathrm{Im}(H^0(\mathscr{O}_{Y'}(-K_{Y'})) \to H^0(\mathscr{O}_L(-K_L)))$ is 4. On the other hand $h^0(\mathscr{O}_{Y'}(-K_{Y'})) = 5$. Hence we have $h^0(\mathscr{O}_{Y'}(-K_{Y'} - L)) = 1$.

Next we treat No. 5.5. Note that a general fiber of $W_1 \to X'$ is a del Pezzo surface of degree 5. Moreover $\mathrm{Bs}|-K_{Y'}| \cap L = (L \cap l_0) \cup (L \cap l_1) \cup (L \cap l_2)$ consists of three points and $-K_L$ is very ample. Thus the dimension of $\mathrm{Im}(H^0(\mathscr{O}_{Y'}(-K_{Y'})) \to H^0(\mathscr{O}_L(-K_L))$ is 4. On the other hand $h^0(\mathscr{O}_{Y'}(-K_{Y'})) = 5$. Hence we have $h^0(\mathscr{O}_{Y'}(-K_{Y'} - L)) = 1$.

Hence in any case, let $\widetilde{E} \in |-K_{Y'} - L|$ be the unique member. The irreducibility of $\widetilde{E}$ can be proved similarly to No. 4.4.

(ii) CLAIM 4.18. *l_i are mutually disjoint.*

Proof. It suffices to prove that l_i $(i \geq 2)$ do not intersect flopping curves for ${W_1}' \dashrightarrow W_1$. This follows from $\mathrm{Bs}|-K_{Y'}| = l_0 \cup \cdots \cup l_{n-1}$. In fact, otherwise there exists a member of $|-K_{{W_1}'}|$ which intersects a flopping curve for ${W_1}' \dashrightarrow W_1$ but does not contain it, a contradiction. □

(2-3) is checked before the proof of (i). We can easily see that other conditions are satisfied.

□

Proof of Theorem 0.21 (C). We prove that there exists m_1 as in (A-1) (3) or (A-2) (3). The assertion about the base locus is put off till Claim 4.21.

First we treat No. 5.4. Let ${F_1}'' \in |2H + L_1|$ be a general member, where L_1 is a fiber of the natural projection $p : {Y_1}' := \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2}) \to X' \simeq \mathbb{P}^1$. Then we show that ${F_1}''$ is a del Pezzo surface of degree 1. Note that $\Phi_{|H|}({Y_1}') \subset \mathbb{P}^4$ is a singular quadric Q with one ODP. Q contain a smooth del Pezzo surface S of degree 1 embedded in $\mathbb{P}^4$ by $|-K_S + l|$, where $l \simeq \mathbb{P}^1$ such that $l^2 = 0$. We may assume that the transform of S on ${Y_1}'$ is isomorphic to S. We denote it also by S. Write $S = aH + bL_1$. Let δ be the section of p corresponding to $\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2} \to \mathscr{O}_{\mathbb{P}^1}$. Note that δ is the exceptional curve for ${Y_1}' \to Q$. Since $S \cdot \delta = 1$, we have $b = 1$. Moreover since $(-K_S)^2 = ((3-a)H - L_1)^2(aH + L_1) = 1$, we have $a = 1$. Hence $S \in |2H + L_1|$ whence by taking ${F_1}''$ generally ${F_1}''$ is a del Pezzo surface of degree 1.

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We can regard ${F_1}''$ as a surface obtained by blowing up $\mathbb{P}^2$ at 8 points and let e_i $(i=1,\dots,8)$ be the exceptional curves, where we may assume that e_1 is a section of $p|_{{F_1}''}$ and e_i $(i\geq 2)$ are components of different 7 degenerate fibers. Let $\pi\colon \widetilde{F_1}'' \to {F_1}''$ be the blow-up at ${F_1}''\cap\delta$ and e_9 π-exceptional divisor (note that ${F_1}''\cap\delta$ consists of one point). Since $-K_{{F_1}''}=(H-L_1)|_{{F_1}''}$ and $\mathrm{Bs}\,|H-L_1|=\delta$, $|-K_{\widetilde{F_1}''}|$ is free. Let ${m_1}':=11l-6e_1-3\sum_{i=2}^{8}e_i-e_9$, where l is the pull-back of a line in $\mathbb{P}^2$.

Claim 4.19. *$|{m_1}'|$ is free.*

Proof. Since ${m_1}'=(3l-2e_1-e_2-e_3-e_4-e_5-e_6-e_7)+(3l-2e_1-e_2-e_3-e_4-e_5-e_6-e_8)+(3l-2e_1-e_2-e_3-e_4-e_5-e_7-e_8)+(2l-e_6-e_7-e_8-e_9)$, $|{m_1}'|$ is nef. Assume that $|{m_1}'|$ is not free. Since ${m_1}'-K_{\widetilde{F_1}''}$ is nef and $({m_1}'-K_{\widetilde{F_1}''})^2>4$, we can apply [Reide88] and obtain a contradiction similarly to the proof of Claim 4.8. □

Let $m_1\in|{m_1}'|$ be a general smooth member and we also denote by m_1 the image of m_1 on ${F_1}''$, which is also smooth. It is easy to check that $g(m_1)=9$ and $(-K_{{Y_1}'}\cdot m_1)=33$.

Next we treat No. 5.5. Let $V\in|H+M_1|$ be a general member, where M_1 is a fiber of the natural projection $p\colon \mathbb{P}(\mathscr{O}^{\oplus 2}\oplus\mathscr{O}(1)^{\oplus 2})\to X'$. Then we show that ${F_1}'':=V\cap{Y_1}''$ is a del Pezzo surface of degree 2 if we take ${Y_1}''\in|2H|$ generally. Note first that $H|_V$ is ample since H is numerically trivial only for horizontal sections of the subvariety S corresponding to $\mathscr{O}^{\oplus 2}\oplus\mathscr{O}(1)^{\oplus 2}\to\mathscr{O}^{\oplus 2}$ and we may assume that V does not contain them. Moreover since $\deg p_*\mathscr{O}_V(H|_V)=3$, we have $p_*\mathscr{O}_V(H|_V)\simeq\mathscr{O}(1)^{\oplus 3}$. Hence $V\simeq\mathbb{P}^2\times\mathbb{P}^1$ and $H|_V$ is a divisor of $(1,1)$-type. Since $H^0(\mathscr{O}(2H))\to H^0(\mathscr{O}_V(2H|_V))$ is surjective, it suffices to prove that a general divisor of $\mathbb{P}^2\times\mathbb{P}^1$ of $(2,2)$-type is a del Pezzo surface. But this is clear.

We can regard ${F_1}''$ as a surface obtained by blowing up $\mathbb{P}^2$ at 7 points and let e_i $(i=1,\dots,7)$ be the exceptional curves, where we may assume that e_1 is a section of $p|_{{F_1}''}$ and e_i $(i\geq 2)$ are components of different 6 degenerate fibers. Let $\pi\colon \widetilde{{F_1}''}\to{F_1}''$ be the blow-up at ${F_1}''\cap S$ and e_j $(j=8,9)$ π-exceptional divisors (note that ${F_1}''\cap S$ consists of two points). Since $-K_{{F_1}''}=(H-M_1)|_{{F_1}''}$ and $\mathrm{Bs}\,|H-M_1|=S$, $|-K_{\widetilde{F_1}''}|$ is free. Let ${m_1}':=7l-4e_1-2\sum_{i=2}^{7}e_i-e_8-e_9$, where l is the pull-back of a line in $\mathbb{P}^2$.

Claim 4.20. *$|{m_1}'|$ is free.*

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Proof. Since ${m_1}' = 2(3l - 2e_1 - \sum_{i=2}^{7} e_i) + (l - e_8 - e_9)$, $|{m_1}'|$ is nef. Assume that $|{m_1}'|$ is not free. Since ${m_1}' - K_{\widetilde{F_1}''}$ is nef and $({m_1}' - K_{\widetilde{F_1}''})^2 > 4$, we can apply [Reide88] and obtain a contradiction similarly to the proof of Claim 4.8. ∎

Let $m_1 \in |{m_1}'|$ be a general smooth member and we also denote by m_1 the image of m_1 on ${F_1}''$, which is also smooth. It is easy to check that $g(m_1) = 3$ and $(-K_{{Y_1}'} \cdot m_1) = 16$.

CLAIM 4.21. $\mathrm{Bs}|-K_{{Y_1}'} - m_1| = m_1 \cup l_1 \cup \cdots \cup l_{n-1}$.

Proof. Consider the exact sequence

$$0 \longrightarrow \mathscr{O}_{{Y_1}'}(-K_{{Y_1}'} - {F_1}'') \longrightarrow \mathscr{O}_{{Y_1}'}(-K_{{Y_1}'}) \longrightarrow \mathscr{O}_{{F_1}''}(-K_{{Y_1}'}) \longrightarrow 0.$$

Since $h^1(\mathscr{O}(-K_{{Y_1}'} - {F_1}'')) = 0$, $H^0(\mathscr{O}_{{Y_1}'}(-K_{{Y_1}'})) \to H^0(\mathscr{O}_{{F_1}''}(-K_{{Y_1}'}))$ is surjective. Note that the base locus of $|-K_{{Y_1}'} - {F_1}''| = |H|_{{Y_1}'} - L_1|$ is $l_1 \cup \cdots \cup l_{n-1}$. Since $|-K'_{Y_1}|_{{F_1}''} - m_1| = |l|$, we have the assertion. ∎

∎

§5. Excluding some possibilities

Next we exclude the cases in Tables 1′–5′. By Corollary 2.2, we may exclude these cases assuming that X has only $\frac{1}{2}(1,1,1)$-singularities.

PROPOSITION 5.1. *Assume that X has only $\frac{1}{2}(1,1,1)$-singularities. Let l be an irreducible component of a flopping curve. Then $\mathcal{N}_{l/Y} \simeq \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-2)$ or $\mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$.*

Proof. Since $|-2K_X|$ is very ample by Corollary 3.3 and $\bar{l} := f(l)$ is a line with respect to $-2K_X$, the natural map $H^0(\mathscr{O}(-2K_Y) \otimes \mathcal{I}_l) \to \mathscr{O}(-2K_Y) \otimes \mathcal{I}_l$ is surjective, where $\mathcal{I}_l$ is the ideal sheaf of l. Hence there is a smooth member S of $|-2K_Y|$ containing l. The assertion follows from this easily. ∎

TABLE 1′. $h = 8$ and $N = 3$.

We prove that this case does not occur. By Proposition 5.1, $\widetilde{E}$ is normal and has only canonical singularities. Let $Y' \dashrightarrow Y_1$ be the anti-flip and $E_1 \subset Y_1$ the strict transform of $\widetilde{E}$. Note that there is a natural morphism $\widetilde{E} \overset{\mu_2}{\to} E_1 \overset{\mu_1}{\to} E$ and the exceptional locus of μ_1 is the union of flopped curves and the exceptional locus of μ_2 is the union of flipped

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curves. Set $\mu := \mu_1 \circ \mu_2$. Let $\mu_3 \colon \widehat{E} \to \widetilde{E}$ be the minimal resolution and $\nu := \mu \circ \mu_3$. Then ν is a composite of 5 times of blow-ups at points. Let F_i $(i = 1, \ldots, 5)$ be the total transforms of the exceptional curves of one point blow-ups and l the total transform of a line of E. As in the proof of (A) of Table 1, a flipping curve l_1 intersects E_1 transversely at a smooth point and $\widetilde{E} \to E_1$ is a two point blow-up at smooth points. Hence we may assume that $\mu_2 \circ \mu_3(F_1) \sim \mu_2 \circ \mu_3(F_3)$ are flopped curves and $\mu_3(F_4)$ and $\mu_3(F_5)$ are flipped curves, which are (-1)-curves contained in the smooth locus of $\widetilde{E}$.

Let γ be a smooth curve on E and $\hat{\gamma}$ (resp. $\tilde{\gamma}$) the strict transform of γ on $\widehat{E}$ (resp. $\widetilde{E}$). Then we can write $\hat{\gamma} = (\nu)^*\gamma - \sum_{i=i_1}^{i_\alpha} F_i - \sum_{j=j_1}^{j_\beta} F_i$, where $i_1 < i_2 \cdots i_\alpha \leq 3 < j_1 \cdots < j_\beta \leq 5$. Then we have

(1) $-K_{Y'} \cdot \widetilde{\gamma} = -K_Y \cdot \gamma + \beta$, and
(2) $\widetilde{E} \cdot \widetilde{\gamma} = E \cdot \gamma + \alpha + 2\beta$.

In fact, (1) follows from [Taka02, Proposition 2.1 (4)], and (2) follows from (1) and $(-K_{\hat{E}}) \cdot \hat{\gamma} = (-K_E) \cdot \gamma - (\alpha + \beta)$.

By Riemann-Roch, we can see that $h^0(2l - \sum_{i=1}^{5} F_i) > 0$. Let m be a member of $|2l - \sum_{i=1}^{5} F_i|$ and $m' := \nu(m)$.

If m' is a reducible (possibly non-reduced) conic, let $m' = m_1 + m_2$ be the irreducible decomposition. We can express $\hat{m}_1$ and $\hat{m}_2$ as $\hat{\gamma}$. If there are at most 2 blow-ups on the strict transform of m_i, then $\hat{m}_i$ is not a (-2)-curve and hence it is not contracted by $f'|_{\widetilde{E}}$. Since $f'|_{\widetilde{E}}(m)$ is a line, one of m_i's must be contracted. Hence there are at least 3 blow-ups on one m_i, say m_1. We use α and β for m_1. Then we have $E' \cdot \tilde{m}_1 = 8 - (3\alpha + 4\beta) \leq 8 - 3\alpha \leq -1$ and $-K_{X'} \cdot f'(m_1) = 9 - 3(\alpha + \beta) \leq 0$. Hence $\tilde{m}_1$ is a fiber of E' and $\alpha = 3$ and $\beta = 0$, i.e., $\hat{m}_1 = \nu^* m_1 - F_1 - F_2 - F_3$.

Next we show that $\hat{m}_2 = \nu^* m_2 - F_4 - F_5$. It suffices to prove that neither F_1, F_2 or F_3 contains F_4 or F_5. If otherwise, one of $\mu_3(F_4)$ and $\mu_3(F_5)$, say $\mu_3(F_4)$ intersects one of $\mu_3(F_1) \sim \mu_3(F_3)$. If a flipping curve intersects an irreducible component a of a flopped curve, then a become a fiber of E' on Y' by [Taka02, Proposition 2.2 (4)]. By $\hat{m}_1 = \nu^* m_1 - F_1 - F_2 - F_3$, the centers of F_4 are not on the strict transform of m_1. Hence the transform of a on E' intersects $\tilde{m}_1$, a contradiction to irreducibility of a fiber of E'. In particular we know that $m_1 \neq m_2$. So we have $\widetilde{E} \cdot \tilde{m}_2 = 2$ and $-K_{Y'} \cdot \tilde{m}_2 = 3$. Hence $E' \cdot \tilde{m}_2 = 0$ and $-K_{X'} \cdot f'(\tilde{m}_2) = 3$. The latter shows that $f'(\tilde{m}_2)$ is a line on X' and hence $\tilde{m}_2 \not\subset E'$. So the former shows that $E' \cap \tilde{m}_2 = \emptyset$.

In particular $\tilde{m}_1 \cap \tilde{m}_2 = \emptyset$. But this is a contradiction since there is no blow-up at the intersection of m_1 and m_2 by $m = \hat{m}_1 + \hat{m}_2$.

If m' is a smooth conic, m' is the strict transform of m. We have $E' \cdot m' = -1$ and $-K_{X'} \cdot f'(m') = 3$. Hence $f'(m')$ is a line on X'. But $f'(m') = C$ since $m' \subset E'$, a contradiction.

TABLE $2'$. $N = 7$.

We deny the possibility of $N = 7$ in Table $2'$ in Theorem 0.3. By the same method, we obtain some properties of No. 2.3 and No. 2.4 in Table 2 of Theorem 0.3.

Let P be a Gorenstein singularity on C. Let $g\colon Z \to X'$ be the blow-up of P and F the exceptional divisor. Since $(P \in X') \simeq (o \in ((xy + zw = 0) \subset \mathbb{C}^4))$ or $(o \in ((xy + z^2 + w^3 = 0) \subset \mathbb{C}^4))$ by [Taka02, Proposition 2.2] and X' is $\mathbb{Q}$-factorial, $\rho(Z) = 2$. Since $-K_{X'}$ is very ample and $-K_Z = g^*(-K_{X'}) - F$, $|-K_Z|$ is free and $(-K_Z)^3 > 0$. Hence Z is a weak Fano 3-fold. Starting by g, we consider the following diagram similar to one in Section 3 and do calculations similar to one there.

$$\begin{array}{ccccc} & Z & \dashrightarrow & Z' & \\ {}^{g}\swarrow & & & & \searrow^{g'} \\ X' & & & & X''. \end{array}$$

Let $\widetilde{F}$ be the strict transform of F on Z'. Then by a similar way, we have $(-K_{Z'})^2\widetilde{F} = 2$, $(-K_{Z'})(\widetilde{F})^2 = -2$ and $(\widetilde{F})^3 = 2 - e'$, where e' is a non-negative integer. Set $d := (-K_{Z'})^3$ and input these into (1-1)–(5-1). Then we obtain the following.

If $N = 5$, then $e' = 6$ and g' is a conic bundle over $\mathbb{P}^2$ with $\deg \Delta' = 6$, where Δ' is the discriminant divisor for g'.

If $N = 6$, then $e' = 5$ and g' is of $(2, 1)$-type and $X'' \simeq \mathbb{P}^3$. Let F' be the exceptional divisor of g'. Then $F' \sim 3(-K_{Z'}) - 4\widetilde{F}$. For the center C' of g', $\deg C' = 8$ and $p_a(C') = 6$.

If $N = 7$, then $e' = 4$ and g' is of $(2, 1)$-type and $X'' \simeq Q_3$. (Since X'' is $\mathbb{Q}$-factorial, X'' is a smooth quadric.) Let F' be the exceptional divisor of g'. Then $F' \sim 2(-K_{Z'}) - 3\widetilde{F}$. For the center C' of g', $\deg C' = 8$ and $p_a(C') = 4$.

Assume that $N = 7$. Then Z' has 5 Gorenstein singular points on the strict transform of C. Since X'' is smooth, C' must have 5 singular points by [Cu, Theorem 4]. But by $p_a(C') = 4$, this is impossible.

TABLE $4'$. $h = 5$ and $N = 6, 7$.

Let l be the fiber containing two $\frac{1}{2}(1,1,1)$-singularities and Q a $\frac{1}{2}(1,1,1)$-singularity. Let $g\colon Z \to Y'$ be the blow-up at Q. Then the transform l' of l is a flipping curve. Let $Z \dashrightarrow Z'$ be the flip. Then Z' has a conic bundle structure $g'\colon Z' \to W$ over $W \simeq \mathbb{F}_2$. There is a natural morphism $\mu\colon W \to X'$. Let Δ' be the strict transform of Δ on W. Note that Δ does not pass through the vertex of X' by [Taka02, Proposition 2.4 (3-1)].

If $\deg \Delta = 2$, then $\Delta' \sim C_0 + 2f$, where C_0 is the negative section and f is a fiber. Since Δ' is disjoint from C_0, $\Delta' \simeq \mathbb{P}^1$. This contradicts [MM85, Proposition 4.7 (1)]. Hence this case does not occur and moreover by Corollary 2.3, the case that $\deg \Delta = 0$ does not occur.

TABLE $4'$. $h = 6$ and $N = 6, 7$.

If $\deg \Delta = 1$, then $\Delta \simeq \mathbb{P}^1$. Hence this case does not occur by [MM85, Proposition 4.7 (1)]. If $\deg \Delta = 2$, then by the same reason, Δ must be a reducible conic. But this contradicts [MM85, Proposition 4.7 (2)].

TABLE $5'$. $h = 4$.

We deny the possibilities of Table $5'$ in Theorem 0.3. By the same method, we obtain some properties of No. 5.1 below.

By Riemann-Roch theorem, we can see $h^0(-K_{Y'} - \widetilde{E}) = 1$. Let $D \in |-K_{Y'} - \widetilde{E}|$ be the unique member. Since $2D \sim F$, $2D$ is a multiple fiber and since the reduced part of any fiber is irreducible, D is irreducible. Since D is not Cartier at $\frac{1}{2}(1,1,1)$-singularities, all $\frac{1}{2}(1,1,1)$-singularities are contained in D. Let Q be a $\frac{1}{2}(1,1,1)$-singularity and $g\colon Z \to Y'$ be the blow-up at Q. Let G be the exceptional divisor and D' the strict transform of D on Z. Set $D' = g^*D - \delta G$. We can prove that $|-K_Z|$ is free outside the transforms of flipped curves. By considering extremal rays over X', we obtain a diagram

$$Z_0 := Z \dashrightarrow Z_1 \dashrightarrow \cdots \dashrightarrow Z_k := Z \xrightarrow{g'} Y''$$

similar to one in [Taka02, Lemma 3.2]. Let G_i (resp. D_i) be the strict transform of G (resp. D') and R_i the extremal ray in $\overline{\mathrm{NE}}(Z_i/X')$ which is other than the ray associated to g if $i = 0$ or K_{Z_i}-negative if $i \geq 1$. If R_0 is a crepant divisorial ray, then the exceptional divisor is D'. By $(-K_Z)^2 D' = (-K_Z)^2(g^*D - \delta G) = \deg D - \delta = 0$, $\delta \in \mathbb{N}$. But since D is not a Cartier divisor, δ cannot be an integer, a contradiction.

CLAIM 3. *$D_i \cdot R_i < 0$.*

Proof. The proof is similar to one of Claim 4.1. □

Hence g' is $K_{Z'}$-negative divisorial contraction of D_k. By calculations similar to [Taka02, Lemma 3.1], we have

$$(1)\qquad (-K_{Z'})^2 D_k = \deg D - \delta - \sum a_i d_i,$$

$$(2)\qquad (-K_{Z'})(D_k)^2 = -2\delta^2 - \sum {a_i}^2 d_i,\ \textit{and}$$

$$(3)\qquad (D_k)^3 = -4\delta^3 - \sum {a_i}^3 d_i - e,$$

where e, a_i and d_i are defined similarly to [Taka02, Lemma 3.1]. Note that a_i is a non-positive integer and $e \leq 0$. Assume that g' is of $(2,1)$-type. Then by

$$(4)\qquad (-K_{Z'} + D_k)^2 D_k = \deg D - \delta(2\delta+1)^2 - \sum d_i a_i (a_i+1)^2 - e = 0.$$

On the other hand,

$$(5)\quad (-K_{Z'} - D_k)^2 D_k = -4(-K_{Z'})(D_k)^2 = 8\delta^2 + 4\sum d_i {a_i}^2 = 8(1-g) - 2m,$$

where g is the genus of $g'(D_k)$ and m is an natural number. Hence $\delta = 1/2$. But this contradicts (4) since $\deg D = 3$ or 4.

Assume that f is of $(2,0)$-type. By [Taka02, Proposition 2.3], we have $(-K_{Z'})(D_k)^2 \geq -2$. So by (2), we have $\delta = 1/2$ and $a_i = -1$ if $a_i \neq 0$. By [Taka02, Proposition 2.3], we have $e = 0$ (i.e., there is no flop while $Z \dashrightarrow Z'$) and if $\deg D = 3$, then g' is of $(2,0)_1$-type and $\sum d_i = 3/2$ and if $\deg D = 4$, then g' is of $(2,0)_5$-type and $\sum d_i = 1$. In any case Y'' is smooth.

If $\deg D = 4$, then $Y'' \to X$ is a quadric bundle. Hence by [Mor82, Theorem (3.5)], we can write $-K_{Y''} \sim 2H + aF'$, where F' is a fiber, H is a divisor such that $H|_{F'}$ is a hyperplane section and a is an integer. Let H' be the transform of ${g'}^* H$ on Y. Then we have $-K_Y \sim 2H' + (2a-3)(-K_Y - E)$ (note that $D \sim -K_Y - \widetilde{E}$). So there is a divisor $\overline{E}$ such that $E \sim 2\overline{E}$, a contradiction. Hence the case $\deg D = 4$ does not occur.

Assume that $\deg D = 3$ below. Since $n = 0$, $-K_Z$ is nef and big and so is $-K_{Y''}$. By calculations similar to [Taka02, Lemma 3.1], we can see that Y'' has a flopping ray and after the flop $Y'' \dashrightarrow Y''^{+}$, there is an extremal contraction $h\colon Y''^{+} \to W$ of $(2,0)_1$-type such that W is A_{18}. Note that the strict transform of G is obtained from G by blow-up three points on the line $l := D' \bigcap G$ and contracting the strict transform of l.

COROLLARY 5.2. *Let X be a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-fold with* (1)–(4) *in Main Assumption* 0.1. *Then* $(-K_X)^3$ *and* $\mathrm{aw}(X)$ *are effectively bounded as in Theorem* 0.3.

Proof. By Theorem 0.3 and Theorem 2.0, we obtain the assertion since $(-K)^3$ and aw are invariant under a deformation. □

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