A CONSTRUCTION OF ASYMPTOTIC SOLUTIONS
AND THE EXISTENCE OF
SMOOTH NULL-SOLUTIONS
FOR A CLASS OF
NON-FUCHSIAN PARTIAL DIFFERENTIAL OPERATORS

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§1. Introduction

Consider a partial differential operator

\[ P = \sum_{j + |\alpha| \leq m} a_{j,\alpha}(t, x) \partial_t^j \partial_x^\alpha, \quad a_{m,0}(t, x) \equiv t^\kappa, \]

where \( \kappa \) is a non-negative integer and \( a_{j,\alpha} \) are real-analytic in a neighborhood of \((0,0) \in \mathbb{R}_+ \times \mathbb{R}^n\).

M. S. Baouendi and C. Goulaoic [1] defined Fuchsian partial differential operators, and proved the unique solvability of the characteristic Cauchy problems in the category of real-analytic (or holomorphic) functions, which is a generalization of the classical Cauchy-Kowalevsky theorem. They also proved a generalization of the Holmgren uniqueness theorem. Especially, from their results it easily follows that if \( P \) is a Fuchsian operator with real-analytic coefficients, then there exist no sufficiently smooth null-solutions. Here, a Schwartz distribution \( u \) in a neighborhood of \((0,0)\) is called a null-solution for \( P \) at \((0,0)\), if \( Pu = 0 \) in a neighborhood of \((0,0)\) and \((0,0) \in \text{supp } u \subset \{ t \geq 0 \} \), where \( \text{supp } u \) denotes the support of \( u \).

The author considered the characteristic Cauchy problems for a class of operators wider than the Fuchsian operators in [3]. In that result, he showed the unique solvability of the characteristic Cauchy problems in the category of functions which are of class \( C^\infty \) with respect to \( t \) and real-analytic with respect to \( x \). He also showed the non-existence of sufficiently smooth null-solutions. (As for
distribution null-solutions, see [4]). This class of operators is defined in terms of four conditions. He gave a conjecture that if the third condition is violated, then there exists a $C^\infty$ null-solution.

In this article, we construct an asymptotic solution of $Pu = 0$ in the form

$$ u(t, x) := \exp \left( - \sum_{j=0}^{M} \frac{\lambda[j](x)}{\mu[j]} t^{-\mu[j]} \right) \cdot t^{l(M+1)(x)} \cdot \sum_{l=0}^{m} \sum_{p=0}^{l} (\log t)^p v_{l,p}(x), $$

where

(i) $M$ is a non-negative integer, and $q$ is a positive integer,

(ii) $\mu[j] (j = 0, 1, \ldots, M)$ are positive rational numbers such that $\mu[0] > \mu[1] > \cdots > \mu[M] > 0$.

(iii) $\lambda[j] (j = 0, 1, \ldots, M + 1)$ and $v_{l,p} (l \geq 0 ; 0 \leq p \leq lm)$ are real-analytic in a fixed open neighborhood of $0 \in \mathbb{R}^n$.

for a class of operators wider than that considered in [3].

Further, using these asymptotic solutions, we prove the conjecture in [3] mentioned above under an additional assumption. The $C^\infty$ null-solution constructed here is one of the most fastly decaying nontrivial solutions of $Pu = 0$.

In Section 2, we give the statements of the main theorems. After giving some preliminaries in Section 3, we prove the main theorems in Sections 4 and 5.

**Notations:**

(i) The set of all integers (resp. nonnegative integers) is denoted by $\mathbb{Z}$ (resp. $\mathbb{N}$). Put $\mathbb{N}/q := \{p/q : p \in \mathbb{N}\}$ for a positive integer $q$, and put $\mathbb{Z}/q$ similarly.

(ii) Put $\partial = t\partial_t$.

(iii) For a bounded domain $\Omega$ in $\mathbb{C}^n$, we denote by $\mathcal{O}(\Omega)$ the set of all holomorphic functions on $\Omega$.

(iv) The space of the Schwartz distributions on $U$ is denoted by $\mathcal{D}'(U)$.

(v) For a complete locally convex topological vector space $E$, put

$$ C_{\text{fiss}}^N([0, T] ; E) := \{ f \in C^N([0, T] ; E) : \frac{d^j f}{dt^j} \bigg|_{t=0} = 0 \text{ for } 0 \leq j \leq N - 1 \}. $$

(vi) Put $(\lambda)_j := \Pi_{i=1}^{j-1}(\lambda - i)$ for $\lambda \in \mathbb{C}$ and $j \in \mathbb{N}$.

(vii) For a commutative ring $R$, the ring of polynomials of $\lambda$ with the coefficients belonging to $R$ is denoted by $R[\lambda]$. The degree of $F \in R[\lambda]$ is denoted by $\deg F$. 

§2. Statement of the main result

Let $q$ be a positive integer, $\Omega$ be a bounded domain in $\mathbb{C}^n$ that includes the origin $0$, and $T$ be a positive real number. Consider a linear partial differential operator of the form (1.1). We assume only the following weaker condition on the coefficients.

\[(A-0)\quad a_{j,a} \in \widehat{\mathcal{F}}_{q}([0, T] ; \mathcal{O}(\Omega)) \quad (j + |\alpha| \leq m),\]

where

\[
\mathcal{F}_{q}([0, T] ; \mathcal{O}(\Omega)) := \{ \phi \in C^{\infty}([0, T] ; \mathcal{O}(\Omega)) : [s \mapsto \phi(s^{q})] \in C^{\infty}([0, T^{1/q}] ; \mathcal{O}(\Omega)) \},
\]

\[
\widehat{\mathcal{F}}_{q}([0, T] ; \mathcal{O}(\Omega)) := \{ \phi \in C^{\infty}((0, T] ; \mathcal{O}(\Omega)) : t^{M} \phi(t) \in \mathcal{F}_{q}([0, T] ; \mathcal{O}(\Omega)) \text{ for some } M \in \mathbb{N} \}.
\]

Let $r(j, \alpha)$ be the generalized vanishing order of $a_{j,a}$ on the hypersurface $\Sigma := \{(0, x) : x < \Xi \Omega \}$, that is

\[(2.1)\quad r(j, \alpha) := \sup \{ r \in \mathbb{Z} / q : t^{-r}a_{j,a} \in \mathcal{F}_{q}([0, T] ; \mathcal{O}(\Omega)) \}.
\]

If $r(j, \alpha) = \infty$, then we redefine $r(j, \alpha) := R$ for a sufficiently large $R$ ($R := \max\{r(j, \alpha) : r(j, \alpha) < \infty\} + 1$ will suffice). Put

\[(2.2)\quad a_{j,\alpha}(t, x) := t^{-r(j,\alpha)}a_{j,a}(t, x) \quad (\in \mathcal{F}_{q}([0, T] ; \mathcal{O}(\Omega))).
\]

Note that if $r(j, \alpha) < R$, then $a_{j,\alpha}(0, x) \neq 0$.

Associating a weight $\omega(j, \alpha) := j - r(j, \alpha)$ to each differential monomial $a_{j,a}(t, x) \partial_{t}^{j} \partial_{x}^{\alpha}$, we draw a Newton polygon $\Delta(P)$ using the points $(j + |\alpha|, -\omega(j, \alpha))$ $(j + |\alpha| \leq m)$ in $(u, v)$-plane as follows.

**Definition 2.1 ([3]).** (1) Put

\[
\Delta(P) := \text{ch} \left( \bigcup_{j + |\alpha| \leq m} \{(u, v) \in \mathbb{R}^{2} : u \leq j + |\alpha|, \ v \geq -\omega(j, \alpha) \} \right),
\]

where $\text{ch}(A)$ denotes the convex hull of $A$. This is called the *Newton polygon* of $P$. 

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(2) Put
\[ \tilde{V} = \tilde{V}(P) := \{(j, \alpha) \in \mathbb{N} \times \mathbb{N}^n : (j + |\alpha|, -\omega(j, \alpha)) \text{ is a vertex of } \Delta(P)\}. \]

(3) Put
\[ \omega = \omega(P) := \max \{\omega(j, \alpha) \in \mathbb{R} : j + |\alpha| \leq m\}, \]
which is the maximum weight of \( P \).

(4) The boundary of \( \Delta(P) \cap ([0, \infty) \times \mathbb{R}) \) is the union of two vertical half-lines and a finite number of compact line segments with distinct slopes. Each of these compact line segments is called a lower side of \( \Delta(P) \). The set of the slopes of the lower sides of \( \Delta(P) \) is denoted by \( S = S(P) (\subseteq \mathbb{Q}) \). For \( \mu \in S(P) \), the lower side of \( \Delta(P) \) with slope \( \mu \) is denoted by \( L_\mu = L_\mu(P) \). Put
\[ I_\mu = I_\mu(P) := \{(j, \alpha) \in \mathbb{N} \times \mathbb{N}^n : (j + |\alpha|, -\omega(j, \alpha)) \in L_\mu(P)\}. \]

Let the right end points of \( L_\mu(P) \) be \( (u, v) \). We put \( d_\mu(P) := u \), and call it the degree of the slope \( \mu \).

If \( 0 \not\in S \), we put \( L_0(P) := \{(0, -\omega(0,0))\} = \{(0, -\omega(P))\} \subseteq \mathbb{R}^2 \), \( I_0(P) := \{(0,0)\} \subseteq \mathbb{N} \times \mathbb{N}^n \), and \( d_0(P) := 0 \).

By the use of these notions, Fuchsian operators in the sense of M. S. Baouendi and C. Goulaouic [1] are characterized as follows. (In fact, they assumed that the
coefficients belong to $C^m([0, T]; \Theta(\Omega))$. This difference is, however, not essential and hence we ignore the difference of the classes of coefficients.)

**Proposition 2.2.** The operator $P$ is Fuchsian if and only if $\omega(P) \geq 0$, $S(P) = \{0\}$, and there exist no $(j, \alpha) \in I_0(P)$ such that $\alpha \neq 0$.

We consider a class of operators wider than the class of Fuchsian operators. First, we assume the following condition.

**(A-1)** For all $\mu \in S(P)$, there exist no $(j, \alpha) \in I_\mu(P)$ such that $\alpha \neq 0$.

**Definition 2.3.** For $\mu \in S(P)$ with $\mu > 0$, we put

$$E_\mu[P](x; \lambda) := \sum_{(j, \alpha) \in I_\mu(P)} \bar{a}_{j,\alpha}(0, x) \lambda^j \in \Theta(\Omega)[\lambda].$$

We also put

$$E_0[P](x; \lambda) := \sum_{(j, \alpha) \in I_\mu(P)} \bar{a}_{j,\alpha}(0, x) \lambda^j \in \Theta(\Omega)[\lambda].$$

The polynomial $E_\mu[P]$ of $\lambda$ is called the indicial polynomial of $P$ associated with the slope $\mu \in S(P) \cup \{0\}$. Note that $d_\mu(P) = \deg E_\mu[P]$.

For $\mu \in S(P) \cup \{0\}$, we consider the following condition.

**(A-2; $\mu$)** If $(j, 0) \in \tilde{V}(P)$ and $j \geq d_\mu(P)$, then $\bar{a}_{j,0}(0,0) \neq 0$.

This is equivalent to the following.

**(A-2; $\mu$)** For every $\nu \in S(P)$ with $\nu \geq \mu$, the coefficient of the top order term of $E_\nu[P](x; \lambda) \in \Theta(\Omega)[\lambda]$ does not vanish at $x = 0$.

**Remark 2.4.** Note that if $(j, 0) \in \tilde{V}(P)$, then $\bar{a}_{j,0}(0, x) \neq 0$. Thus, the condition $(A-2; \mu)$ is a kind of non-degeneracy at $x = 0$. Further, the condition $(A-2; \mu)$ for $\mu > 0$ is weaker than the condition $(A-2; 0)$, and $(A-2; 0)$ is equivalent to $(A-2)$ in [3].

Now, the following is one of the three main theorems in this article.

**Theorem 2.5.** Assume that $P$ satisfies $(A-0)$ and $(A-1)$. Let $\mu_0 \in S(P) \cap N/q$, $\mu_0 > 0$, and assume the condition $(A-2; \mu_0)$. If $\lambda_0$ is a simple root of $E_{\mu_0}[P](0; \lambda) = 0$, then there exist

(i) $M \in N,$
(ii) \( \mu[j] \in \mathbb{N}/q (j = 0, 1, \ldots, M) \), where \( \mu_0 = \mu[0] > \mu[1] > \cdots > \mu[M] > 0 \).

(iii) a subdomain \( \Omega_0 \) of \( \Omega \) including 0,

(iv) \( \lambda[j] \in \partial(\Omega_0) \) \( (j = 0, 1, \ldots, M + 1) \), where \( \lambda[0](0) = \lambda_0 \),
such that the following holds.

For an arbitrarily given \( v_{0,0}(x) \in \partial(\Omega_0) \), there exists \( v_{1,p}(x) \in \partial(\Omega_0) \) \( (l \geq 0; 0 \leq p \leq |m|) \) such that a formal series

\[
(2.3) \quad u(t, x) := \exp \left( -\sum_{j=0}^{M} \frac{\lambda[j](x)}{\mu[j]} t^{-\mu[j]} \right) \cdot \sum_{l=0}^{\infty} \sum_{p=0}^{l/m} (\log t)^p v_{1,p}(x)
\]

is an asymptotic solution of \( Pu = 0 \). That is, for every \( N \in \mathbb{N} \) there holds

\[
(2.4) \quad t^{-1(M+1)q} \cdot \exp \left( \sum_{j=0}^{M} \frac{\lambda[j](x)}{\mu[j]} t^{-\mu[j]} \right) \cdot P \left( \exp \left( -\sum_{j=0}^{M} \frac{\lambda[j](x)}{\mu[j]} t^{-\mu[j]} \right) \right) \\
\times t^{1(M+1)q} \cdot \sum_{l=0}^{N} \sum_{p=0}^{l/m} (\log t)^p v_{1,p}(x) = o(t^{N/q-r_0}),
\]

with some \( r_0 \in \mathbb{N} \).

This theorem shall be proved in Section 4. We shall also give a proposition which corresponds to the case of \( \mu_0 = 0 \) and \( M = -1 \).

Remark 2.6. Even if \( \mu_0 \in S(P) \) but \( \mu_s \notin \mathbb{N}/q \), we can retake another \( q \) such that \( \mu_0 \in \mathbb{N}/q \) and (A-0) is satisfied. Hence, we can always apply this theorem with this new \( q \).

Next, we consider the following condition for \( \mu \in S(P) \).

(A-6: \( \mu \)) If \( \nu \in S(P) \) and \( \nu > \mu \), then all non-zero roots \( \lambda \) of \( \mathbb{C}[P](0; \lambda) = 0 \) satisfy \( \text{Re} \lambda < 0 \). Further, there exists \( \lambda_0 \in \mathbb{C} \) which satisfies the following.

(i) \( \text{Re} \lambda_0 > 0 \),

(ii) \( \lambda_0 \) is a simple root of \( \mathbb{C}[P](0; \lambda) = 0 \) and the other roots \( \lambda \) satisfy \( \text{Re} \lambda < \text{Re} \lambda_0 \).

Remark 2.7. In this section, we define only the conditions (A-0), (A-1), (A-2: \( \mu \)), and (A-6: \( \mu \)). This apparently strange numbering is for the consistency with [3]. We shall introduce another condition (A-3) in Section 5.

Using the theorem above, we can show the existence theorem of smooth null-solutions, which is the second of the main theorems.
THEOREM 2.8. Assume the conditions (A-0), (A-1), (A-2; μ₀), and (A-6; μ₀) for some μ₀ ∈ S(P) with μ₀ > 0. Then, P has a C∞ null-solution at (0,0).

The C∞ null-solution given in this theorem is one of the most fastly decaying nontrivial solutions as t → + 0. In fact, we have the following theorem, which is the last of the main theorems.

THEOREM 2.9. Assume the conditions (A-0), (A-1), (A-2; μ₀), and (A-6; μ₀) for some μ₀ ∈ S(P) with μ₀ > 0. Assume that u is a C solution of Pu = 0 for t > 0. If there exist δ > Re λₜ and C₀ > 0 such that the inequality

| u(t, x) | ≤ C₀ exp(- δ t⁻μ₀)

holds for t > 0 in a neighborhood of (0,0), then u = 0 for t > 0 in a neighborhood of (0,0).

Theorems 2.8 and 2.9 shall be proved in Section 5.

Finally, let us consider a typical example.

EXAMPLE 2.10. First, we consider the following ordinary differential operator decomposed into first order operators.

$$P₀ := t^d (t^r g - λ_1(t, x)) \cdots (t^r g - λ_r(t, x)) (\partial_r - \tilde{λ}_{r+1}(t, x)) \cdots (\partial_r - \tilde{λ}_m(t, x)),$$

where m, r, d ∈ N, 0 ≤ r ≤ m, k_j ∈ N(1 ≤ j ≤ r) and λ_j, \tilde{λ}_j ∈ C∞([0, T]; \partial(Ω)) (1 ≤ j ≤ r; r + 1 ≤ l ≤ m). Assume that λ_j(0, x) ≠ 0 (1 ≤ j ≤ r) and k_1 ≥ k_2 ≥ ⋯ ≥ k_r ≥ 0. For this operator, S(P₀) = \{k_1, \ldots, k_r, 0\} if r < m, and S(P₀) = \{k_1, \ldots, k_m\} if r = m. The condition (A-1) is trivially satisfied, and the condition (A-2; μ) is “if k_j > μ then λ_j(0, 0) ≠ 0”. We can also show that

$$\mathcal{C}_μ [P₀](x; λ) = \prod_{j: k_j > μ} (- λ_j(0, x)) \cdot \prod_{j: k_j = μ} (λ - λ_j(0, x)) \cdot \lambda^{h(μ) + m - r}$$

for μ ∈ S(P₀) with μ > 0, where h(μ) is the number of k_j's that satisfy k_j < μ. Thus, the condition (A-6; μ₀) for μ₀ > 0 is the following.

If k_j > μ₀ then Re λ_j(0, 0) < 0. Further, there exists j₀ such that

(i) k_j₀ = μ₀,

(ii) Re λ_j₀(0, 0) > 0,

(iii) If k_j = μ₀ and j ≠ j₀, then Re λ_j(0, 0) < Re λ_j₀(0, 0).

Next, we consider a partial differential operator. Put μ_j := 0 (1 ≤ j ≤ m - r) and μ_m-r+j := k_{r+1-j} (1 ≤ j ≤ r). Also put ω_j := d + \sum_{l=1}^{j} μ_l (0 ≤ j ≤ m).

Consider an operator
$P = P_0 + \sum_{j=0}^{m} t^{j+1} B_j(t, x ; \vartheta, \partial_x),$

where $B_j(t, x ; \vartheta, \partial_x) = \sum_{|\alpha| \leq 1} b_{j, \alpha}(t, x) \partial_x^\alpha \vartheta^{j-|\alpha|}$ and $b_{j, \alpha} \in C^\infty([0, T] ; \Theta(\Omega)).$

Then, $P$ satisfies the condition (A-1), and there hold $\Delta(P) = \Delta(P_0)$, $S(P) = S(P_0)$, $\mathcal{C}_u[P] = \mathcal{C}_u[P_0]$. (See Lemma 3.1.) Hence, $P$ satisfies the condition (A-2; $\mu_0$) (resp. (A-6; $\mu_0$)), if and only if $P_0$ satisfies (A-2; $\mu_0$) (resp. (A-6; $\mu_0$)).

### §3. Preliminaries

In this section, we give some preliminaries for the proofs of the main theorems.

Let $P$ be an operator (1.1) satisfying (A-0). By $t^j \partial_t^j = \vartheta(\vartheta - 1) \ldots (\vartheta - j + 1) = (\vartheta)_j$, we can easily show the following lemma, which is useful in our arguments.

**Lemma 3.1.** We can rewrite $P$ as

$$P = \sum_{j+|\alpha| \leq m} b_{j, \alpha}(t, x) \vartheta^j \partial_x^\alpha,$$

with $b_{j, \alpha} \in \hat{\mathcal{F}}_q([0, T] ; \Theta(\Omega))$. For this $b_{j, \alpha}$, we define the generalized vanishing order

$$r'(j, \alpha) := \sup \{r \in \mathbb{Z} / q : t^r b_{j, \alpha} \in \hat{\mathcal{F}}_q([0, T] ; \Theta(\Omega)) \}.$$

For $\mu \geq 0$, we put $\omega_\mu(P) := \max(-r'(j, \alpha) + \mu(j + |\alpha|) : j + |\alpha| \leq m)$.

Then, we have

$$\Delta(P) = \text{ch}\left( \bigcup_{j+|\alpha| \leq m} \left\{ (u, v) \in \mathbb{R}^2 : u \leq j + |\alpha|, v \geq r'(j, \alpha) \right\} \right),$$

$$\hat{V}(P) = \left( (j, \alpha) \in \mathbb{N} \times \mathbb{N}^n : (j + |\alpha|, r'(j, \alpha)) \text{ is a vertex of } \Delta(P) \right),$$

$$\omega(P) = \max(-r'(j, \alpha) \in \mathbb{R} : j + |\alpha| \leq m) = \omega_\mu(P),$$

$$I_\mu(P) = \left\{ (j, \alpha) \in \mathbb{N} \times \mathbb{N}^n : -r'(j, \alpha) + \mu(j + |\alpha|) = \omega_\mu(P) \right\}.$$

Further, the condition (A-1) is stated as follows:

(A-1) For every $\mu \in S(P)$, if $-r'(j, \alpha) + \mu(j + |\alpha|) = \omega_\mu(P)$, then $\alpha = 0$.

Under (A-1), there holds

$$\| \mathcal{C}_u[P](x ; \lambda) \| = \sum_{j=0}^{m} \{ b_{j, \alpha}(t, x) t^{\omega_\mu(P) - j} \} \left| t = 0 \right| \lambda^j$$

$$\begin{cases} [t^{\omega_\mu(P)} e^{it\varphi x} P(e^{-it\varphi x})] \left| t = 0 \right| (\mu > 0), \\ [t^{\omega_\mu(P)} t^{-j} P(t^j)] \left| t = 0 \right| (\mu = 0), \end{cases}$$
and the condition \((A-2; \mu)\) is stated as follows:

\((A-2; \mu)\) If \((j, 0) \in \hat{V}(P)\) and \(j \geq d_u(P)\), then \(\{b_{j,0}(t,0)\} |_{t=0} \neq 0\).

It is convenient to consider the operator in the form (3.1) rather than the form (1.1).

**Remark 3.2.** For \(\mu \geq 0\), we can define \(C\mu(P)\) by (3.2), even if \(\mu \not\in S\). If \(\mu \in S\) and \(\mu > 0\), then \(C\mu(P)\) has more than one term as a polynomial of \(\lambda\). If \(\mu \not\in S\) and \(\mu > 0\), then \(C\mu(P)\) has only one term.

The key tool for the proofs of main theorems is the following type of transformation of operators.

**Lemma 3.3.** Assume that an operator \(P\) of the form (1.1) (or (3.1)) satisfies the conditions \((A-0)\) and \((A-1)\). Let \(\mu \in S(P) \cap \mathbb{N} / q\), \(\mu > 0\), and assume \((A-2; \mu)\). Let \(\lambda_1\) be a simple root of \(C\mu(P)(0; \lambda) = 0\). Take a subdomain \(\Omega'\) of \(\Omega\) including 0 and \(\lambda(x) \in \partial(\Omega')\) so that they satisfy \(\lambda(0) = \lambda_1\) and \(C\mu(P)(x; \lambda(x)) \equiv 0\) on \(\Omega'\). If we put
\[ P' := \exp \left( \frac{\lambda(x)}{\mu} t^{-u} \right) \cdot P \cdot \exp \left( -\frac{\lambda(x)}{\mu} t^{-u} \right), \]

then \( P' \) is an operator on \([0, T] \times \Omega'\) of the form (1.1) and satisfies the following:

(a) The operator \( P' \) satisfies (A-0) and (A-1).
(b) \( S(P') \cap (\mu, \infty) = S(P) \cap (\mu, \infty) \).
(c) \( \mathcal{C}_\nu[P'](x; \cdot) = \mathcal{C}_\nu[P](x; \cdot) \) for every \( \nu > \mu \) and \( x \in \Omega' \).
(d) There holds \( \mathcal{C}_\nu[P'](x; \lambda) = \mathcal{C}_\nu[P](x; \lambda + \lambda(x)) \). Further, if \( d_\mu(P) > 1 \), then \( \mu \in S(P') \); if \( d_\mu(P) = 1 \), then \( \mu \notin S(P') \).
(e) There exists \( \mu' < \mu \) such that \( \mu' \in \mathbb{N}/q \) and \( S(P') \cap [0, \mu) = \{ \mu' \} \).
(f) \( d_{\mu'}(P') = 1 \) and \( P' \) satisfies (A-2; \( \mu' \)).

The upper part of the dotted line is \( \Delta(P) \).

The upper part of the real line is \( \Delta(P') \).

**Figure 3.** \( \Delta(P') \) and \( \Delta(P) \)

**Proof.** First, note that
\[
\exp\left(\frac{\lambda(x)}{\mu} t^{-\mu}\right) \odot \delta \cdot \exp\left(-\frac{\lambda(x)}{\mu} t^{-\mu}\right) = \delta + \lambda(x) t^{-\mu},
\]
(3.3)
\[
\exp\left(\frac{\lambda(x)}{\mu} t^{-\mu}\right) \odot \partial_x \cdot \exp\left(-\frac{\lambda(x)}{\mu} t^{-\mu}\right) = \partial_x + \frac{-\lambda(x)}{\mu} t^{-\mu}.
\]

From these, it is easy to see that \( P' \) is an operator of the form (3.1) and satisfies the conditions \((A-0), (A-1), \) and \((A-2; \mu). \) It is also easy to see that there hold the conclusions \((b), (c). \) Further, we have \( \mathcal{C}_{\mu}[P'](x; \lambda) = \mathcal{C}_{\mu}[P](x; \lambda + \lambda(x)). \) Since \( \mathcal{C}_{\mu}[P'](x; 0) \equiv 0 \) and since \( \left(\partial_\mu \mathcal{C}_{\mu}[P']\right)(0; 0) = \left(\partial_\mu \mathcal{C}_{\mu}[P]\right)(0; \lambda_1) \neq 0, \) we have \( (1, 0, \ldots, 0) \in \tilde{V}(P') \subset \mathbb{N} \times \mathbb{N}^n. \) Hence, if \( d_\mu(P) > 1, \) then \( \mu \in S(P'). \) Further, there exists \( \mu' \in \mathbb{N}/q \) such that \( \mu' < \mu, \) \( S(P') \cap [0, \mu] = \{\mu'\}, \) and \( d_{\mu'}(P') = 1. \) The condition \((A-2; \mu)\) and the fact that \( \left(\partial_\mu \mathcal{C}_{\mu}[P']\right)(0; 0) \neq 0 \) imply \((A-2; \mu'). \)

By an iterative use of this lemma, we have the following.

**Proposition 3.4.** Assume that \( P \) satisfies \((A-0)\) and \((A-1).\) Let \( \mu_0 \in S(P) \cap \mathbb{N}/q, \mu_0 > 0, \) and assume \((A-2; \mu_0). \) Let \( \lambda_0 \) be a simple root of \( \mathcal{C}_{\mu_0}[P](0; \lambda) = 0. \) Then, there exist

\begin{enumerate}[(i)]
  \item \( M \in \mathbb{N}, \)
  \item \( \mu[j] \in \mathbb{N}/q (j = 0, 1, \ldots, M), \) where \( \mu_0 = \mu[0] > \mu[1] > \cdots > \mu[M] > 0, \)
  \item a subdomain \( \Omega_{M+1} \) of \( \Omega \) including 0,
  \item \( \lambda[j] \in \mathcal{O}(\Omega_{M+1}) (j = 0, 1, \ldots, M), \) where \( \lambda[0](0) = \lambda_0, \)
\end{enumerate}

such that the operator

\[
P^{(M+1)} := \exp\left(\sum_{j=0}^M \frac{\lambda[j](x)}{\mu[j]} t^{-\mu[j]}\right) \cdot P \cdot \exp\left(-\sum_{j=0}^M \frac{\lambda[j](x)}{\mu[j]} t^{-\mu[j]}\right)
\]

is an operator on \([0, T] \times \Omega_{M+1}\) of the form (1.1) and satisfies the following:

\begin{enumerate}[(a)]
  \item The operator \( P^{(M+1)} \) satisfies \((A-0)\) and \((A-1).\)
  \item \( S(P^{(M+1)}) \cap (\mu_0, \infty) = S(P) \cap (\mu_0, \infty). \)
  \item \( \mathcal{C}_{\mu}[P^{(M+1)}](x; \cdot) = \mathcal{C}_{\mu}[P](x; \cdot) \) for every \( \nu > \mu_0 \) and \( x \in \Omega_{M+1}. \)
  \item There holds \( \mathcal{C}_{\mu_0}[P^{(M+1)}](x; \lambda) = \mathcal{C}_{\mu_0}[P](x; \lambda + \lambda[0](x)). \) If \( d_{\mu_0}(P) > 1, \) then \( \mu_0 \in S(P^{(M+1)}); \) if \( d_{\mu_0}(P) = 1, \) then \( \mu_0 \not\in S(P^{(M+1)}). \)
  \item \( S(P^{(M+1)}) \cap [0, \mu_0] = \{0\}. \)
  \item \( d_{\mu}(P^{(M+1)}) = 1 \) and \( P^{(M+1)} \) satisfies \((A-2; 0). \)
\end{enumerate}
The upper part of the dotted line is $\Delta(P) = \Delta(P^{(0)})$.
The upper part of the real line is $\Delta(P^{(j)})$ $(1 < j < M)$.
The upper part of the bold real line is $\Delta(P^{(M+1)})$.

**Figure 4.** $\Delta(P) = \Delta(P^{(0)})$ and $\Delta(P^{(M+1)}) \subset \cdots \subset \Delta(P^{(i)})$

**Proof.** Since $\lambda_0$ is a simple root, we can take a subdomain $\Omega_1$ of $\Omega$ including 0 and $\lambda[0](x) \in \mathcal{O}(\Omega_1)$ such that they satisfy $\lambda[0](0) = \lambda_0$ and $\mathcal{O}_0(P)(x; \lambda[0](x)) \equiv 0 \text{ on } \Omega_1$.

Put $P^{(0)} := P$ and $\mu[0] := \mu_\psi$. If we put

$$P^{(1)} := \exp\left(\frac{\lambda[0](x)}{\mu[0]} - t^{-\mu[0]}\right) \cdot P^{(0)} \cdot \exp\left(-\frac{\lambda[0](x)}{\mu[0]} t^{-\mu[0]}\right),$$

then by Lemma 3.3, the operator $P^{(1)}$ is also an operator of the form (1.1) on $[0, T] \times \Omega_1$ and satisfies the following:

(a) The operator $P^{(1)}$ satisfies (A-0) and (A-1).
(b) \( S(P^{(1)}) \cap (\mu[0], \infty) = S(P^{(0)}) \cap (\mu[0], \infty) \).

(c) \( \mathcal{C}_\nu(P^{(1)}) (x ; \cdot) = \mathcal{C}_\nu(P^{(0)}) (x ; \cdot) \) for every \( \nu > \mu[0] \) and \( x \in \Omega_1 \).

(d) There holds \( \mathcal{C}_{\mu[0]}(P^{(1)}) (x ; \lambda) = \mathcal{C}_{\mu[0]}(P^{(0)}) (x ; \lambda + \lambda[0](x)) \). If \( d_{\mu[0]}(P^{(0)}) > 1 \), then \( \mu[0] \in S(P^{(1)}) \); if \( d_{\mu[0]}(P^{(0)}) = 1 \), then \( \mu[0] \in \not S(P^{(1)}) \).

(e) There exists \( \mu[1] < \mu[0] \) such that \( \mu[1] \in \mathbb{N}/q \) and \( S(P^{(1)}) \cap [0, \mu[0]) = \{ \mu[1] \} \).

(f) \( d_{\mu[1]}(P^{(1)}) = 1 \) and \( P^{(1)} \) satisfies (A-2; \( \mu[1] \)).

By (f), we have \( \mathcal{C}_{\mu[1]}(P^{(1)}) (x ; \lambda) = a[1](x) \lambda - b[1](x) \) for some \( a[1], b[1] \in \mathcal{O}(\Omega_2) \) with \( a[1](0) \neq 0 \).

If \( \mu[1] = 0 \), then put \( M = 0 \). Consider the case when \( \mu[1] > 0 \). We can take a subdomain \( \Omega_2 \) of \( \Omega_1 \) including 0 such that \( a[1](x) \neq 0 \) on \( \Omega_2 \), and hence we can take \( \lambda[1] \in \mathcal{O}(\Omega_2) \) such that \( \mathcal{C}_{\mu[1]}(P^{(1)}) (x ; \lambda[1](x)) \equiv 0 \) on \( \Omega_2 \).

If we put

\[
P^{(2)} := \exp\left( \frac{\lambda[1](x)}{\mu[1]} \cdot t^{-\mu[1]} \right) \ast P^{(1)} \ast \exp\left( \frac{-\lambda[1](x)}{\mu[1]} \cdot t^{-\mu[1]} \right),
\]

then by Lemma 3.3 and by \( d_{\mu[1]}(P^{(1)}) = 1 \), the operator \( P^{(2)} \) is also an operator of the form (1.1) and satisfies the following:

(a) The operator \( P^{(2)} \) satisfies (A-0) and (A-1).

(b) \( S(P^{(2)}) \cap (\mu[1], \infty) = S(P^{(1)}) \cap (\mu[1], \infty) \).

(c) \( \mathcal{C}_\nu(P^{(2)}) (x ; \cdot) = \mathcal{C}_\nu(P^{(1)}) (x ; \cdot) \) for every \( \nu > \mu[1] \) and \( x \in \Omega_2 \).

(d) There holds \( \mathcal{C}_{\mu[1]}(P^{(2)}) (x ; \lambda) = \mathcal{C}_{\mu[1]}(P^{(1)}) (x ; \lambda + \lambda[1](x)) = a[1](x) \lambda \), and \( \mu[1] \in \not S(P^{(2)}) \).

(e) There exists \( \mu[2] < \mu[1] \) such that \( \mu[2] \in \mathbb{N}/q \) and \( S(P^{(2)}) \cap [0, \mu[1]) = \{ \mu[2] \} \).

(f) \( d_{\mu[2]}(P^{(2)}) = 1 \) and \( P^{(2)} \) satisfies (A-2; \( \mu[2] \)).

We can continue this procedure unless \( \mu[j] = 0 \). Since \( \mu[j] \in \mathbb{N}/q \) and \( \mu[0] > \mu[1] > \cdots \geq 0 \), we necessarily reach \( \mu[M+1] = 0 \). □

The following lemma is used to construct each term of infinite series in asymptotic solutions.

**Lemma 3.5.** Let \( Q(x ; \lambda) \in \mathcal{O}(\Omega)[\lambda] \) and \( \Lambda \in \mathcal{O}(\Omega) \). Assume that \( Q(x ; \Lambda(x)) \neq 0 \) on \( \Omega \). Then, we can solve the equation

\[
Q(x ; \theta) v = t^{\Lambda(x)} \sum_{p=0}^{L} g_p(x) \left( \log t \right)^p, \quad g_p \in \mathcal{O}(\Omega) \quad (0 \leq p \leq L)
\]
as \( v = t^{A(x)} \sum_{p=0}^{L} v_p(x) (\log t)^p \), \( v_p \in \mathcal{C}(\Omega) \) (0 \( \leq p \leq L \)).

**Proof.** By an easy calculation, we have

\[
Q(x; \theta)(t^{A(x)}(\log t)^p) = \sum_{j=0}^{\infty} \binom{p}{j} (\partial_j^i Q)(x; \Lambda(x)) \cdot t^{A(x)}(\log t)^{p-j}.
\]

Hence, (3.4) is equivalent to

\[
Q(x; \Lambda(x)) \cdot v_p(x) + \sum_{j=1}^{L-p} \binom{p+j}{j} (\partial_j^i Q)(x; \Lambda(x)) \cdot v_{p+j}(x) = g_p(x) \quad (p = 0, 1, \ldots, L).
\]

Thus, by \( Q(x; \Lambda(x)) \neq 0 \), we can uniquely determine \( v_L, v_{L-1}, \ldots, v_0 \).

---

**§4. Proof of Theorem 2.5**

In this section, we prove Theorem 2.5. First, we give the existence of an asymptotic solution with no exponential factor, which corresponds to the case \( \mu_0 = 0 \) and \( M = -1 \) in Theorem 2.5. Although we use only the case when \( \deg_{x} \mathcal{C}_0[P] = 1 \) in the proof of main theorems, this proposition has its own value.

**Proposition 4.1.** Assume that \( P \) satisfies (A-0), (A-1), and (A-2; 0). Let \( \lambda(x) \in \mathcal{C}(\Omega_0) \) satisfy

(i) \( \mathcal{C}_0[P](x; \lambda(x)) \equiv 0 \) on \( \Omega_0 \),

(ii) \( \mathcal{C}_0[P](x; \lambda(x) + 1/q) \neq 0 \) on \( \Omega_0 \) for \( l \in \mathbb{N} \setminus \{0\} \),

for some subdomain \( \Omega_0 \) of \( \Omega \) including 0. Then, for an arbitrarily given \( v_{0,0}(x) \in \mathcal{C}(\Omega_0) \), there exist \( v_{i,p}(x) \in \mathcal{C}(\Omega_0) \) (\( i \geq 0 \); 0 \( \leq p \leq lm \)) such that

(4.1)

\[
\begin{align*}
\quad u(t, x) &:= t^{i(x)} \cdot \sum_{i=0}^{\infty} \sum_{p=0}^{lm} (\log t)^p v_{i,p}(x) \\
\quad &\text{is an asymptotic solution of } Pu = 0. \quad \text{That is}
\end{align*}
\]

\[
t^{i(x)}P \left( \sum_{i=0}^{N} \sum_{p=0}^{lm} (\log t)^p v_{i,p}(x) \right) = o(t^{N/q-\omega(P)}),
\]

for every \( N \in \mathbb{N} \).

**Proof.** We can formally expand \( P \) with respect to \( t \) as
\[ P = t^{-\omega} \left( \mathcal{C}_0[P](x ; \varrho) + \sum_{h=1}^{\infty} B_h(x, \partial_x ; \varrho) t^{h/q} \right), \]

where \( B_h(x, \partial_x ; \varrho) = \sum_{|\alpha| \leq m} b_{h,\alpha}(x) \partial_x^\alpha \varrho^l \) with \( b_{h,\alpha} \in \mathcal{O}(\Omega) \) and \( \omega := \omega(P) \).

Hence, we have only to find \( v_{i,p} \) that satisfy

\[ \mathcal{C}_0[P](x ; \varrho) \left( t^{i(x)+1/q} \sum_{p=0}^{km} (\log t)^p v_{i,p}(x) \right) = - \sum_{h=0}^{l-1} B_{l-h}(x, \partial_x ; \varrho) \left( t^{i(x)+1/q} \sum_{p=0}^{km} (\log t)^p v_{h,p}(x) \right) (l \in \mathbb{N}). \]

Since

\[ \mathcal{C}_0[P](x ; \varrho) \left( t^{i(x)} v_{0,0}(x) \right) = \mathcal{C}_0[P](x ; \lambda(x)) \cdot t^{i(x)} v_{0,0}(x) = 0, \]

\[ \partial_x (t^{i(x)+1/q} (\log t)^p v(x)) = t^{i(x)+1/q} (\log t)^p (\partial_x v)(x) + t^{i(x)+1/q} (\log t)^{p+1} (\partial_x \lambda)(x) v(x), \]

and since

\[ \mathcal{C}_0[P](x ; \lambda(x) + l/q) \neq 0 \ (l \geq 1) \quad \text{on} \quad \Omega_0, \]

we can get \( v_{i,p} \) with an arbitrarily given \( v_{0,0} \) by applying Lemma 3.5.

**Proof of Theorem 2.5.** We can apply Proposition 3.4 to \( P \). By (f) of the proposition, we have

\[ \mathcal{C}_0[P^{(M+1)}](x ; \lambda) = a[M+1](x) \lambda - b[M+1](x) \]

for some \( a[M+1], b[M+1] \in \mathcal{O}(\Omega_{M+1}) \) with \( a[M+1](0) \neq 0 \). Hence, we can take a subdomain \( \Omega_0 \) of \( \Omega_{M+1} \) including 0 such that \( a[M+1](x) \neq 0 \) on \( \Omega_0 \). We can take \( \lambda[M+1] \in \mathcal{O}(\Omega_0) \) such that \( \mathcal{C}_0[P^{(M+1)}](x ; \lambda[M+1](x)) = 0 \) and \( \mathcal{C}_0[P^{(M+1)}](x ; \lambda[M+1](x) + l/q) \neq 0 \) on \( \Omega_0 \) for \( l \in \mathbb{N} \setminus \{0\} \).

By applying Proposition 3.4 to \( P^{(M+1)} \), we can construct an asymptotic solution

\[ v = t^{i(M+1)/x} \cdot \sum_{l=0}^{\infty} t^{l/q} \sum_{p=0}^{km} (\log t)^p v_{l,p}(x) \]

of \( P^{(M+1)} v = 0 \) for an arbitrarily given \( v_{0,0} \in \mathcal{O}(\Omega_0) \).

Thus, the proof of Theorem 2.5 is completed.
§5. Proof of Theorems 2.8 and 2.9

In this section, we prove Theorems 2.8 and 2.9.

First, we introduce another condition (A-3).

(A-3) If \( \mu \in S(P) \) and \( \mu > 0 \), then all the non-zero roots \( \lambda \) of \( \mathcal{C}_\mu[P](0; \lambda) = 0 \) satisfy \( \operatorname{Re}\lambda < 0 \).

From the results in [3], we easily get the following theorem, which shall be used later.

**Theorem 5.1.** Assume the conditions (A-0), (A-1), (A-2; 0), and (A-3). Then, there exist \( N_0 \in \mathbb{N}, T_0 > 0 \), and a domain \( \Omega_0 \) including 0 for which the following holds:

1. For every \( N \geq N_0 \) and every \( f \in \mathcal{C}[\sigma^{(P)}]([0, T]; \theta(\Omega)) \), there exists a unique \( u \in \mathcal{C}([0, T_0]; \theta(\Omega_0)) \) such that \( Pu = f \) on \([0, T_0] \times \Omega_0\).
2. If \( u \in t^{N_0} \times \mathcal{C}([0, T]; \theta'(\Omega \cap \mathbb{R}^n)) \) and \( Pu = 0 \) for \( t > 0 \) in a neighborhood of \((0,0)\), then \( u = 0 \) for \( t > 0 \) in a neighborhood of \((0,0)\). Especially, there exists no sufficiently smooth null-solution for \( P \) at \((0,0)\).

In (2) of this theorem, the domain where \( u = 0 \) may depend not only on the domain where \( Pu = 0 \) but also on \( u \) itself. As for solutions in \( \mathcal{C}([0, T]; \mathcal{C}(\Omega \cap \mathbb{R}^n)) \), however, we can show the existence of a common domain of uniqueness, by a standard argument as follows.

**Corollary 5.2.** Assume the same assumptions as in the theorem above. Then there exists \( N_0 \in \mathbb{N} \) such that for every \( T' \in (0, T) \) and every open neighborhood \( U' \) of \( 0 \in \mathbb{R}^n \), there exist \( T'' \in (0, T') \) and an open neighborhood \( U'' \) of \( 0 \) for which the following holds. If \( u \in t^{N_0} \times \mathcal{C}([0, T]; \mathcal{C}(\Omega \cap \mathbb{R}^n)) \) and \( Pu = 0 \) on \((0, T') \times U'\), then \( u = 0 \) on \((0, T'') \times U''\).

**Proof.** Put \( K := \{u \in t^{N_0} \times \mathcal{C}([0, T]; \mathcal{C}(\Omega \cap \mathbb{R}^n)) : Pu = 0 \text{ on } (0, T') \times U'\} \). This is a closed subspace of a Fréchet space \( t^{N_0} \times \mathcal{C}([0, T]; \mathcal{C}(\Omega \cap \mathbb{R}^n)) \), and hence it is also a Fréchet space. Let \( \{T_n\}_{n \in \mathbb{N}} \) be a decreasing sequence of positive real numbers converging to 0 and let \( \{U_n\}_{n \in \mathbb{N}} \) be a fundamental system of open neighborhoods of 0. Put \( L_n := \{u \in K : u = 0 \text{ on } (0, T_n) \times U_n\} \), which are closed subspaces of \( K \). By Theorem 5.1-(2), there holds \( K = \bigcup_{n=0}^\infty L_n \). Since a Fréchet space is a Baire space, there exists an \( n \) such that \( L_n \) has an inner point, that is \( L_n = K \). \( \square \)
Now, we give a proof of Theorem 2.8.

**Proof of Theorem 2.8.** We may assume that $\mu_0 \in \mathbb{N}/q$ without loss of generality, and we can apply Proposition 3.4 to $P$. The operator $P^{(M+1)}$ satisfies (A-0), (A-1) and (A-2; 0). By the assumption (A-6; $\mu_0$) for $P$ and by the conditions (c), (d), (e) in Proposition 3.4, the operator $P^{(M+1)}$ satisfies (A-3). Further, as we have shown in the proof of Theorem 2.5, the operator $P^{(M+1)}$ has a formal solution (4.2) with $v_{0,0} \equiv 1$.

If we put

$$v_N := t^{(M+1)(x)} \sum_{l=0}^{q^N} \sum_{p=0}^{l_m} (\log t)^p v_{l,p}(x)$$

and $g_N := P^{(M+1)}(v_N)$ for sufficiently large $N \in \mathbb{N}$, then we have

$$g_N \in C^2_{\text{flat}}([0, T_0] ; \partial(\Omega_0)),$$

where $\Omega_0$ is a subdomain of $\Omega$ including 0 and $r_0 \in \mathbb{N}$, both independent of $N$. By Theorem 5.1, we get $w_N \in C^{(M+1)}([0, T_0] ; \partial(\Omega_0))$ such that $P^{(M+1)}(w_N) = -g_N$, where $T_0 > 0$ and $\Omega'_0$ is a subdomain of $\Omega_0$ including 0. Thus, $v := v_N + w_N$ satisfies $P^{(M+1)}(v) = 0$ and $t^{-\lambda_{M+1}(x)} v(t, x) \to 1(t \to + 0)$. Note that Corollary 5.2 implies that $v$ is independent of $N$ for sufficiently large $N$ in a neighborhood of $(0,0)$.

Since $\Re \lambda[0](0) > 0$ by the assumption, we can easily show that

$$u(t, x) := \exp \left( - \sum_{j=0}^{M} \frac{\lambda[j](x)}{\mu[j]} t^{-\mu[j]} \right) \cdot v(t, x)$$

belongs to $C^\infty_{\text{flat}}([0, T_0] ; \partial(\Omega'_0))$. Thus, $u$ is a $C^\infty$ null-solution for $P$.

Next, we give a proof of Theorem 2.9.

**Proof of Theorem 2.9.** If we take $\delta'$ as $\delta > \delta' > \Re \lambda_0$, and if we put $v := \exp(\delta' t^{-\mu_0}/\mu_0) u$, then we have $v \in C^0([0, T] ; \partial(\Omega_0 \cap R^n))$ for every $N \in \mathbb{N}$ with some domain $\Omega_0$ and $T > 0$. We also have

$$0 = \hat{P} \left( \exp \left( - \frac{\delta'}{\mu_0} t^{-\mu_0} \right) v \right) = \exp \left( - \frac{\delta'}{\mu_0} t^{-\mu_0} \right) \hat{P} v,$$

that is, $\hat{P} v = 0$, where $\hat{P} := \exp(\delta' t^{-\mu_0}/\mu_0) \cdot P \cdot \exp(- \delta' t^{-\mu_0}/\mu_0)$. We have only to show that $v = 0$ for $t > 0$ in a neighborhood of $(0,0)$.

By an argument similar to and easier than that in the proof of Lemma 3.3, the
operator $\tilde{P}$ is an operator of the form (1.1) and satisfies the following:

(a) The operator $\tilde{P}$ satisfies (A-0), (A-1), and (A-2; $\mu_0$).
(b) $S(\tilde{P}) \cap (\mu_0, \infty) = S(P) \cap (\mu_0, \infty)$.
(c) $\mathcal{E}_\nu[\tilde{P}](x ; \cdot) = \mathcal{E}_\nu[P](x ; \cdot)$ for every $\nu > \mu_0$ and $x \in \Omega_0$.
(d) $\mathcal{E}_{\nu_0}^{\prime}[\tilde{P}](x ; \lambda) = \mathcal{E}_{\mu_0}^{\prime}[P](x ; \lambda + \delta')$.
(e) $S(\tilde{P}) \cap [0, \mu_0] = \{\mu_0\}$.

By (d) and the condition (A-6; $\mu_0$) for $P$, all the roots $\lambda$ of $\mathcal{E}_{\nu_0}^{\prime}[\tilde{P}](0 ; \lambda) = 0$ satisfy $\text{Re} \lambda < 0$. This and the conditions (c), (e) imply that the operator $\tilde{P}$ satisfies (A-3). Further, also by (d), we have $\mathcal{E}_{\nu_0}^{\prime}[\tilde{P}](0 ; 0) \neq 0$. This and the assumption (A-2; $\mu_0$) imply (A-2; 0). Thus, we can apply Theorem 5.1 to $\tilde{P}$, and hence, we have $v = 0$ for $t > 0$ in a neighborhood of (0,0). □

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