FINITE GROUPS IN WHICH SYLOW 2-SUBGROUPS ARE ABELIAN AND CENTRALIZERS OF INVOLUTIONS ARE SOLVABLE

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Introduction. The purpose of this paper is to establish the following theorem:

THEOREM 1. *Let* $\mathfrak{G}$ *be a finite group with abelian Sylow* 2-*subgroups in which the centralizer of every involution is solvable. Then either* $\mathfrak{G}$ *is solvable or else* $\mathfrak{G}/\mathbf{O}(\mathfrak{G})$ *is isomorphic to a subgroup of* $\mathrm{P\Gamma L}(2, q)$ *containing* $\mathrm{PSL}(2, q)$, *where either* $q \equiv 3$ *or* $5 \pmod 8$, $q \geqslant 5$, *or* $q = 2^n$, $n \geqslant 2$.

As an immediate corollary, we obtain

THEOREM 2. *If* $\mathfrak{G}$ *is a simple group with abelian Sylow* 2-*subgroups in which the centralizer of every involution is solvable, then* $\mathfrak{G}$ *is isomorphic to* $\mathrm{PSL}(2, q)$, *where either* $q \equiv 3$ *or* $5 \pmod 8$ *and* $q \geqslant 5$ *or* $q = 2^n$ *and* $n \geqslant 2$.

The proof of Theorem 1 is carried out by induction on the order of $\mathfrak{G}$. Combined with a number of known results, Theorem 1 is easily derived as a consequence of the following theorem:

THEOREM 3. *There exists no finite simple group* $\mathfrak{G}$ *which satisfies the following conditions:*

(a) *A Sylow* 2-*subgroup* $\mathfrak{S}$ *of* $\mathfrak{G}$ *is abelian.*

(b) *The centralizer of every involution of* $\mathfrak{G}$ *is solvable.*

(c) $\mathfrak{S}$ *is not generated by two elements.*

(d) *If* $\mathfrak{S}$ *is elementary of order* 8, *then* $|\mathbf{N}(\mathfrak{S})/\mathbf{C}(\mathfrak{S})| = 7$.

(e) *There exists a distinct conjugate* $\mathfrak{S}_1$ *of* $\mathfrak{S}$ *such that* $\mathfrak{S}_1 \cap \mathfrak{S} \neq 1$.

(*f*) *A proper subgroup* $\mathfrak{H}$ *of* $\mathfrak{G}$ *is either solvable or else* $\mathfrak{H}/\mathbf{O}(\mathfrak{H})$ *is isomorphic to a subgroup of* $\mathrm{P\Gamma L}(2, q)$ *containing* $\mathrm{PSL}(2, q)$, *where* $q \equiv 3$ *or* $5 \pmod 8$, $q \geqslant 5$ *or* $q = 2^n$, $n \geqslant 2$.

(g) *The normalizer of some non-identity solvable subgroup of* $\mathfrak{G}$ *is non-solvable.*

Thus the bulk of the paper is devoted to the proof of Theorem 3. Conditions (a), (b), and (e) of the theorem imply directly that $\mathfrak{S}$ normalizes, but does not centralize, a p-subgroup of $\mathfrak{G}$ for some odd prime p. The set σ of all such odd primes p, which is therefore non-empty, plays a central role in the paper; and Theorem 3 is established by showing, on the other hand, that σ must be empty. This result is ultimately obtained by applying the main results of

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"On the Maximal Subgroups of Finite Simple Groups" **(5)**, which give sufficient conditions for a simple group $\mathfrak{G}$ to satisfy the *uniqueness condition* for some prime p (for the definition, see Section 1 below).

The application of these results follows rather closely the arguments of Sections 8 and 9 of "The Characterization of Finite Groups with Dihedral Sylow 2-Subgroups" **(6)**. However, it turns out that we actually require a slight extension of the results of **(5)**, which is implicitly contained in that paper. This generalization involves a slight weakening of the concept of p-constraint which was introduced in **(5)**, and is described in detail in Section 1.

We remark also that the prime 3 plays a somewhat exceptional role, owing to the fact that in certain situations we cannot verify for the prime 3 all the conditions which must be met in order to be able to apply the main results of **(5)**. These exceptions occur only when $\mathfrak{G}$ involves subgroups $\mathfrak{H}$ such that $\mathfrak{H}/\mathbf{O}(\mathfrak{H})$ is isomorphic to either $P\Gamma L(2, 8)$ or to a subgroup of $P\Gamma L(2, 3^t)$ containing $PSL(2, 3^t)$ with $t > 1$.

Finally we shall follow as closely as possible the notation of **(4; 5; 6; 12)**; and it will be assumed that the reader is familiar with the more standard parts of this notation.

1. Weak p-constraint. The principal results of **(5)** will be of fundamental importance to us in the paper. These results consist in giving a set of conditions on a simple group $\mathfrak{G}$ in order that $\mathfrak{G}$ satisfy the *uniqueness condition* for a particular prime p, as this term is defined in **(6**, Section 8**)**. If $\mathfrak{P}$ is an S_p-subgroup of a group $\mathfrak{G}$ such that $\mathscr{SCN}_3(\mathfrak{P})$ is non-empty, we say that $\mathfrak{G}$ satisfies the uniqueness condition for the prime p provided $\mathfrak{G}$ possesses a unique subgroup $\mathfrak{M}$ which is maximal subject to containing an element of $\mathscr{A}_i(\mathfrak{P})$, $i = 1, 2, 3$, or 4 and $p \in \pi_s(\mathfrak{M})$. Here

$$\mathscr{A}_1(\mathfrak{P}) = \{\mathfrak{P}_0 | \mathfrak{P}_0 \subseteq \mathfrak{P} \text{ and } \mathfrak{A} \subseteq \mathfrak{P}_0 \text{ for some } \mathfrak{A} \text{ in } \mathscr{SCN}_3(\mathfrak{P})\},$$

and

$$\mathscr{A}_i(\mathfrak{P}) = \{\mathfrak{P}_0 | \mathfrak{P}_0 \subseteq \mathfrak{P}, \mathfrak{P}_0 \text{ contains a subgroup } \mathfrak{P}_1 \text{ of type } (p, p) \text{ such that for each } P \text{ in } \mathfrak{P}_1^{\#}, \mathbf{C}_{\mathfrak{P}}(P) \in \mathscr{A}_{i-1}(\mathfrak{P})\},$$

for $i = 2, 3, 4$. Theorems C, D, and E of **(5)** give sufficient conditions for a simple group $\mathfrak{G}$ to satisfy the uniqueness condition for a particular prime p, while Theorems A and B give results which are needed in the applications to verify the hypotheses of Theorem C.

However, the conditions of Theorems C, D, and E of **(5)** are not entirely sufficient for the applications to the present paper. As a consequence we require a slightly weaker set of conditions which are sufficient to imply the same conclusions. This extension will consist in replacing the condition of p-constraint throughout **(5)** by a somewhat weaker condition. As we shall see, the arguments in **(5)** all remain valid, with possible minor modifications in some of their statements, when this weaker condition is used in place of the original definition of p-constraint. We wish to emphasize that the results of this section are in themselves completely independent of the balance of the paper.

By definition, a group $\mathfrak{G}$ is said to be p-constrained if for any p-subgroup $\mathfrak{P} \neq 1$ of $\mathfrak{G}$, $\mathbf{C}(\mathfrak{P}^*)$ is solvable, where $\mathfrak{P}^*$ is an S_p-subgroup of $\mathbf{O}_{p',p}(\mathbf{C}(\mathfrak{P})\mathfrak{P})$. It will be convenient to say that $\mathfrak{G}$ is p-constrained *with respect to* $\mathfrak{P}$ if this condition holds for a particular non-trivial p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$. In this terminology, $\mathfrak{G}$ is p-constrained provided $\mathfrak{G}$ is p-constrained with respect to each non-trivial p-subgroup of $\mathfrak{G}$. We note that in **(5)** the use of the concept of p-constraint was limited exclusively to primes p for which $\mathscr{SCN}_3(p)$ was non-empty. Our generalization consists in showing that it is unnecessary to demand that $\mathfrak{G}$ be p-constrained with respect to every non-trivial p-subgroup, but only with respect to certain ones. To this end, we now make the following definition:

DEFINITION. *A group $\mathfrak{G}$ will be called weakly p-constrained for any prime p for which $\mathscr{SCN}_3(p)$ is non-empty provided that $\mathfrak{G}$ is p-constrained with respect to any non-trivial p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ such that* $\mathbf{N}(\mathfrak{P})$ *contains an element of $\mathscr{A}_4(\tilde{\mathfrak{P}})$ for some S_p-subgroup $\tilde{\mathfrak{P}}$ of $\mathfrak{G}$.*

Remark. Since any subgroup of $\tilde{\mathfrak{P}}$ of type (p, p, p) lies in $\mathscr{A}_3(\tilde{\mathfrak{P}})$ by **(4**, Lemma 24.2**)**, it follows that for $\mathfrak{G}$ to be weakly p-constrained, it is necessary that $\mathfrak{G}$ be p-constrained with respect to each non-trivial p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ whose normalizer contains a subgroup of type (p, p, p). This should serve to point up the very close relation between the concepts of p-constraint and weak p-constraint. Furthermore, it shows that in any argument of **(5)** in which each of the "critical" groups occurring therein possesses a subgroup of type (p, p, p) (or more generally an element of $\mathscr{A}_4(\tilde{\mathfrak{P}})$), the given conclusion will hold and the given argument will remain valid if the term "p-constraint" is replaced throughout by "weak p-constraint." This observation is essentially all that is needed to establish the generalization which we seek.

Corresponding to this generalization of the concept of p-constraint, we alter the definition of "weakly p-tame" as given in **(5**, Definition 7**)** by replacing the assumption that $\mathfrak{G}$ be p-constrained by the condition that $\mathfrak{G}$ be weakly p-constrained. Furthermore, we use this new definition of weakly p-tame to redefine the concepts of p-tame, strongly p-tame, and τ-tame, as given in **(5**, Definitions 8, 9, and 10**)**. We shall now examine briefly the various statements and proofs of **(5)** in order to demonstrate that the principal results of that paper remain valid when these new definitions are used in place of the original ones.

First of all, in **(5**, Lemma 3.3**)** the subgroup $\mathfrak{H}$ of the simple group $\mathfrak{G}$ contains a p-subgroup $\mathfrak{P}$ of index at most p in an S_p-subgroup $\tilde{\mathfrak{P}}$ of $\mathfrak{G}$. Hence if $\mathscr{SCN}_3$ $(\tilde{\mathfrak{P}})$ is non-empty, then $\mathfrak{P} \in \mathscr{A}_4(\tilde{\mathfrak{P}})$. Thus this lemma remains valid with weak p-constraint replacing p-constraint in the hypothesis. Similarly, the very important Lemma 3.4 of **(5)** remains valid provided we add to the hypothesis the assertion that the p-subgroup $\mathfrak{P}_1$ of the lemma contain an element of $\mathscr{A}_4(\tilde{\mathfrak{P}})$ for some S_p-subgroup $\tilde{\mathfrak{P}}$ or $\mathfrak{G}$.

It should, of course, be understood that it is the modified statements of these lemmas which are to be applied in the balance of **(5)** in carrying out the desired generalization.

Next consider the various results in **(5**, sections 4 and 5**)**. A reading of these two sections will reveal that in any argument in which the concept of p-constraint is invoked, the corresponding p-subgroup has a normalizer which contains an element $\mathfrak{A}$ of $\mathscr{SCN}_3\ (p)$. It follows at once from this fact that all the results of these sections continue to hold with our new definitions of weakly p-tame and of p-tame. Now consider **(5**, Theorem 2**)**, which gives a sufficient condition for a simple group $\mathfrak{G}$ to satisfy E^n_{pq}. If $\mathfrak{A} \in \mathscr{U}(p)$, then $\mathbf{C}(\mathfrak{A})$ contains an element of $\mathscr{SCN}_3\ (\tilde{\mathfrak{P}})$ for some S_p-subgroup $\tilde{\mathfrak{P}}$ of $\mathfrak{G}$, and hence $\mathfrak{A} \in \mathscr{A}_2(\tilde{\mathfrak{P}})$. Similarly if $\mathfrak{B} \in \mathscr{U}(q)$, then $\mathfrak{B} \in \mathscr{A}_2(\tilde{\mathfrak{Q}})$ for some S_q-subgroup $\tilde{\mathfrak{Q}}$ of $\mathfrak{G}$. Furthermore, it follows likewise from the definition of $\mathscr{F}(p)$ that any element of $\mathscr{F}(p)$ lies in $\mathscr{A}_4(\tilde{\mathfrak{P}})$ for some S_p-subgroup $\tilde{\mathfrak{P}}$ of $\mathfrak{G}$. But now to see that **(5**, Theorem 2**)** remains valid, we have only to observe that whenever we invoked p-constraint or q-constraint in the course of its proof, the normalizer of the corresponding p- or q-subgroup contained either an element of $\mathscr{U}(p)$, of $\mathscr{F}(p)$, or a subgroup of type (p, p, p), or correspondingly a subgroup of $\mathscr{U}(q)$ or a subgroup of type (q, q, q).

To show that **(5**, Theorem 3**)** also remains valid, it is necessary only to observe that $\mathbf{N}(\mathfrak{Q}_1)$ contains an element of $\mathscr{A}_4(\mathfrak{Q})$, $\mathfrak{Q}$ an S_q-subgroup of $\mathfrak{G}$ containing an S_q-subgroup $\mathfrak{Q}^*$ of $\mathbf{N}(\mathfrak{Q}_1)$; for from this it will follow that each of the critical subgroups involved in the proof of Theorem 3 contains an element of $\mathscr{A}_4(\mathfrak{Q})$. Here $\mathfrak{Q}_1$ is a maximal element of $\mathsf{M}(\mathfrak{P}; q)$, $\mathfrak{P}$ an S_p-subgroup of $\mathfrak{G}$, and $\mathfrak{P}$ does not centralize $\mathfrak{Q}_1$. But by **(5**, Lemma 5.3**)** (which as we have shown above continues to hold), it follows that $\mathfrak{Q}_1$ is non-cyclic; and this implies at once that $\mathbf{N}(\mathfrak{Q}_1)$ contains an element of $\mathscr{A}_4(\mathfrak{Q})$.

Finally Theorem A and the other results of Section 8 of **(5)** continue to hold, for the subgroups of $\mathfrak{G}$ involved in the various arguments of this section always contain Sylow subgroups for the appropriate primes. Likewise in the proofs of Theorems B, C, and D and the various lemmas of Section 10 of **(5)**, one sees that the critical subgroups again always contain either an S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ or an element $\mathfrak{A}$ of $\mathscr{SCN}_3(\mathfrak{P})$; so these theorems also continue to hold. Finally in the proof of Theorem E and the other lemmas of Section 11 of **(5)**, the arguments involve subgroups of $\mathfrak{G}$ which in each case contain an element of $\mathscr{A}_4(\mathfrak{P})$, so that these results remain valid, too.

Summarizing, then, we have the following theorem.

THEOREM 4. *Theorems A–E of* **(5)** *continue to hold if in the definition of weakly p-tame, the assumption of p-constraint is replaced by that of weak p-constraint, and if corresponding modifications are made in the definitions of p-tame, strongly p-tame, and τ-tame.*

Finally similar remarks apply to Lemmas 8.1–8.4 of **(6)**, which continue to hold when the above changes are made. In applying any of the generalized results of **(5)** or Section 8 of **(6)** in the present paper, we shall, for simplicity, continue to refer to the corresponding statements of **(5)** or **(6)**, even though it is the appropriate modification of such statements that is intended in each case.

2. A_0-groups and A_1-groups. In proving Theorem 3 it will be essential for us to establish first a number of properties of solvable groups with abelian S_2-subgroups and also of non-solvable groups $\mathfrak{H}$ in which $\mathfrak{H}/\mathbf{O}(\mathfrak{H})$ is isomorphic to a subgroup of $\mathrm{P\Gamma L}(2, q)$ containing $\mathrm{PSL}(2, q)$, where either $q \equiv 3$ or 5 (mod 8) or $q = 2^n$, $n \geqslant 2$. It will therefore be convenient to adopt the following terminology:

Definition. We call $\mathfrak{H}$ an A_0-group if $\mathfrak{H}$ is a solvable group with an abelian S_2-subgroup. We call $\mathfrak{H}$ an A_1-group provided:

(i) *$\mathfrak{H}$ is non-solvable.*

(ii) *An S_2-subgroup of $\mathfrak{H}$ is abelian.*

(iii) *$\mathfrak{H}/\mathbf{O}(\mathfrak{H})$ is isomorphic to a subgroup of* $\mathrm{P\Gamma L}(2, q)$ *containing* $\mathrm{PSL}(2, q)$.

(iv) *$\mathbf{C}_{\mathfrak{H}}(T)$ is solvable for any involution T in $\mathfrak{H}$.*

Since an S_2-subgroup of $\mathfrak{H}$ is abelian, either $q \equiv 3$ or 5 (mod 8) or $q = 2^n$. Furthermore, $q > 3$ since $\mathfrak{H}$ is non-solvable by assumption. As in **(6**, Section 3**)**, we call q the *characteristic* of the A_1-group $\mathfrak{H}$. We note that since $\mathrm{PSL}(2, 4)$ and $\mathrm{PSL}(2, 5)$ are isomorphic, an A_1-group of characteristic 4 is also of characteristic 5, and conversely. In all other cases the characteristic is unique. If $\mathfrak{H}$ is an A_1-group of *odd* characteristic, then an S_2-subgroup of $\mathfrak{H}$ is a four-group and consequently $\mathfrak{H}$ is a non-solvable D-group in the sense of **(6**, Section 3**)**. In particular, many of the results of Sections 3, 4, and 8 of **(6)** hold in this case. In the present section we shall extend these results to the class of A_0- and A_1-groups.

By analogy with Lemmas 3.1 and 3.3 of **(6)**, we first list in two successive lemmas the properties of $\mathrm{PSL}(2, 2^n)$ and $\mathrm{P\Gamma L}(2, 2^n)$ which we shall need. Proofs of the various statements are either given explicitly in Dickson **(2)** or Dieudonné **(3)** or else can be derived directly from their results and the proofs of Lemmas 3.1 and 3.3 of **(6)**.

LEMMA 2.1. *Set $\mathfrak{H} = \mathrm{PSL}(2, q)$, where $q = 2^n$, $n \geqslant 1$. Then the following hold:*

(i) $|\mathfrak{H}| = q(q^2 - 1)$.

(ii) *$\mathfrak{H}$ is simple if $q > 2$. If $q = 2$, $\mathfrak{H}$ is isomorphic to the symmetric group S_3. If $q = 4$, then $\mathfrak{H}$ is isomorphic to* $\mathrm{PSL}(2, 5)$.

(iii) *An S_2-subgroup $\mathfrak{S}$ of $\mathfrak{H}$ is elementary of order q. $\mathfrak{S}$ is disjoint from its conjugates. If $\mathfrak{T}$ is any non-trivial subgroup of $\mathfrak{S}$, then $\mathbf{C}_{\mathfrak{H}}(\mathfrak{T}) = \mathfrak{S}$. $\mathbf{N}_{\mathfrak{H}}(\mathfrak{S})$ is a Frobenius group of order $(q-1)q$, and contains a cyclic group of order $q - 1$ which acts transitively on the involutions of $\mathfrak{S}$.*

(iv) *If* $\mathfrak{T}$ *is any* 2*-subgroup of* $\mathfrak{H}$ *of order at least* 4, *then* $\mathsf{M}_{\mathfrak{H}}(\mathfrak{T}; 2')$ *is trivial.*

(v) $\mathfrak{H}$ *contains cyclic Hall subgroups* $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$ *of orders* $q - 1$ *and* $q + 1$ *respectively.* $\mathbf{N}_{\mathfrak{H}}(\mathfrak{R}_i)$ *is a dihedral group of order* $2|\mathfrak{R}_i|$, $i = 1, 2$. *If* $X \in R_i^{\#}$, *then* $\mathbf{C}_{\mathfrak{H}}(X) = \mathfrak{R}_i$, $i = 1, 2$. $\mathfrak{R}_i$ *is disjoint from its conjugates,* $i = 1, 2$.

(vi) *If* $q = 2^n > 4$, *then the only non-solvable subgroups of* $\mathfrak{H}$ *are isomorphic to* PSL$(2, 2^m)$ *with* $m|n$.

(vii) *If* X *is an element of* $\mathfrak{H}$ *of prime power order* $p^t > 1$, p *odd, there exists a conjugate* X' *of* X *in* $\mathfrak{H}$ *such that* $\langle X, X'\rangle$ *is not a* p*-group.*

(viii) *There are no non-trivial central extensions of* $\mathfrak{H}$ *by a group of odd order when* $q \geqslant 4$. *If* $\mathfrak{K}$ *is a non-solvable group such that* $\mathfrak{K}/\mathbf{O}(\mathfrak{K})$ *is isomorphic to* $\mathfrak{H}$, *and* $\mathbf{C}_{\mathfrak{K}}(\mathbf{O}(\mathfrak{K})) \not\subseteq \mathbf{O}(\mathfrak{K})$, *then* $\mathfrak{K}$ *contains a normal subgroup* $\mathfrak{L}$ *isomorphic to* $\mathfrak{H}$, *and* $\mathfrak{K} = \mathfrak{L} \times \mathbf{O}(\mathfrak{K})$.

LEMMA 2.2. *Let* $\mathfrak{H} =$ PSL$(2, q)$ *and* $\mathfrak{H}^* =$ PΓL$(2, q)$, $q = 2^n$, $n \geqslant 1$. *Then the following hold:*

(i) $\mathfrak{H}^* = \mathfrak{H}\mathfrak{F}$, *where* $\mathfrak{H} \lhd \mathfrak{H}^*$, $\mathfrak{F}$ *is cyclic of order* n, *and* $\mathfrak{H} \cap \mathfrak{F} = 1$. $\mathfrak{F}$ *normalizes subgroups of* $\mathfrak{H}$ *of orders* $q - 1$, q, *and* $q + 1$ *respectively.*

(ii) *If* $\mathfrak{F}_0 \subseteq \mathfrak{F}$ *and* $|\mathfrak{F}_0| = k$, *then* $\mathbf{C}_{\mathfrak{H}}(\mathfrak{F}_0)$ *is isomorphic to* PSL$(2, 2^m)$, *where* $m = n/k$. *An* S_2*-subgroup of* $\mathfrak{H}\mathfrak{F}_0$ *is abelian if and only if* $|\mathfrak{F}_0|$ *is odd.*

(iii) *If* $\mathfrak{S}$ *is an* S_2*-subgroup of* $\mathfrak{H}$ *normalized by* $\mathfrak{F}$ *and if* $\mathfrak{T}$ *is a subgroup of* $\mathfrak{S}$ *of order at least* 4, *then the subgroups* $\mathbf{C}_{\mathfrak{F}}(\mathfrak{T})^S$ *with* S *in* $\mathfrak{S}$ *are the only maximal elements of* $\mathsf{M}_{\mathfrak{H}^*}(\mathfrak{T}; 2')$.

(iv) *The* S_p*-subgroups of* $\mathfrak{H}^*$ *are cyclic or metacyclic for any odd prime* p.

(v) *If* $\mathfrak{H}$ *is isomorphic to a normal subgroup* $\mathfrak{L}$ *of a group* $\mathfrak{K}$ *in which* $\mathbf{C}_{\mathfrak{K}}(\mathfrak{L}) = 1$, *then* $\mathfrak{K}$ *is isomorphic to a subgroup of* $\mathfrak{H}^*$ *containing* $\mathfrak{H}$.

(vi) *If* X *is an element of* $\mathfrak{H}^*$ *of prime power order* $p^t > 1$, p *odd, then there exists a conjugate* X' *of* X *in* $\mathfrak{H}^*$ *such that* $\langle X, X'\rangle$ *is not a* p*-group.*

(vii) *Let* $\mathfrak{F}_0$ *be a subgroup of* $\mathfrak{F}$ *of order* $p^t > 1$, p *an odd prime, and let* $\mathfrak{R}$ *be a cyclic subgroup of* $\mathfrak{H}$ *of order* $q - 1$ *or* $q + 1$ *normalized by* $\mathfrak{F}_0$. *Then either* $\mathfrak{F}_0$ *does not centralize* $\mathbf{O}_{p'}(\mathfrak{R})$ *or* $p^t = 3$, $q = 8$, *and* $|\mathfrak{R}| = 9$.

(viii) *Let* $\mathfrak{F}_0$ *be a subgroup of* $\mathfrak{F}$ *of odd prime order* p, *let* $\mathfrak{R}$ *be a cyclic subgroup of* $\mathfrak{H}$ *of order* $q - 1$ *or* $q + 1$ *normalized by* $\mathfrak{F}_0$, *and let* $\mathfrak{R}_0$ *be a subgroup of* $\mathfrak{R}$ *not contained in* $\mathbf{C}_{\mathfrak{H}}(\mathfrak{F}_0)$. *Then if* $n > p$, *or equivalently if* $\mathbf{C}_{\mathfrak{H}}(\mathfrak{F}_0)$ *is non-solvable, we have* $\langle \mathbf{C}_{\mathfrak{H}}(\mathfrak{F}_0), \mathfrak{R}_0\rangle = \mathfrak{H}$.

Our next lemma is a consequence of a theorem of Huppert **(10)**.

LEMMA 2.3. *Let* $\mathfrak{H}$ *be an* A_0*-group, let* $\mathfrak{S}$ *be an* S_2*-subgroup of* $\mathfrak{H}$, *and assume that the following conditions hold:* (a) $\mathbf{O}(\mathfrak{H}) = 1$, (b) $\mathfrak{S}$ *is elementary, and* (c) $\mathfrak{H}$ *has one class of involutions. Then:*

(i) $\mathfrak{H}$ *is isomorphic to a subgroup of the one-dimensional affine group of semi-linear transformations over* GF(q), *where* $q = |\mathfrak{S}|$.

(ii) $\mathfrak{S} \lhd \mathfrak{H}$ *and* $\mathfrak{H}$ *possesses a cyclic subgroup which acts regularly on* $\mathfrak{S}$ *by conjugation and which has order at least* $(|\mathfrak{S}| - 1)/d$, *where* $d = (|\mathfrak{S}| - 1, m(\mathfrak{S}))$.

(iii) *For any involution* T *in* $\mathfrak{S}$, $|\mathbf{C}_{\mathfrak{H}}(T)| = |\mathfrak{S}|w$, *where* $w|m(\mathfrak{S})$.

Proof. By assumption, $\mathfrak{H}$ is solvable and $\mathfrak{S}$ is abelian. Since $\mathbf{O}(\mathfrak{H}) = 1$, it follows from **(9**, Lemma 1.2.3**)** that $\mathfrak{S} \lhd \mathfrak{H}$ and that $\mathbf{C}_{\mathfrak{H}}(\mathfrak{S}) = \mathfrak{S}$. Furthermore, since $\mathfrak{H}$ has one class of involutions and $\mathfrak{S}$ is elementary, $\mathfrak{H}$ can be represented as a doubly transitive permutation group on $|\mathfrak{S}|$ letters. Since $\mathbf{C}_{\mathfrak{H}}(\mathfrak{S}) = \mathfrak{S}$, this representation is faithful. But now we can apply a result of Huppert **(10)** concerning doubly transitive solvable groups to conclude that $\mathfrak{H}$ is isomorphic to a subgroup of the one-dimensional affine group $\mathfrak{H}^*$ of semi-linear transformations over $\mathrm{GF}(q)$, $q = |\mathfrak{S}|$. Without loss we may identify $\mathfrak{H}$ with its image in $\mathfrak{H}^*$.

Now $\mathfrak{H}^*$ consists of all transformations $x' = ax^\alpha + b$, where $a, b \in \mathrm{GF}(q)$, $a \neq 0$, and α is an element of the Galois group of $\mathrm{GF}(q)$ over $\mathrm{GF}(2)$. Hence $|\mathfrak{H}^*| = q(q-1)m$, where $q = 2^m$. Furthermore, the set of all transformations $x' = ax^\alpha$ forms a subgroup $\mathfrak{X}^*$ of $\mathfrak{H}^*$ of order $(q-1)m$; and $\mathfrak{X}^* = \mathfrak{N}^*\mathfrak{M}^*$, where $\mathfrak{N}^*$ is a cyclic normal subgroup of $\mathfrak{X}^*$ of order $q - 1$ consisting of the transformations of the form $x' = ax$, and $\mathfrak{M}^*$ is cyclic of order m consisting of the transformations of the form $x' = x^\alpha$. Also $\mathfrak{S}$ consists of the transformations of the form $x' = x + b$ and $\mathfrak{S}\mathfrak{N}^*$ is a Frobenius group of order $q(q-1)$. Since $\mathfrak{M}^*$ fixes the involution $x' = x + 1$ and since all involutions of $\mathfrak{H}^*$ are conjugate, we conclude that $|\mathbf{C}_{\mathfrak{H}^*}(T)| = |\mathfrak{S}|m$ for any involution T in $\mathfrak{S}$. Thus $|\mathbf{C}_{\mathfrak{H}}(T)| = |\mathfrak{S}|w$, where w divides $m = m(\mathfrak{S})$, and (iii) holds.

Finally $\mathfrak{H} = \mathfrak{S}\mathfrak{X}$, where $\mathfrak{X} \subseteq \mathfrak{X}^*$. Since $\mathfrak{H}$ is doubly transitive, $|\mathfrak{X}|$ is a multiple of $q - 1$, and hence $|\mathfrak{X} \cap \mathfrak{N}^*| \geqslant q - 1/d$, where $d = (q-1, m)$. Since $\mathfrak{X} \cap \mathfrak{N}^*$ is cyclic and acts regularly on $\mathfrak{S}$, (ii) also holds and the lemma is proved.

LEMMA 2.4. *Let $\mathfrak{H}$ be an A_0-group, let $\mathfrak{S}$ be an S_2-subgroup of $\mathfrak{H}$, and let $\mathfrak{T}$ be a non-trivial 2-subgroup of $\mathfrak{S}$. Then:*

(i) $\mathfrak{H} = \mathbf{O}(\mathfrak{H})\mathfrak{K}$, *where* $\mathfrak{K} = \mathbf{N}_{\mathfrak{H}}(\mathfrak{S})$. $\mathfrak{H}$ *has 2-length* 1.

(ii) $\mathbf{C}_{\mathfrak{H}}(\mathfrak{T})$ *acts transitively on the maximal elements of* $\mathsf{M}_{\mathfrak{H}}(\mathfrak{T}; p)$ *for any odd prime* p.

(iii) *If* $\mathfrak{P}$ *is a maximal element of* $\mathsf{M}_{\mathfrak{H}}(\mathfrak{T}; p)$, *then* $\mathfrak{P} = (\mathfrak{P} \cap \mathbf{O}(\mathfrak{H}))\mathbf{C}_{\mathfrak{P}}(\mathfrak{T})$ *and* $[\mathfrak{P}, \mathfrak{T}] \subseteq \mathbf{O}(\mathfrak{H})$. *Also* $\mathfrak{P}$ *is permutable with an S_2-subgroup of* $\mathfrak{H}$ *containing* $\mathfrak{T}$.

(iv) *Let* $\mathfrak{P}$ *be a maximal element of* $\mathsf{M}_{\mathfrak{H}}(\mathfrak{S}; p)$. *Then* $\mathfrak{P} \subseteq \mathbf{O}(\mathfrak{H})$. *If* $\mathfrak{S} \subseteq [\mathfrak{H}, \mathfrak{H}]$, *then either* $\mathfrak{S}$ *centralizes* $\mathfrak{P}$ *or* $\mathfrak{S}/\mathbf{C}_{\mathfrak{S}}(\mathfrak{P})$ *is non-cyclic and* $\mathscr{SCN}_3(\mathfrak{P})$ *is non-empty.*

(v) *If* $\mathfrak{K} = \mathbf{N}_{\mathfrak{H}}(\mathfrak{S})$, *then* $\mathbf{C}_{\mathfrak{H}}(\mathfrak{T}) \subseteq \mathbf{O}(\mathfrak{H})\mathbf{C}_{\mathfrak{K}}(\mathfrak{T})$.

(vi) *If* $\mathfrak{H}$ *has one class of involutions, then* $\mathfrak{S}$ *is homocyclic of type* $(2^a, \ldots, 2^a)$ *on* $m(\mathfrak{S})$ *generators. In this case, if* $\mathfrak{P}$ *is a maximal element of* $\mathsf{M}_{\mathfrak{H}}(\mathfrak{T}; p)$, *an S_2-subgroup of* $\mathbf{N}_{\mathfrak{H}}(\mathfrak{P})$ *is homocyclic of type* $(2^a, 2^a, \ldots, 2^a)$ *on* $k \leqslant m(\mathfrak{S})$ *generators.*

Proof. Since $\mathfrak{S}$ is abelian, **(9**, Lemma 1.2.3**)** implies that $\mathfrak{H}$ has 2-length 1. But then $\mathfrak{H} = \mathbf{O}(\mathfrak{H})\mathbf{N}_{\mathfrak{H}}(\mathfrak{S})$ by Sylow's theorem, yielding (i).

Next let $\mathfrak{P}$ be a maximal element of $\text{И}_{\mathfrak{H}}(\mathfrak{T}; p)$. Then by (i), $\mathfrak{T} \subseteq \mathbf{O}_{2',2}(\mathfrak{H})$, and consequently $\gamma\mathfrak{T}\mathfrak{P} \subseteq \mathfrak{P} \cap \mathbf{O}_{2',2}(\mathfrak{H}) \subseteq \mathfrak{P} \cap \mathbf{O}(\mathfrak{H})$. This in turn implies that $\mathfrak{P} = (\mathfrak{P} \cap \mathbf{O}(\mathfrak{H}))\mathbf{C}_{\mathfrak{P}}(\mathfrak{T})$. In particular, $\mathfrak{P} \subseteq \mathfrak{L} = \mathbf{O}(\mathfrak{H})\mathbf{C}_{\mathfrak{H}}(\mathfrak{T})$. Since an S_p-subgroup of $\mathbf{C}_{\mathfrak{H}}(\mathfrak{T})$ normalizes some $\mathfrak{T}$-invariant S_p-subgroup of $\mathbf{O}(\mathfrak{H})$, maximality of $\mathfrak{P}$ implies that $\mathfrak{P}$ is an S_p-subgroup of $\mathfrak{L}$. Since $\mathfrak{L}$ contains an S_2-subgroup of $\mathfrak{H}$, the final assertion of (iii) now follows from $D_{2,p}$ in $\mathfrak{L}$.

To prove (ii), let $\mathfrak{P}_1$ be a second maximal element of $\text{И}_{\mathfrak{H}}(\mathfrak{T}; p)$. Then also $\mathfrak{P}_1 \subseteq \mathfrak{L}$, and the images $\bar{\mathfrak{P}}$ and $\bar{\mathfrak{P}}_1$ of $\mathfrak{P}$ and $\mathfrak{P}_1$ respectively in $\bar{\mathfrak{L}} = \mathfrak{L}/\mathbf{O}(\mathfrak{H})$ are each S_p-subgroups of $\bar{\mathfrak{L}}$. Since $\mathfrak{L} = \mathbf{O}(\mathfrak{H})\mathbf{C}_{\mathfrak{H}}(\mathfrak{T})$, it follows that $\mathfrak{P}_1^X \subseteq \mathbf{O}(\mathfrak{H})\mathfrak{P}$ for some X in $\mathbf{C}_{\mathfrak{H}}(\mathfrak{T})$. But then $\mathfrak{P}$ and $\mathfrak{P}_1^X$ are each $\mathfrak{T}$-invariant S_p-subgroups of $\mathbf{O}(\mathfrak{H})\mathfrak{P}$ and consequently are conjugate by an element in $\mathbf{C}_{\mathfrak{H}}(\mathfrak{T})$, thus proving (ii).

Now let $\mathfrak{P}$ denote a maximal element of $\text{И}_{\mathfrak{H}}(\mathfrak{S}; p)$. Since $\mathfrak{S} \subseteq \mathbf{O}_{2',2}(\mathfrak{H})$, $\mathfrak{P}$ centralizes $\mathfrak{S}$ (mod $\mathbf{O}_{2'}(\mathfrak{H}) = \mathbf{O}(\mathfrak{H})$), and therefore $\mathfrak{P} \subseteq \mathbf{O}(\mathfrak{H})$ by **(9**, Lemma 1.2.3**)**. Suppose next that $\mathfrak{S} \subseteq [\mathfrak{H}, \mathfrak{H}]$ and that $\mathfrak{P} \not\subseteq \mathbf{C}(\mathfrak{S})$. Now $\mathfrak{P}$ is an S_p-subgroup of $\mathbf{O}(\mathfrak{H})$ and hence $\mathfrak{H} = \mathbf{O}(\mathfrak{H})\mathfrak{N}$, where $\mathfrak{N} = \mathbf{N}_{\mathfrak{H}}(\mathfrak{P})$. This implies that $\mathfrak{S} \subseteq [\mathfrak{N}, \mathfrak{N}]$. Set $\mathfrak{C} = \mathbf{C}_{\mathfrak{N}}(\mathfrak{P})$, in which case $\mathfrak{C} \lhd \mathfrak{N}$. If $\mathfrak{S}/\mathfrak{S} \cap \mathfrak{C}$ were cyclic, then an S_2-subgroup of $\mathfrak{N}/\mathfrak{C}$ would be cyclic and consequently $\mathfrak{N}/\mathfrak{C}$ would have a normal 2-complement by Burnside's Transfer Theorem. But then $\mathfrak{N}$ would possess a normal subgroup of index 2 contrary to $\mathfrak{S} \subseteq [\mathfrak{N}, \mathfrak{N}]$. Thus $\mathfrak{S}/\mathfrak{S} \cap \mathfrak{C} = \mathfrak{S}/\mathbf{C}_{\mathfrak{S}}(\mathfrak{P})$ is non-cyclic, as asserted.

To complete the proof of (iv), we show next that $\mathscr{SCN}_3(\mathfrak{P})$ is non-empty. Let $\mathfrak{D}$ be a subgroup of $\mathfrak{P}$ chosen in accordance with **(4**, Lemma 8.2**)**. Then $\text{cl}(\mathfrak{D}) \leqslant 2$, $\mathfrak{D} \lhd \mathfrak{N}$, and $\mathfrak{S}$ does not centralize $\mathfrak{D}$. Assume by way of contradiction that $\mathscr{SCN}_3(\mathfrak{P})$, and hence also $\mathscr{SCN}_3(\mathfrak{D})$, is empty. Since $\text{cl}(\mathfrak{D}) \leqslant 2$, this implies that $\bar{\mathfrak{D}} = \mathfrak{D}/\mathbf{D}(\mathfrak{D})$ is elementary of order p or p^2. If $\bar{\mathfrak{N}} = \mathfrak{N}/\mathbf{D}(\mathfrak{D})$ and $\bar{\mathfrak{C}} = \mathbf{C}_{\bar{\mathfrak{N}}}(\bar{\mathfrak{D}})$, it follows that $\bar{\mathfrak{N}}/\bar{\mathfrak{C}}$ is isomorphic to a subgroup of $GL(2, p)$. But $\mathfrak{S}$ does not centralize $\mathfrak{D}$ (mod $\mathbf{D}(\mathfrak{D})$) and consequently $\bar{\mathfrak{N}}/\bar{\mathfrak{C}}$ has even order. However, it is easy to see that any subgroup of $GL(2, p)$ with abelian S_2-subgroups possesses a normal 2-complement; cf. **(6**, Lemma 3.4**)**. Thus $\bar{\mathfrak{N}}$, and consequently also $\mathfrak{N}$, possesses a normal subgroup of index 2, which is a contradiction. Thus $\mathscr{SCN}_3(\mathfrak{D})$ is non-empty, and (iv) holds.

Set $\bar{\mathfrak{H}} = \mathfrak{H}/\mathbf{O}(\mathfrak{H})$ and let $\bar{\mathfrak{T}}$ be the image of $\mathfrak{T}$ in $\bar{\mathfrak{H}}$. Then by **(6**, Lemma 1.4(iv)**)**, $\mathbf{C}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{T}})$ is the image of $\mathbf{C}_{\mathfrak{H}}(\mathfrak{T})$ in $\bar{\mathfrak{H}}$. But by (i), $\mathfrak{H} = \mathbf{O}(\mathfrak{H})\mathfrak{K}$, where $\mathfrak{K} = \mathbf{N}_{\mathfrak{H}}(\mathfrak{S})$ and consequently $\mathfrak{K}$ maps on $\bar{\mathfrak{H}}$. It follows at once that

$$\mathbf{C}_{\mathfrak{H}}(\mathfrak{T}) \subseteq \mathbf{O}(\mathfrak{H})\mathbf{C}_{\mathfrak{K}}(\mathfrak{T}),$$

proving (v).

Finally to prove (vi), we assume that $\mathfrak{H}$ has one class of involutions. Since $\mathfrak{S}$ is abelian, two elements of $\mathfrak{S}$ conjugate in $\mathfrak{H}$ are already conjugate in $\mathfrak{K} = \mathbf{N}_{\mathfrak{H}}(\mathfrak{S})$. Thus $\mathfrak{K}$ has one class of involutions. This implies that no proper subgroup of $\Omega_1(\mathfrak{S})$ is characteristic in $\mathfrak{S}$. But then if $\mathfrak{S}$ is of exponent 2^a, we must have $\Omega^{a-1}(\mathfrak{S}) = \Omega_1(\mathfrak{S})$; and we conclude at once that $\mathfrak{S}$ is homocyclic of type $(2^a, 2^a, \ldots, 2^a)$.

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Finally let $\mathfrak{P}$ be a maximal element of $\mathsf{M}_{\mathfrak{H}}(\mathfrak{T}; p)$, set $\bar{\mathfrak{H}} = \mathfrak{H}/\mathbf{O}(\mathfrak{H})$ and let $\bar{\mathfrak{T}}, \bar{\mathfrak{S}}, \bar{\mathfrak{P}}$ be the respective images of $\mathfrak{T}$, $\mathfrak{S}$, and $\mathfrak{P}$ in $\bar{\mathfrak{H}}$. Then $\bar{\mathfrak{S}} \lhd \bar{\mathfrak{H}}$. Since $\bar{\mathfrak{S}}$ is an abelian p'-group, it follows that $\bar{\mathfrak{S}} = \bar{\mathfrak{S}}_1 \times \bar{\mathfrak{S}}_2$, where $\bar{\mathfrak{S}}_1 = \mathbf{C}_{\bar{\mathfrak{S}}}(\bar{\mathfrak{P}})$ and $\bar{\mathfrak{S}}_2 = [\bar{\mathfrak{S}}, \bar{\mathfrak{P}}]$. But then $\bar{\mathfrak{S}}_1$ is homocyclic of type $(2^a, 2^a, \ldots, 2^a)$ on $k \leqslant m(\mathfrak{S}) = m(\bar{\mathfrak{S}})$ generators. Furthermore, since $\bar{\mathfrak{S}} \cap \mathbf{N}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{P}})$ centralizes $\bar{\mathfrak{P}}$, $\bar{\mathfrak{S}}_1$ is an S_2-subgroup of $\mathbf{N}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{P}})$. Let $\mathfrak{H}_1$ denote the inverse image of $\mathbf{N}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{P}})$ in $\mathfrak{H}$. Then an S_2-subgroup of $\mathfrak{H}_1$ is isomorphic to $\bar{\mathfrak{S}}_1$. But $\mathfrak{T} \subseteq \mathfrak{H}_1$ and $\mathfrak{P} \subseteq \mathbf{O}(\mathfrak{H}_1)$, whence $\mathfrak{P}$ is an S_p-subgroup of $\mathbf{O}(\mathfrak{H}_1)$. By Sylow's Theorem, $\mathbf{N}_{\mathfrak{H}_1}(\mathfrak{P})$ contains an S_2-subgroup of $\mathfrak{H}_1$. But clearly $\mathfrak{H}_1 = \mathbf{N}_{\mathfrak{H}}(\mathbf{O}(\mathfrak{H})\mathfrak{P})$ and consequently $\mathbf{N}_{\mathfrak{H}}(\mathfrak{P}) \subseteq \mathfrak{H}_1$. Thus $\mathbf{N}_{\mathfrak{H}}(\mathfrak{P}) = \mathbf{N}_{\mathfrak{H}_1}(\mathfrak{P})$, and we conclude that an S_2-subgroup of $\mathbf{N}_{\mathfrak{H}}(\mathfrak{P})$ is homocyclic of type $(2^a, 2^a, \ldots, 2^a)$ on $k \leqslant m(\mathfrak{S})$ generators. Thus (vi) hold, and all parts of the lemma are proved.

LEMMA 2.5. *Let $\mathfrak{H}$ be an A_1-group, let $\mathfrak{S}$ be an S_2-subgroup of $\mathfrak{H}$, let $\mathfrak{T}$ be a subgroup of $\mathfrak{S}$ of order at least* 4, *and assume that* $|\mathfrak{S}| > 4$. *Then the following conditions hold:*

(i) *$\mathfrak{S}$ is elementary. $\mathfrak{H}$ has characteristic $2^{m(\mathfrak{S})}$. $\mathbf{N}_{\mathfrak{H}}(\mathfrak{S})$ contains an element which acts transitively on the involutions of $\mathfrak{S}$; in particular, $\mathfrak{H}$ has one class of involutions.*

(ii) *Any two maximal elements of $\mathsf{M}_{\mathfrak{H}}(\mathfrak{T}; p)$ are conjugate by an element of $\mathbf{C}_{\mathfrak{H}}(\mathfrak{T})$, where p is an odd prime.*

(iii) *If $\mathfrak{P}$ is a maximal element of $\mathsf{M}_{\mathfrak{H}}(\mathfrak{T}; p)$, then $\mathfrak{P} = (\mathfrak{P} \cap \mathbf{O}(\mathfrak{H}))\mathbf{C}_{\mathfrak{P}}(\mathfrak{T})$ and $[\mathfrak{P}, \mathfrak{T}] \subseteq \mathbf{O}(\mathfrak{H})$. $\mathfrak{P}$ is permutable with an S_2-subgroup of $\mathfrak{H}$ containing $\mathfrak{T}$. $\mathbf{N}_{\mathfrak{H}}(\mathfrak{P})$ is an A_1-group of characteristic $2^{m(\mathfrak{S})/k} \geqslant 4$, where $k = |\mathfrak{P}/\mathfrak{P} \cap \mathbf{O}(\mathfrak{H})|$.*

(iv) *Every element of $\mathsf{M}(\mathfrak{S})$ lies in $\mathbf{O}(\mathfrak{H})$.*

(v) *If $\mathfrak{K} = \mathbf{N}_{\mathfrak{H}}(\mathfrak{S})$, then $\mathbf{C}_{\mathfrak{H}}(\mathfrak{T}) \subseteq \mathbf{O}(\mathfrak{H})\mathbf{C}_{\mathfrak{K}}(\mathfrak{T})$.*

(vi) *Assume that $\mathbf{O}(\mathfrak{H})$ is a p-group contained in the centre of $\mathfrak{H}$. Then $\mathfrak{H}$ possesses a normal subgroup $\mathfrak{H}_1$ which is isomorphic to a subgroup of* $\mathrm{P\Gamma L}(2, 2^{m(\mathfrak{S})})$ *containing* $\mathrm{PSL}(2, 2^{m(\mathfrak{S})})$ *and which contains an S_p-subgroup of $\mathfrak{H}$.*

Proof. In proving (i), we may clearly assume that $\mathbf{O}(\mathfrak{H}) = 1$, in which case it follows from the definition of an A_1-group that $\mathfrak{H}$ is isomorphic to a subgroup of $\mathrm{P\Gamma L}(2, q)$ containing $\mathrm{PSL}(2, q)$, where $q = p^n$ and either $p = 2$ or $q \equiv 3$ or 5 (mod 8). Hence by (**6**, Lemma 3.3(i)) and Lemma 2.2 (i) and (ii), we have $\mathfrak{H} = \mathfrak{L}\mathfrak{F}$, where $\mathfrak{L}$ is isomorphic to $\mathrm{PSL}(2, p^n)$ or $\mathrm{PGL}(2, p^n)$, $\mathfrak{L} \lhd \mathfrak{H}$, $\mathfrak{L} \cap \mathfrak{F} = 1$, and $\mathfrak{F}$ is cyclic of order dividing n. Suppose first that $q = p^n$ is odd. In this case an S_2-subgroup of $\mathrm{PGL}(2, q)$ is dihedral of order at least 8. Since the S_2-subgroup $\mathfrak{S}$ of $\mathfrak{H}$ is abelian, $\mathfrak{L}$ must therefore be isomorphic to $\mathrm{PSL}(2, q)$. Since $q \equiv 3$ or 5 (mod 8) and $|\mathfrak{L}| = \frac{1}{2}q(q-1)(q+1)$, it follows that $|\mathfrak{S} \cap \mathfrak{L}| = 4$. But $|\mathfrak{S}| > 4$ by assumption, and hence $\mathfrak{F}$ necessarily has even order. Since $|\mathfrak{F}|$ divides n, $n = 2m$ for some integer m, and we conclude from the formula for $|\mathfrak{L}|$ that $|\mathfrak{L}|$ is divisible by 8. This contradiction shows that p must be equal to 2. In this case, $\mathrm{PGL}(2, q) = \mathrm{PSL}(2, q)$. By Lemma 2.2(i), $\mathfrak{F}$ normalizes an S_2-subgroup $\mathfrak{S}_0$ of $\mathfrak{L}$; and $\mathfrak{S}_0$ is elementary of order 2^n. Let $\mathfrak{F}_0$ be an S_2-subgroup of $\mathfrak{F}$ and suppose that $\mathfrak{F}_0 \neq 1$. Then by Lemma 2.2(ii),

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$|\mathbf{C}_{\mathfrak{S}_0}(\mathfrak{F}_0)| = 2^{n/k}$, where $k = |\mathfrak{F}_0|$. But this implies that $\mathfrak{S}_0 \mathfrak{F}_0$ is non-abelian, contrary to the fact that the S_2-subgroups of $\mathfrak{H}$ are abelian. Thus $\mathfrak{F}_0 = 1$, and we conclude that $\mathfrak{S}_0$ is an S_2-subgroup of $\mathfrak{H}$. Hence $\mathfrak{S} \subseteq \mathfrak{L}$, $\mathfrak{S}$ is elementary of order 2^n, $n = m(\mathfrak{S})$, and $\mathfrak{H}$ is of characteristic $2^{m(\mathfrak{S})}$. Furthermore, $\mathbf{N}_{\mathfrak{L}}(\mathfrak{S})$ contains a cyclic subgroup of order $q - 1$ which acts transitively on the elements of $\mathfrak{S}^{\#}$ by Lemma 2.1(iii), and consequently all parts of (i) are established.

In proving the remaining parts of the lemma, we drop the assumption $\mathbf{O}(\mathfrak{H}) = 1$, and set $\bar{\mathfrak{H}} = \mathfrak{H}/\mathbf{O}(\mathfrak{H})$. Let $\bar{\mathfrak{T}}$ and $\bar{\mathfrak{S}}$ be the images of $\mathfrak{T}$ and $\mathfrak{S}$ in $\bar{\mathfrak{H}}$. By (i), $\bar{\mathfrak{H}} = \bar{\mathfrak{L}}\bar{\mathfrak{F}}$, where $\bar{\mathfrak{L}}$ is isomorphic to PSL$(2, q)$, where $q = 2^{m(\mathfrak{S})}$, $\bar{\mathfrak{L}} \lhd \bar{\mathfrak{H}}$, $\bar{\mathfrak{F}}$ is cyclic of odd order, and $\bar{\mathfrak{L}} \cap \bar{\mathfrak{F}} = 1$. Furthermore, we can assume that $\bar{\mathfrak{F}}$ is chosen so as to normalize $\bar{\mathfrak{S}}$. Finally we denote by $\mathfrak{M}$ the inverse image of $\bar{\mathfrak{S}}\bar{\mathfrak{F}}$ in $\mathfrak{H}$. Since $\mathbf{O}(\bar{\mathfrak{S}}\bar{\mathfrak{F}})$ centralizes $\bar{\mathfrak{S}}$, it follows at once from Lemma 2.2(ii) that $\mathbf{O}(\bar{\mathfrak{S}}\bar{\mathfrak{F}}) = 1$, and consequently $\mathbf{O}(\mathfrak{H}) = \mathbf{O}(\mathfrak{M})$. Furthermore, by Lemma 2.2(iii), every element of $\mathsf{M}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{T}}; 2')$ lies in $\bar{\mathfrak{S}}\bar{\mathfrak{F}}$.

Now let $\mathfrak{P}$ be a maximal element of $\mathsf{M}_{\mathfrak{H}}(\mathfrak{T}; p)$ and let $\bar{\mathfrak{P}}$ be its image in $\bar{\mathfrak{H}}$. Then $\bar{\mathfrak{P}} \subseteq \mathbf{C}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{T}}) \subseteq \bar{\mathfrak{S}}\bar{\mathfrak{F}}$ and hence $\mathfrak{P} \subseteq \mathfrak{M}$. Thus $\mathfrak{M}$ contains every maximal element of $\mathsf{M}_{\mathfrak{H}}(\mathfrak{T}; p)$. Since $\mathfrak{M}$ is solvable, (ii) follows at once from Lemma 2.4(ii) applied to $\mathfrak{M}$; and since $\mathbf{O}(\mathfrak{M}) = \mathbf{O}(\mathfrak{H})$, the first assertion of (iii) follows in the same way from Lemma 2.4(iii). Furthermore, since $\bar{\mathfrak{P}} \subseteq \bar{\mathfrak{S}}\bar{\mathfrak{F}}$, $\mathbf{C}_{\bar{\mathfrak{L}}}(\bar{\mathfrak{P}})$ is isomorphic to PSL$(2, 2^m)$, where $m = m(\mathfrak{S})/k$ and $k = |\bar{\mathfrak{P}}|$ by Lemma 2.2(ii). But $\bar{\mathfrak{P}}$ centralizes $\bar{\mathfrak{T}}$, $\bar{\mathfrak{T}} \subseteq \bar{\mathfrak{L}}$, and $|\bar{\mathfrak{T}}| \geqslant 4$ by hypothesis. Hence $m \geqslant 2$, and consequently $\mathbf{C}_{\bar{\mathfrak{L}}}(\bar{\mathfrak{P}})$ is non-solvable and so is an A_1-group of characteristic 2^m. Now let $\mathfrak{H}_1$ be the inverse image of $\mathbf{C}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{P}})$ in $\mathfrak{H}$. Then $\mathfrak{H}_1$ is an A_1-group of characteristic 2^m and $\mathfrak{P} \subseteq \mathbf{O}(\mathfrak{H}_1)$. Since $\mathfrak{T} \subseteq \mathfrak{H}_1$ and $\mathfrak{P}$ is a maximal element of $\mathsf{M}_{\mathfrak{H}}(\mathfrak{T}; p)$, $\mathfrak{P}$ is an S_p-subgroup of $\mathbf{O}(\mathfrak{H}_1)$; and it follows at once from Sylow's theorem that $\mathbf{N}_{\mathfrak{H}_1}(\mathfrak{P})$ is an A_1-group of characteristic 2^m. But clearly $\mathbf{C}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{P}}) = \mathbf{N}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{P}})$, and consequently $\mathbf{N}_{\mathfrak{H}}(\mathfrak{P})$ maps into $\mathbf{C}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{P}})$. Thus $\mathbf{N}_{\mathfrak{H}}(\mathfrak{P}) \subseteq \mathfrak{H}_1$, and we conclude that $\mathbf{N}_{\mathfrak{H}}(\mathfrak{P}) = \mathbf{N}_{\mathfrak{H}_1}(\mathfrak{P})$ is an A_1-group of characteristic 2^m. Since $m = m(S)/k \geqslant 2$ and $k = |\bar{\mathfrak{P}}| = |\mathfrak{P}/\mathfrak{P} \cap \mathbf{O}(\mathfrak{H})|$, the final assertion of (iii) is proved.

By Lemma 2.2(iii), $\mathsf{M}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{S}})$ is trivial, and hence every element of $\mathsf{M}_{\mathfrak{H}}(\mathfrak{S}; p)$ lies in $\mathbf{O}(\mathfrak{H})$. Thus (iv) also holds.

Since $\mathbf{C}_{\mathfrak{H}}(\mathfrak{T})$ maps into $\mathbf{C}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{T}})$ and $\mathbf{C}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{T}}) \subseteq \bar{\mathfrak{S}}\bar{\mathfrak{F}}$, we have $\mathbf{C}_{\mathfrak{H}}(\mathfrak{T}) \subseteq \mathfrak{M}$. Since $\mathfrak{S} \subseteq \mathfrak{M}$, Lemma 2.4(v) implies that $\mathbf{C}_{\mathfrak{H}}(\mathfrak{T}) \subseteq \mathbf{O}(\mathfrak{H})\mathbf{C}_{\mathfrak{K}_1}(\mathfrak{T})$, where $\mathfrak{K}_1 = \mathbf{N}_{\mathfrak{M}}(\mathfrak{S})$, and (v) also holds.

Finally we establish (vi), which is the analogue of (6, Lemma 3.10) for A_1-groups of characteristic 2^n. Let $\bar{\mathfrak{H}} = \mathfrak{H}/\mathbf{O}(\mathfrak{H}) = \bar{\mathfrak{L}}\bar{\mathfrak{F}}$, where $\bar{\mathfrak{L}}$, $\bar{\mathfrak{F}}$ have the usual meaning, and let $\mathfrak{L}$ and $\mathfrak{F}$ denote respectively the inverse images of $\bar{\mathfrak{L}}$ and $\bar{\mathfrak{F}}$. Then $\mathbf{O}(\mathfrak{H}) = \mathbf{O}(\mathfrak{L})$ and by our hypothesis, $\mathbf{O}(\mathfrak{L}) \subseteq \mathbf{Z}(\mathfrak{L})$. Since $\mathfrak{L}/\mathbf{O}(\mathfrak{L}) = \bar{\mathfrak{L}}$ is isomorphic to PSL$(2, 2^{m(\mathfrak{S})})$, it follows from Lemma 2.1(viii) that $\mathfrak{L} = \mathfrak{H}_0 \times \mathbf{O}(\mathfrak{H})$, where $\mathfrak{H}_0$ is isomorphic to $\bar{\mathfrak{L}}$. Next let $\mathfrak{P}$ be an S_p-subgroup of $\mathfrak{F}$. Since $\mathbf{O}(\mathfrak{H}) \subseteq \mathbf{Z}(\mathfrak{F})$ and $\mathfrak{F}/\mathbf{O}(\mathfrak{H}) = \bar{\mathfrak{F}}$ is cyclic, $\mathfrak{F}$ is abelian and consequently $\mathfrak{P} \lhd \mathfrak{F}$. Thus $\mathfrak{H}_1 = \mathfrak{H}_0\mathfrak{P} \lhd \mathfrak{H}$ and $\mathfrak{H}_1$ contains an S_p-subgroup of $\mathfrak{H}$. Finally since $\mathbf{O}(\mathfrak{H})$ is a p-group, $\mathfrak{P} \cap \mathbf{O}(\mathfrak{H}) = 1$, and $\mathfrak{H}_1$ is isomorphic to a subgroup of $P\Gamma L(2, 2^{m(\mathfrak{S})})$ containing $PSL(2, 2^{m(\mathfrak{S})})$. This completes the proof of the lemma.

LEMMA 2.6. *Let $\mathfrak{H}$ be an A_1-group of characteristic q and let $\mathfrak{S}$ be an S_2-subgroup of $\mathfrak{H}$. Then the following conditions hold:*

(i) *If $\mathfrak{H}$ contains an A_1-subgroup of characteristic $q_1 > 5$, then q_1 divides q.*

(ii) *If $q > 5$, and q is odd, then $\mathfrak{H} = \langle \mathbf{C}_{\mathfrak{H}}(T) | T \in \mathfrak{S}^{\#} \rangle$.*

(iii) *$\mathfrak{H}$ has no normal subgroups of index 2.*

(iv) $\mathfrak{S} \subseteq [\mathbf{N}_{\mathfrak{H}}(\mathfrak{S}), \mathbf{N}_{\mathfrak{H}}(\mathfrak{S})]$.

(v) *If $\mathfrak{P}$ is a normal p-subgroup of $\mathfrak{H}$ such that $\mathbf{C}_{\mathfrak{H}}(\mathfrak{P}) \not\subseteq \mathbf{O}(\mathfrak{H})$, then $\mathbf{C}_{\mathfrak{H}}(\mathfrak{P})$ is an A_1-group and contains an S_2-subgroup of $\mathfrak{H}$.*

(vi) *Suppose q is odd and let $\mathfrak{P}$ be a maximal element of $\mathsf{V}_{\mathfrak{H}}(\mathfrak{S}; p)$. If either $q = p^n$ or $\mathfrak{S} \subseteq [\mathbf{N}_{\mathfrak{H}}(\mathfrak{P}), \mathbf{N}_{\mathfrak{H}}(\mathfrak{P})]$, then $\mathfrak{P} = (\mathfrak{P} \cap \mathbf{O}(\mathfrak{H}))\mathbf{C}_{\mathfrak{P}}(\mathfrak{S})$.*

(vii) *Suppose $q = p^n$ is odd and $q > 5$. Then there exists a prime $r \neq p$ such that $\mathfrak{S}$ does not centralize a maximal element of $\mathsf{V}_{\mathfrak{H}}(\mathfrak{S}; r)$.*

Proof. In proving (i), we can clearly assume that $\mathbf{O}(\mathfrak{H}) = 1$. Then $\mathfrak{H} = \mathfrak{L}\mathfrak{F}$, where $\mathfrak{L} \lhd \mathfrak{H}$, $\mathfrak{L}$ is isomorphic to $PSL(2, q)$, $q \equiv 3$ or $5 \pmod 8$ or $q = 2^n$, $\mathfrak{F}$ is cyclic of odd order and $\mathfrak{L} \cap \mathfrak{F} = 1$. Hence if $\mathfrak{H}_1$ is an A_1-subgroup of $\mathfrak{H}$ of characteristic q_1, $\mathfrak{H} \cap \mathfrak{L} = \mathfrak{L}_1$ is an A_1-group of the same characteristic q. Since $q_1 > 5$, clearly $q > 5$. If q is odd, then $q = p^n$ and $q_1 = p^m$ with $m|n$ by **(6**, Lemma 3.1(viii)**)**. On the other hand, if $q = 2^n$, then $q_1 = 2^m$ with $m|n$ by Lemma 2.2(vi). This proves (i).

Since $\mathbf{O}(\mathfrak{H}) = \langle \mathbf{C}_{\mathbf{O}(\mathfrak{H})}(T) | T \in \mathfrak{S}^{\#} \rangle$, (ii) will follow if we can show that $\bar{\mathfrak{H}} = \langle \mathbf{C}_{\bar{\mathfrak{H}}}(\bar{T}) | \bar{T} \in \bar{\mathfrak{S}}^{\#} \rangle$, where $\bar{\mathfrak{H}} = \mathfrak{H}/\mathbf{O}(\mathfrak{H})$ and $\bar{\mathfrak{S}}$ is the image of $\mathfrak{S}$ in $\bar{\mathfrak{H}}$. Since $\mathbf{O}(\bar{\mathfrak{H}}) = 1$, it suffices to prove (ii) under the additional assumption that $\mathbf{O}(\mathfrak{H}) = 1$. Thus $\mathfrak{H} = \mathfrak{L}\mathfrak{F}$ as above, where now $q \equiv 3$ or $5 \pmod 8$. By **(6**, Lemma 3.3(i)**)**, we may assume that $\mathfrak{S}$ centralizes $\mathfrak{F}$. Hence we need only show that $\mathfrak{L} = \langle \mathbf{C}_{\mathfrak{L}}(T) | T \in \mathfrak{S}^{\#} \rangle$. Since $q > 5$, our conditions imply that, in fact, $q \geqslant 11$. The desired conclusion now follows from **(6**, Lemma 3.1(ix)**)**. This proves (ii).

Let $\mathfrak{H}_1$ be a normal subgroup of $\mathfrak{H}$ of index 2. Since $\mathfrak{H}$ is non-solvable, so also is $\mathfrak{H}_1$, and hence $\mathfrak{H}_1$ is an A_1-group. Therefore $|\mathfrak{S} \cap \mathfrak{H}_1| \geqslant 4$ and so $|\mathfrak{S}| > 4$. But by Lemma 2.5(i), $\mathfrak{S}$ is elementary and $\mathfrak{H}$ has one class of involutions. This implies that $\mathfrak{H}_1$ contains all involutions of $\mathfrak{S}$ and hence that $\mathfrak{S} \subseteq \mathfrak{H}_1$, contrary to the fact that $|\mathfrak{H} : \mathfrak{H}_1| = 2$ and $\mathfrak{S}$ is an S_2-subgroup of $\mathfrak{H}$. Thus (iii) holds.

If $|\mathfrak{S}| > 4$, then by Lemma 2.5(i), $\mathbf{N}_{\mathfrak{H}}(\mathfrak{S})$ possesses an element which acts transitively on the involutions of $\mathfrak{S}$ and also $\mathfrak{S}$ is elementary. It follows at once from this that $\mathfrak{S} \subseteq [\mathbf{N}_{\mathfrak{H}}(\mathfrak{S}), \mathbf{N}_{\mathfrak{H}}(\mathfrak{S})]$. On the other hand, if $|\mathfrak{S}| = 4$, then $\mathbf{N}_{\mathfrak{H}}(\mathfrak{S})$ contains a 3-element which cyclically permutes the three involutions of $\mathfrak{S}$; otherwise $\mathbf{N}_{\mathfrak{H}}(\mathfrak{S}) = \mathbf{C}_{\mathfrak{H}}(\mathfrak{S})$ and $\mathfrak{H}$ would have a normal 2-complement. Thus (iv) holds in this case as well.

Now let $\mathfrak{P}$ be a normal p-subgroup of $\mathfrak{H}$ such that $\mathfrak{C} = \mathbf{C}_{\mathfrak{H}}(\mathfrak{P}) \not\subseteq \mathbf{O}(\mathfrak{H})$. Set $\bar{\mathfrak{H}} = \mathfrak{H}/\mathbf{O}(\mathfrak{H})$ and let $\bar{\mathfrak{H}} = \bar{\mathfrak{L}}\bar{\mathfrak{F}}$, where $\bar{\mathfrak{L}}$ is isomorphic to $PSL(2, q)$,

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$\bar{\mathfrak{L}} \lhd \bar{\mathfrak{H}}$, $\bar{\mathfrak{F}}$ is cyclic, and $\bar{\mathfrak{L}} \cap \bar{\mathfrak{F}} = 1$. Furthermore, $|\bar{\mathfrak{F}}|$ is odd since $\mathfrak{H}$ has no normal subgroups of index 2. Now $\mathfrak{C} \lhd \mathfrak{H}$ and consequently the image $\bar{\mathfrak{C}}$ of $\mathfrak{C}$ in $\bar{\mathfrak{H}}$ is normal in $\bar{\mathfrak{H}}$ and $\bar{\mathfrak{C}} \neq 1$. Since $\bar{\mathfrak{L}}$ is simple and $\mathbf{C}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{L}}) = 1$, it follows that $\bar{\mathfrak{L}} \subseteq \bar{\mathfrak{C}}$. Since also $\bar{\mathfrak{L}}$ contains an S_2-subgroup of $\bar{\mathfrak{H}}$, we conclude that $\mathfrak{C}$ is an A_1-group and that $\mathfrak{C}$ contains an S_2-subgroup of $\mathfrak{H}$, proving (v).

Assume next that q is odd and let $\mathfrak{P}$ be a maximal element of $\mathscr{U}_{\mathfrak{H}}(\mathfrak{S}; p)$ for some prime p. Once again set $\bar{\mathfrak{H}} = \mathfrak{H}/\mathbf{O}(\mathfrak{H})$ and let $\bar{\mathfrak{P}}$, $\bar{\mathfrak{S}}$ be the images of $\mathfrak{P}$ and $\mathfrak{S}$ in $\bar{\mathfrak{H}}$. Then $\bar{\mathfrak{H}} = \bar{\mathfrak{L}}\bar{\mathfrak{F}}$, where $\bar{\mathfrak{L}}$ and $\bar{\mathfrak{F}}$ are as in the preceding paragraph. We may also assume, in view of **(6**, Lemma 3.3(i)**)**, that $\bar{\mathfrak{F}}$ centralizes $\bar{\mathfrak{S}}$. Suppose first that $q = p^n$. Then $\mathscr{U}_{\bar{\mathfrak{L}}}(\bar{\mathfrak{S}})$ is trivial by **(6**, Lemma 3.1(vii)**)**, and consequently $\bar{\mathfrak{F}}$ is the unique maximal element of $\mathscr{U}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{S}})$ by **(6**, Lemma 3.3(ii)**)**. Hence $\bar{\mathfrak{P}} \subseteq \bar{\mathfrak{F}} \subseteq \mathbf{C}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{S}})$. We conclude at once from this that $\mathfrak{P} = (\mathfrak{P} \cap \mathbf{O}(\mathfrak{H}))\mathbf{C}_{\mathfrak{P}}(\mathfrak{S})$. Thus (vi) holds in this case.

Assume next that $\mathfrak{S} \subseteq [\mathbf{N}_{\mathfrak{H}}(\mathfrak{P}), \mathbf{N}_{\mathfrak{H}}(\mathfrak{P})]$, whence $\bar{\mathfrak{S}} \subseteq [\mathbf{N}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{P}}), \mathbf{N}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{P}})]$. We shall argue that $\bar{\mathfrak{P}} \cap \bar{\mathfrak{L}} = 1$, so assume the contrary. Since $\bar{\mathfrak{P}} \cap \bar{\mathfrak{L}} \in \mathscr{U}_{\bar{\mathfrak{L}}}(\bar{\mathfrak{S}}; p)$, **(6**, Lemma 3.1(vii)**)** implies that $q \neq p^n$. Hence $\bar{\mathfrak{P}} \cap \bar{\mathfrak{L}}$ is cyclic by **(6**, Lemma 3.1(v)**)**. Since $\bar{\mathfrak{P}} \cap \bar{\mathfrak{L}} \lhd \mathbf{N}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{P}})$ and $\bar{\mathfrak{S}} \subseteq [\mathbf{N}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{P}}), \mathbf{N}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{P}})]$, it follows that $\bar{\mathfrak{S}}$ centralizes $\bar{\mathfrak{P}} \cap \bar{\mathfrak{L}}$. But $\bar{\mathfrak{S}} = \mathbf{C}_{\bar{\mathfrak{L}}}(\bar{\mathfrak{S}})$ by **(6**, Lemma 3.1(iv)**)**, yielding a contradiction. Thus $\bar{\mathfrak{P}} \cap \bar{\mathfrak{L}} = 1$. Since $\bar{\mathfrak{S}} \subseteq \bar{\mathfrak{L}}$, $\gamma\bar{\mathfrak{S}}\bar{\mathfrak{P}} \subseteq \bar{\mathfrak{L}} \cap \bar{\mathfrak{P}} = 1$, and therefore $\bar{\mathfrak{S}}$ centralizes $\bar{\mathfrak{P}}$. This proves (vi).

Finally to prove (vii), assume $q = p^n$ is odd and $q > 5$. If

$$\bar{\mathfrak{H}} = \mathfrak{H}/\mathbf{O}(\mathfrak{H}) = \bar{\mathfrak{L}}\bar{\mathfrak{F}}$$

and if $\bar{\mathfrak{S}}$ denotes as above the image of $\mathfrak{S}$ in $\bar{\mathfrak{H}}$, it will clearly suffice to show that $\bar{\mathfrak{S}}$ does not centralize a maximal element of $\mathscr{U}_{\bar{\mathfrak{L}}}(\bar{\mathfrak{S}}; r)$ for some prime $r \neq p$. But since $\bar{\mathfrak{L}}$ is isomorphic to PSL(2, q), it follows from **(6**, Lemma 3.1 (iii)**)** that $|\mathbf{C}_{\bar{\mathfrak{L}}}(\bar{T})| = q - \delta$, where $\delta = \pm 1$ and $\delta \equiv q \pmod 4$, for any $\bar{T}$ in $\bar{\mathfrak{S}}^{\#}$. Furthermore, since $|\bar{\mathfrak{S}}| = 4$, $(q - \delta)/4$ is necessarily odd, and since $q = p^n > 5$ by assumption, it follows that $|\mathbf{C}_{\bar{\mathfrak{L}}}(\bar{T})|$ is divisible by an odd prime $r \neq p$. But by **(6**, Lemma 3.1(iii)**)**, $\mathbf{C}_{\bar{\mathfrak{L}}}(\bar{T})$ is a dihedral group, and we conclude that $\bar{\mathfrak{S}}$ does not centralize an S_r-subgroup of $\mathbf{C}_{\bar{\mathfrak{L}}}(\bar{T})$. This proves (vii), and completes the proof of the lemma.

The next lemma gives a slight extension of Lemma 2.4(iv).

Lemma 2.7. *Let $\mathfrak{H}$ be an A_i-group, $i = 0$ or 1, and let $\mathfrak{S}$ be an S_2-subgroup of $\mathfrak{H}$. If $\mathfrak{H}$ is an A_0-group, assume in addition that $\mathfrak{S} \subseteq [\mathfrak{H}, \mathfrak{H}]$. Then if $\mathfrak{P}$ is a normal p-subgroup of $\mathfrak{H}$, p odd, either $\mathfrak{S}$ centralizes $\mathfrak{P}$ or $\mathscr{SCN}_3(\mathfrak{P})$ is non-empty.*

Proof. Let $\mathfrak{D}$ be a subgroup of $\mathfrak{P}$ chosen in accordance with **(4**, Lemma 8.2**)**. If $\mathscr{SCN}_3(\mathfrak{D})$ is empty, it follows as in the proof of Lemma 2.4(iv) that $\bar{\mathfrak{H}} = \mathfrak{H}/\mathbf{C}_{\mathfrak{H}}(\mathfrak{D})$ is isomorphic to a subgroup of GL(2, p) and that $\bar{\mathfrak{H}}$ possesses a normal 2-complement. Hence either $\mathfrak{S} \subseteq \mathbf{C}_{\mathfrak{H}}(\mathfrak{D})$ or $\bar{\mathfrak{H}}$, and consequently also $\mathfrak{H}$, has a normal subgroup of index 2. However, by Lemma 2.6(iii), the latter case is not possible if $\mathfrak{H}$ is an A_1-group; and it is also excluded by our hypothesis if $\mathfrak{H}$ is an A_0-group. Thus $\mathfrak{S} \subseteq \mathbf{C}_{\mathfrak{H}}(\mathfrak{D})$, and therefore $\mathfrak{S}$ centralizes $\mathfrak{P}$ by **(4**, Lemma 8.2**)**. The lemma follows immediately.

LEMMA 2.8. *Let $\mathfrak{H}$ be an A_1-group of characteristic r^n, let $\mathfrak{T}$ be a 2-subgroup of $\mathfrak{H}$ of order at least 4, and let $\mathfrak{P}$ be a maximal element of $\mathsf{N}_{\mathfrak{H}}(\mathfrak{T}; p)$ for some prime p. Assume, in addition, that $\mathfrak{P} = (\mathfrak{P} \cap \mathbf{O}(\mathfrak{H}))\mathbf{C}_{\mathfrak{P}}(\mathfrak{T})$; and set $k = |\mathfrak{P}/\mathfrak{P} \cap \mathbf{O}(\mathfrak{H})|$. Then k divides n and we have the following two cases:*

(i) *$r^{n/k} > 3$. Then $\mathbf{N}_{\mathfrak{H}}(\mathfrak{P})$ is an A_1-group of characteristic $r^{n/k}$.*

(ii) *$r^{n/k} = 3$. Then $r = 3$ and $n = k$, and $\mathfrak{P}$ possesses a maximal subgroup $\mathfrak{P}^*$ containing $\mathfrak{P} \cap \mathbf{O}(\mathfrak{H})$ such that $\mathbf{N}_{\mathfrak{H}}(\mathfrak{P}^*)$ is an A_1-group of characteristic 3^p. Furthermore, if $p \geqslant 5$, then $\mathfrak{P}$ is an S_p-subgroup of $\mathfrak{H}$.*

Proof. If $r = 2$, (i) follows from the final assertion of Lemma 2.5(iii). Hence we may assume that r is an odd prime. Set $\bar{\mathfrak{H}} = \mathfrak{H}/\mathbf{O}(\mathfrak{H})$, and let $\bar{\mathfrak{P}}$, $\bar{\mathfrak{T}}$ be the images of $\mathfrak{P}$ and $\mathfrak{T}$ in $\bar{\mathfrak{H}}$. Then by **(6**, Lemma 3.3(i)**)**, and Lemma 2.6(iii), $\bar{\mathfrak{H}} = \bar{\mathfrak{L}}\bar{\mathfrak{F}}$, where $\bar{\mathfrak{L}}$ is isomorphic to PSL$(2, r^n)$, $\bar{\mathfrak{L}} \lhd \bar{\mathfrak{H}}$, $\bar{\mathfrak{F}}$ is cyclic of odd order, and $\bar{\mathfrak{L}} \cap \bar{\mathfrak{F}} = 1$. Furthermore, we may assume that $\bar{\mathfrak{F}}$ is chosen so as to centralize $\bar{\mathfrak{T}}$. Now our hypothesis implies that $\bar{\mathfrak{P}}$ centralizes $\bar{\mathfrak{T}}$, and hence $\bar{\mathfrak{P}} \subseteq \bar{\mathfrak{F}}$ by **(6**, Lemma 3.3(i)**)**. Since $\mathfrak{P}$ is a maximal element of $\mathsf{N}_{\mathfrak{H}}(\mathfrak{T}; p)$, $\bar{\mathfrak{P}}$ is a maximal element of $\mathsf{N}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{T}}; p)$, and therefore $\bar{\mathfrak{P}}$ is, in fact, an S_p-subgroup of $\bar{\mathfrak{F}}$. Applying **(6**, Lemma 3.3(i)**)** once again, it follows that $\bar{\mathfrak{L}}_1 = \mathbf{C}_{\bar{\mathfrak{L}}}(\bar{\mathfrak{P}})$ is isomorphic to PSL$(2, r^{n/k})$, where $k = |\bar{\mathfrak{P}}|$. If $r^{n/k} > 3$, then $\bar{\mathfrak{L}}_1$ is an A_1-group. In this case, we set $\mathfrak{H}_1$ equal to the inverse image of $\mathbf{N}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{P}}) = \bar{\mathfrak{L}}_1\bar{\mathfrak{F}}$ in $\mathfrak{H}$. Then $\mathfrak{H}_1$ is also an A_1-group of characteristic $r^{n/k}$ and $\mathfrak{P} \subseteq \mathbf{O}(\mathfrak{H}_1)$. Furthermore, $\mathfrak{P}$ is an S_p-subgroup of $\mathbf{O}(\mathfrak{H}_1)$, and so $\mathbf{N}_{\mathfrak{H}_1}(\mathfrak{P})$ is an A_1-group of the same characteristic $r^{n/k}$ by Sylow's Theorem. Since $\mathbf{N}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{P}})$ is the image of $\mathfrak{H}_1$ in $\bar{\mathfrak{H}}$, we see that $\mathbf{N}_{\mathfrak{H}}(\mathfrak{P}) \subseteq \mathfrak{H}_1$, whence $\mathbf{N}_{\mathfrak{H}}(\mathfrak{P}) = \mathbf{N}_{\mathfrak{H}_1}(\mathfrak{P})$ is an A_1-group of characteristic $r^{n/k}$. Since $k = |\bar{\mathfrak{P}}| = |\mathfrak{P}/\mathfrak{P} \cap \mathbf{O}(\mathfrak{H})|$, we conclude that (i) holds.

Suppose finally that $r^{n/k} = 3$, in which case $r = 3$ and $n = k$. Since $\mathfrak{H}$ is an A_1-group, it is non-solvable, and consequently $\bar{\mathfrak{L}}$ is not isomorphic to PSL(2, 3). Thus $n > 1$, and so $n = k = p^s = |\bar{\mathfrak{P}}|$, where $s \geqslant 1$. In particular, $\bar{\mathfrak{P}} = \bar{\mathfrak{F}}$. Let $\bar{\mathfrak{P}}^*$ be the unique maximal subgroup of $\bar{\mathfrak{P}}$ and set $\bar{\mathfrak{L}}^* = \mathbf{C}_{\bar{\mathfrak{L}}}(\bar{\mathfrak{P}}^*)$. Then by **(6**, Lemma 3.3(i)**)**, $\bar{\mathfrak{L}}^*$ is isomorphic to PSL$(2, 3^p)$, and $\bar{\mathfrak{P}}$ normalizes, but does not centralize, $\bar{\mathfrak{L}}^*$. Setting $\bar{\mathfrak{H}}^* = \mathbf{N}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{P}}^*)$, it follows that $\bar{\mathfrak{H}}^* = \bar{\mathfrak{L}}^*\bar{\mathfrak{P}}$ and that $\bar{\mathfrak{P}}^*$ is an S_p-subgroup of $\mathbf{O}(\bar{\mathfrak{H}}^*)$. Now let $\mathfrak{H}^*$, $\mathfrak{P}^*$ denote respectively the inverse images of $\bar{\mathfrak{H}}^*$ in $\mathfrak{H}$ and $\bar{\mathfrak{P}}^*$ in $\mathfrak{P}$. Then $\mathfrak{H}^*$ is an A_1-group of characteristic 3^p, $\mathfrak{P}^*$ is an S_p-subgroup of $\mathbf{O}(\mathfrak{H}^*)$, $|\mathfrak{P} : \mathfrak{P}^*| = p$, and $\mathfrak{P} \cap \mathbf{O}(\mathfrak{H}) \subseteq \mathfrak{P}^*$. It follows now as in the preceding case that $\mathbf{N}_{\mathfrak{H}}(\mathfrak{P}^*) = \mathbf{N}_{\mathfrak{H}^*}(\mathfrak{P}^*)$ is an A_1-group of characteristic 3^p, thus completing the proof of the first assertion of (ii). Furthermore, in this case, $\mathfrak{H}$ is of characteristic 3^n, where $n = p^s$ and hence $\bar{\mathfrak{L}}$ is isomorphic to PSL$(2, 3^n)$, whence $|\bar{\mathfrak{L}}| = \frac{1}{2}\, 3^{p^s}(3^{p^{2s}} - 1)$. But then if $p \geqslant 5$, $|\bar{\mathfrak{L}}|$ is prime to p, and we conclude at once that $\mathfrak{P}$ is an S_p-subgroup of $\mathfrak{H}$. The lemma is proved.

In order to be able to apply the main results of **(5)** later in the paper (including the modifications made in the preceding section), we need some results concerning p-stability, p-restriction, and p-reduction for the class of A_0- and A_1-groups. If $\mathfrak{H}$ is an A_1-group of odd characteristic, then $\mathfrak{H}$ is a D-group, and hence the conditions under which $\mathfrak{H}$ is p-stable, p-restricted, and p-reductive are given in **(6**, Propositions 6, 7, and 8**)**. We shall now derive analogous results when $\mathfrak{H}$ is an A_0-group or an A_1-group of characteristic 2^n, $n \geqslant 3$. We note that the definitions of p-restriction and p-reduction are made only for groups $\mathfrak{H}$ in which $\mathbf{O}_{p'}(\mathfrak{H}) = 1$. The precise definitions of each of these concepts is given in **(5**, Section 2**)** and is repeated in **(6**, Section 4**)**.

Furthermore, the related concept of p-restriction with respect to a subgroup $\mathfrak{Z}$ of $\mathbf{O}_p(\mathfrak{H})$ is defined in **(6**, Section 4**)**.

PROPOSITION 1. *Let $\mathfrak{H}$ be an A_0-group or an A_1-group of characteristic 2^n, $n \geqslant 3$, and let p be an odd prime in $\pi_s(\mathfrak{H})$. Then*

(i) *$\mathfrak{H}$ is p-stable.*

(ii) *If $\mathbf{O}_{p'}(\mathfrak{H}) = 1$, then either $\mathfrak{H}$ is p-restricted and p-reductive or $p = 3$ and $\mathfrak{H}/\mathbf{O}(\mathfrak{H})$ is isomorphic to* $P\Gamma L(2, 8)$.

Proof. The proofs of these results follow those of **(6**, Propositions 6, 7, and 8**)** very closely. Hence we shall limit ourselves to giving an outline of the arguments. We first discuss p-stability. In view of the proof of **(6**, Proposition 6**)**, it will suffice to establish the following assertion: Let $\mathfrak{H}$ be an A_0-group or an A_1-group of characteristic 2^n, $n \geqslant 3$, in which $\mathbf{O}_p(\mathfrak{H}) = 1$, and assume that $\mathfrak{H}$ is a linear group of transformations of a vector space $\mathfrak{B}$ over $GF(p)$, p odd; then $\mathfrak{H}$ is a p-stable linear group—that is, every non-identity p-element of $\mathfrak{H}$ acts on $\mathfrak{B}$ with non-quadratic minimal polynomial.

Since the S_2-subgroups of $\mathfrak{H}$ are abelian, the desired conclusion follows directly from **(9**, Theorem B**)** if $\mathfrak{H}$ is an A_0-group. Hence we may assume that $\mathfrak{H}$ is an A_1-group. In this case the argument parallels that of **(6**, Lemma 4.2**)**. If $\mathfrak{K} = \mathbf{C}_{\mathfrak{H}}(\mathbf{O}_{p'}(\mathfrak{H})) \subseteq \mathbf{O}(\mathfrak{H})$, then the assertion follows again from **(9**, Theorem B**)**. On the other hand, if $\mathfrak{K} \not\subseteq \mathbf{O}(\mathfrak{H})$, we find that $\mathfrak{K}$ is an A_1-group of the same characteristic as $\mathfrak{H}$ and that $\mathbf{O}(\mathfrak{H}) \subseteq \mathbf{Z}(\mathfrak{K})$. Now application of Lemma 2.5(vi) yields the existence of a normal subgroup $\mathfrak{K}_1$ of $\mathfrak{K}$ which is isomorphic to a subgroup of $P\Gamma L(2, 2^n)$ containing $PSL(2, 2^n)$ and which contains an S_p-subgroup of $\mathfrak{K}$. Hence it suffices to prove the desired assertion for $\mathfrak{K}_1$. But now if X is any p-element of $\mathfrak{K}_1$, it follows from Lemma 2.2(vi) that there exists a conjugate X' of X in $\mathfrak{K}_1$ such that $\mathfrak{H}_0 = \langle X, X' \rangle$ is not a p-group. Since the S_2-subgroups of $\mathfrak{H}_0$ are abelian, the last paragraph of the argument of **(6**, Lemma 4.2**)** applies without change to yield that X has a non-quadratic minimal polynomial on $\mathfrak{B}$. Thus $\mathfrak{H}$ is p-stable in all cases.

We next treat p-restriction. Let $\mathfrak{Z}$ be a non-identity subgroup of $\mathbf{Z}(\mathbf{O}_p(\mathfrak{H}))$ such that $\mathfrak{Z} \lhd \mathfrak{H}$ and $\mathbf{O}_p(\mathfrak{H}/\mathbf{C}_{\mathfrak{H}}(\mathfrak{Z})) = 1$. Then by definition, $\mathfrak{H}$ will be p-restricted, provided $\mathfrak{H}$ is p-restricted with respect to $\mathfrak{Z}$ for each such subgroup $\mathfrak{Z}$. As at the beginning of the proof of **(6**, Proposition 7**)** this will be the case if $\bar{\mathfrak{H}} = \mathfrak{H}/\mathbf{C}_{\mathfrak{H}}(\mathfrak{Z})$ is a p-restricted linear group acting on $\mathfrak{B} = \Omega_1(\mathfrak{Z})$—that is, for every S_p-subgroup or abelian p-subgroup $\bar{\mathfrak{P}} \neq 1$ of $\bar{\mathfrak{H}}$, $\bar{\mathfrak{P}}$ contains a normal

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subgroup $\bar{\mathfrak{P}}_1$ such that $\bar{\mathfrak{P}}/\bar{\mathfrak{P}}_1$ is cyclic and such that a generator of $\bar{\mathfrak{P}}/\bar{\mathfrak{P}}_1$ has minimal polynomial of degree greater than 2 on $\mathbf{C}_{\mathfrak{B}}(\bar{\mathfrak{P}}_1)$. We note that $\bar{\mathfrak{H}}$ is an A_0- or an A_1-group according as $\mathfrak{H}$ is an A_0- or an A_1-group, and in the latter case that $\bar{\mathfrak{H}}$ is of the same characteristic as $\mathfrak{H}$ and that $\bar{\mathfrak{H}}/\mathbf{O}(\bar{\mathfrak{H}})$ is isomorphic to $P\Gamma L(2, 8)$ if and only if the same is true of $\mathfrak{H}/\mathbf{O}(\mathfrak{H})$. Furthermore, we have $\mathbf{O}_p(\bar{\mathfrak{H}}) = 1$. In carrying out the proof, we drop the superscripts for simplicity of notation.

Let then $\mathfrak{P} \neq 1$ be either an S_p-subgroup of $\mathfrak{H}$ or an abelian p-subgroup of $\mathfrak{H}$. Consider first the case that $\mathbf{O}_{p'}(\mathfrak{H}) \neq 1$ and that $\mathfrak{P} \not\subseteq \mathbf{C}_{\mathfrak{H}}(\mathbf{O}_{p'}(\mathfrak{H}))$. Then arguing as in the corresponding case of **(6**, Proposition 7**)**, we find in all cases that $\mathfrak{H}$ is a p-restricted linear group. We note that since the S_2-subgroups of $\mathfrak{H}$ are abelian **(6**, Lemma 4.4**)** (which is used in the argument) applies without exception. As in **(6)**, we are thus reduced to the case that $\mathfrak{P} \subseteq \mathfrak{L} = \mathbf{C}_{\mathfrak{H}}(\mathbf{O}_{p'}(\mathfrak{H}))$ $\not\subseteq \mathbf{O}(\mathfrak{H})$.

We now apply Lemma 2.5(vi) as in the case of p-stability; we conclude that $\mathfrak{P} \subseteq \mathfrak{L}_1$, where $\mathfrak{L}_1 \lhd \mathfrak{L}$ and $\mathfrak{L}_1$ is isomorphic to a subgroup of $P\Gamma L(2, 2^n)$ containing $PSL(2, 2^n)$. Since $\mathfrak{L}_1$ is a p-stable linear group acting on $\mathfrak{B}$, the desired conclusion follows with $\mathfrak{P}_1 = 1$ if $\mathfrak{P}$ is cyclic; hence we may also assume that $\mathfrak{P}$ is non-cyclic. Now by Lemma 2.2(i), $\mathfrak{L}_1 = \mathfrak{L}_0\mathfrak{F}$, where $\mathfrak{L}_0$ is isomorphic to $PSL(2, 2^n)$, $\mathfrak{L}_0 \lhd \mathfrak{L}_1$, $\mathfrak{F}$ is cyclic, and $\mathfrak{L}_0 \cap \mathfrak{F} = 1$. We may assume that $\mathfrak{F}$ is chosen so that $\mathfrak{P} = (\mathfrak{P} \cap \mathfrak{L}_0)(\mathfrak{P} \cap \mathfrak{L})$. Since the S_p-subgroups of $\mathfrak{L}_0$ are cyclic, we have $\mathfrak{P} \cap \mathfrak{L}_0 \neq 1$ and $\mathfrak{P} \cap \mathfrak{F} \neq 1$. Furthermore, $\mathfrak{P} \cap \mathfrak{F}$ normalizes $\mathfrak{R} = \mathbf{C}_{\mathfrak{L}_0}(\mathfrak{P} \cap \mathfrak{L}_0)$. Now $\mathfrak{R}$ is cyclic of order $2^n \pm 1$ by Lemma 2.1(v). Suppose that either $p \neq 3$ or $2^n \neq 8$. Then $\mathfrak{P} \cap \mathfrak{F}$ does not centralize $\mathbf{O}_{p'}(\mathfrak{R})$ by Lemma 2.2(vii). But then the final paragraph of the proof of **(6**, Proposition 7**)** applies without change to show that $\mathfrak{H}$ is a p-restricted linear group in any of these cases. We conclude that if $\mathfrak{H}$ is not a p-restricted linear group, then $p = 3$, $2^n = 8$, and $\mathfrak{P}$ is non-cyclic, in which case $\mathfrak{H}/\mathbf{O}(\mathfrak{H})$ is isomorphic to $P\Gamma L(2, 8)$. Thus (ii) holds with regard to p-restriction.

Finally we consider the concept of p-reduction. Let $\mathfrak{P}$ be an S_p-subgroup of $\mathfrak{H}$, let $\mathfrak{B}$ be a subgroup of $\mathfrak{P}$ such that $\mathfrak{B}$ is generated by elementary subgroups $\mathfrak{B}_i$, $i = 1, 2, \ldots, s$, with the additional property that $\mathfrak{B} = \mathbf{V}(\mathrm{ccl}_{\mathfrak{H}}(\mathfrak{B}); \mathfrak{P})$. Let $\mathfrak{P}_0$ be a normal subgroup of $\mathfrak{P}$ contained in $\mathbf{O}_p(\mathfrak{H})$ and set $\mathfrak{L} = \mathbf{C}_{\mathfrak{H}}(\mathfrak{P}_0)$. Suppose that $\mathfrak{H} \neq \mathfrak{L}\mathbf{N}_{\mathfrak{H}}(\mathfrak{B})$. For $\mathfrak{H}$ to be p-reductive, we must show under these circumstances that there exists a subgroup $\mathfrak{K}$ of $\mathfrak{H}$ satisfying certain prescribed conditions which are stated just before **(6**, Proposition 8**)**. Set $\mathfrak{P}^* = \mathfrak{P} \cap \mathbf{S}(\mathfrak{H})$, $\mathfrak{H}_1 = \mathbf{N}_{\mathfrak{H}}(\mathfrak{P}^*)$, and $\mathfrak{H}_0 = \mathbf{S}(\mathfrak{H})\mathfrak{P}$. Arguing as at the beginning of the proof of **(6**, Proposition 8**)**, it follows that $\mathfrak{H}_i \neq (\mathfrak{L} \cap \mathfrak{H}_i)\mathbf{N}_{\mathfrak{H}_i}(\mathfrak{B})$ for at least one value of $i = 0$ or 1 and that $\mathbf{O}_{p'}(\mathfrak{H}_i) = 1$ for both $i = 0, 1$. Hence it suffices to prove the existence of the required subgroup $\mathfrak{K}$ under the additional assumption that $\mathfrak{H} = \mathfrak{H}_i$, $i = 0$ or 1.

If $\mathfrak{H} = \mathfrak{H}_0$, then $\mathfrak{H}$ is an A_0-group. Since we have already shown that every A_0-group is p-stable, the argument of the corresponding case of **(6**, Proposition 8**)** applies without change to yield the existence of $\mathfrak{K}$. Hence we may assume that $\mathfrak{H} = \mathfrak{H}_1$ and that $\mathfrak{H}$ is not solvable. Then $\mathfrak{P}^*$ is an S_p-subgroup of $\mathbf{O}(\mathfrak{H}) = \mathbf{S}(\mathfrak{H})$. Setting $\bar{\mathfrak{H}} = \mathfrak{H}/\mathbf{O}(\mathfrak{H})$, we have $\bar{\mathfrak{H}} = \bar{\mathfrak{M}}\bar{\mathfrak{F}}$, where $\bar{\mathfrak{M}}$ is isomorphic to $\mathrm{PSL}(2, 2^n)$, $\bar{\mathfrak{F}}$ is cyclic of odd order, and $\bar{\mathfrak{M}} \cap \bar{\mathfrak{F}} = 1$. Furthermore, we can assume without loss that $\bar{\mathfrak{P}} = (\bar{\mathfrak{P}} \cap \bar{\mathfrak{M}})(\bar{\mathfrak{P}} \cap \bar{\mathfrak{F}})$, where $\bar{\mathfrak{P}}$ is the image of $\mathfrak{P}$ in $\bar{\mathfrak{H}}$. If $\bar{\mathfrak{L}}$ denotes the image of $\mathfrak{L}$ in $\bar{\mathfrak{H}}$, then as in the proof of (**6**, Proposition 8), we may assume that $\bar{\mathfrak{M}} \not\subseteq \bar{\mathfrak{L}}$. Furthermore, since $\mathfrak{H}$ is p-stable, we may also assume, as in the same proof, that $|\mathfrak{B}_i/\mathfrak{B}_i \cap \mathfrak{P}^*| \geqslant p^2$ for some $i = 1, 2, \ldots, s$.

Finally let $\bar{\mathfrak{B}}_i$ be the image of $\mathfrak{B}_i$ in $\bar{\mathfrak{P}}$ and set $\bar{\mathfrak{B}}_0 = \bar{\mathfrak{B}}_i \cap \bar{\mathfrak{F}}$. Then by Lemma 2.1(v), we have $|\bar{\mathfrak{B}}_i| = p^2$, $|\bar{\mathfrak{B}}_0| = p$, and $\bar{\mathfrak{P}} \cap \bar{\mathfrak{M}} \neq 1$. Now $\bar{\mathfrak{B}}_0$ normalizes a subgroup $\bar{\mathfrak{R}} = \mathbf{C}_{\bar{\mathfrak{M}}}(\bar{\mathfrak{P}} \cap \bar{\mathfrak{M}})$, and by Lemma 2.1(v), $\bar{\mathfrak{R}}$ is cyclic of order $2^n + \delta$, where $\delta = \pm 1$. Furthermore, $\bar{\mathfrak{M}}_0 = \mathbf{C}_{\bar{\mathfrak{M}}}(\bar{\mathfrak{B}}_0)$ is isomorphic to $\mathrm{PSL}(2, 2^m)$, where $mp = n$. Since p divides $2^n + \delta$, it follows that p divides $(2^n + \delta)/(2^m + \delta)$. Since $|\bar{\mathfrak{R}}| = 2^n + \delta$ and $|\bar{\mathfrak{R}} \cap \bar{\mathfrak{M}}_0| = 2^m + \delta$ and since $\bar{\mathfrak{P}}$ is an S_p-subgroup of $\bar{\mathfrak{R}}$, this implies that $\bar{\mathfrak{P}} \cap \bar{\mathfrak{M}} \not\subseteq \bar{\mathfrak{M}}_0$. Hence if $m > 1$, or equivalently if $\bar{\mathfrak{M}}_0$ is non-solvable, then it follows from Lemma 2.2(viii) that $\bar{\mathfrak{M}} = \langle \bar{\mathfrak{M}}_0, \bar{\mathfrak{P}} \cap \bar{\mathfrak{M}} \rangle$. In this case the final two paragraphs of the proof of (**6**, Proposition 8) apply without change to yield the existence of the required subgroup $\mathfrak{K}$.

There remains then the case $m = 1$ and $\bar{\mathfrak{M}}_0$ is isomorphic to $\mathrm{PSL}(2, 2)$. Now $\bar{\mathfrak{B}}_0$ centralizes $\Omega_1(\bar{\mathfrak{P}} \cap \bar{\mathfrak{M}})$, and consequently $\bar{\mathfrak{P}} \cap \bar{\mathfrak{M}}_0 \neq 1$. But $|\bar{\mathfrak{M}}_0| = 6$, whence $p = 3$, $2^n = 8$, and $\mathfrak{H}/\mathbf{O}(\mathfrak{H})$ is isomorphic to $\mathrm{P\Gamma L}(2, 8)$. Thus (ii) holds and the proposition is proved.

Finally we prove the following elementary lemma.

LEMMA 2.9. *Let P be a p-group, p odd, acted on by a four-group $\mathfrak{T}$, and let $\mathfrak{N}$ be a $\mathfrak{T}$-invariant normal subgroup of $\mathfrak{P}$ such that $\mathfrak{P} = \mathfrak{N}\mathbf{C}_{\mathfrak{P}}(\mathfrak{T})$. If $\mathfrak{T}$ centralizes $\mathbf{Z}(\mathfrak{P})$, then $\mathfrak{T}$ centralizes $\mathbf{Z}(\mathfrak{N})$.*

Proof. Let T_1, T_2, T_3 be the involutions of $\mathfrak{T}$ and let

$$\mathbf{Z}(\mathfrak{N}) = \mathfrak{Z}_0 \times \mathfrak{Z}_1' \times \mathfrak{Z}_2' \times \mathfrak{Z}_3'$$

be the $\mathfrak{T}$-decomposition of $\mathbf{Z}(\mathfrak{N})$. Since $\mathbf{Z}(\mathfrak{N}) \lhd \mathfrak{P}\mathfrak{T}$, $\mathfrak{P}_0 = \mathbf{C}_{\mathfrak{P}}(\mathfrak{T})$ normalizes each $\mathfrak{Z}_i'$. Since $\mathfrak{P} = \mathfrak{N}\mathfrak{P}_0$, it follows that $\mathbf{C}_{\mathfrak{Z}_i'}(\mathfrak{P}_0) \subseteq \mathbf{Z}(\mathfrak{P})$. But $\mathfrak{T}$ centralizes $\mathbf{Z}(\mathfrak{P})$ by assumption, and consequently $\mathbf{C}_{\mathfrak{Z}_i'}(\mathfrak{P}_0) = 1$. This forces $\mathfrak{Z}_i' = 1$, $i = 1, 2, 3$, whence $\mathbf{Z}(\mathfrak{N}) = \mathfrak{Z}_0$ and $\mathfrak{T}$ centralizes $\mathbf{Z}(\mathfrak{N})$.

3. Summary of known results and first reductions of the theorem.

The following five known theorems cover particular cases of Theorem 1:

THEOREM A (Gorenstein and Walter, **6**). *If $\mathfrak{G}$ is a simple group with an abelian S_2-subgroup of type* $(2, 2)$, *then $\mathfrak{G}$ is isomorphic to* $\mathrm{PSL}(2, q)$, $q \geqslant 5$ *and* $q \equiv 3, 5 \pmod 8$.

THEOREM B (Brauer, **1**). *If $\mathfrak{G}$ is a simple group with an abelian S_2-subgroup of type* $(2^n, 2^n)$, *then* $n = 1$.

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THEOREM C (Brauer, to appear). *If* $\mathfrak{G}$ *is a simple group with an abelian* S_2*-subgroup* $\mathfrak{S}$ *of type* (2, 2, 2), *and if* $\mathbf{C}(T)$ *is solvable for every involution* T *in* $\mathfrak{G}$, *then* $|\mathbf{N}(\mathfrak{S})/\mathbf{C}(\mathfrak{S})| = 7$.

THEOREM D (Suzuki, **11**). *If* $\mathfrak{G}$ *is a simple group with an abelian* S_2*-subgroup* $\mathfrak{S}$ *and if* $\mathfrak{S}$ *is disjoint from its conjugates, then* $\mathfrak{G}$ *is isomorphic to* $\mathrm{PSL}(2, 2^n)$, $n \geqslant 2$.

THEOREM E (Thompson, **12**). *If* $\mathfrak{G}$ *is a simple group with abelian* S_2*-subgroups and if the normalizer of every non-identity solvable subgroup of* $\mathfrak{G}$ *is solvable, then* $\mathfrak{G}$ *is isomorphic to* $\mathrm{PSL}(2, q)$, $q \geqslant 4$, *where either* $q = 2^n$ *or* $q \equiv 3, 5 \pmod 8$.

With the aid of these results, we can obtain the following reduction in the proof of Theorem 1:

PROPOSITION 2. *Let* $\mathfrak{G}$ *be a group of least order satisfying the hypotheses, but not the conclusion of Theorem* 1. *Then we have:*

(i) $\mathfrak{G}$ *is simple.*

(ii) *If* $\mathfrak{S}$ *is an* S_2*-subgroup of* $\mathfrak{G}$, *then* $m(\mathfrak{S}) \geqslant 3$. *Furthermore, if* $\mathfrak{S}$ *is elementary of order* 8, *then* $|\mathbf{N}(\mathfrak{S})/\mathbf{C}(\mathfrak{S})| = 7$.

(iii) *Every proper subgroup of* $\mathfrak{G}$ *is an* A_i*-group,* $i = 0$ *or* 1.

(iv) *There exists a distinct conjugate* $\mathfrak{S}_1$ *of* $\mathfrak{S}$ *such that* $\mathfrak{S}_1 \cap \mathfrak{S} \neq 1$.

(v) *The normalizer of some non-identity solvable subgroup of* $\mathfrak{G}$ *is non-solvable.*

Proof. Let $\mathfrak{H}$ be a proper subgroup of $\mathfrak{G}$. Then an S_2-subgroup of $\mathfrak{H}$ is abelian, and the minimality of $\mathfrak{G}$ implies that either $\mathfrak{H}$ is solvable or else $\mathfrak{H}/\mathbf{O}(\mathfrak{H})$ is isomorphic to a subgroup of $\mathrm{P\Gamma L}(2, q)$ containing $\mathrm{PSL}(2, q)$, where either $q \equiv 3$ or $5 \pmod 8$, $q \geqslant 5$, or $q = 2^n$, $n \geqslant 2$. Furthermore, $\mathbf{C}_{\mathfrak{H}}(T)$ is solvable for any involution T in $\mathfrak{H}$. It follows therefore from the definition that $\mathfrak{H}$ is either an A_0- or an A_1-group. Thus (iii) holds.

We show next that $\mathfrak{G}$ is simple. It is immediate that our hypotheses carry over to $\mathfrak{G}/\mathbf{O}(\mathfrak{G})$. Hence, if $\mathbf{O}(\mathfrak{G}) \neq 1$, then our minimal choice of $\mathfrak{G}$ implies that the conclusion of Theorem 1 holds for $\mathfrak{G}/\mathbf{O}(\mathfrak{G})$, and hence also holds for $\mathfrak{G}$, a contradiction. Thus $\mathbf{O}(\mathfrak{G}) = 1$. Suppose next that $\mathbf{O}_2(\mathfrak{G}) \neq 1$. Then $\mathfrak{C} = \mathbf{C}(\mathbf{O}_2(\mathfrak{G}))$ is solvable since $\mathfrak{C} \subseteq \mathbf{C}(T)$ for any involution T of $\mathbf{O}_2(\mathfrak{G})$. Furthermore, since an S_2-subgroup of $\mathfrak{G}$ is abelian, $\mathfrak{S} \subseteq \mathfrak{C}$. Thus $\mathfrak{C} \lhd \mathfrak{G}$, $\mathfrak{C}$ is solvable, and $|\mathfrak{G}/\mathfrak{C}|$ is odd. But then $\mathfrak{G}/\mathfrak{C}$ is solvable, and hence $\mathfrak{G}$ is solvable, which is not the case. Therefore $\mathbf{O}_2(\mathfrak{G}) = 1$, and we conclude that $\mathbf{S}(\mathfrak{G}) = 1$.

Suppose now that $\mathfrak{G}$ is not simple and let $\mathfrak{H}$ be a minimal normal subgroup of $\mathfrak{G}$. Since $\mathbf{S}(\mathfrak{H})$ char $\mathfrak{H} \lhd \mathfrak{G}$, we have $\mathbf{S}(\mathfrak{H}) \subseteq \mathbf{S}(\mathfrak{G}) = 1$. Thus by the first paragraph of the proof, $\mathfrak{H}$ is an A_1-group. Minimality of $\mathfrak{H}$ forces $\mathfrak{H}$ to be isomorphic to $\mathrm{PSL}(2, q)$, $q \equiv 3, 5 \pmod 8$, $q \geqslant 5$, or $q = 2^n$, $n \geqslant 2$. Our hypothesis implies that $|\mathbf{C}(\mathfrak{H})|$ is odd. Since $\mathbf{C}(\mathfrak{H}) \lhd \mathfrak{G}$, it follows that $\mathbf{C}(\mathfrak{H}) \subseteq \mathbf{S}(\mathfrak{G}) = 1$. But now we can apply Lemma 2.2(v), to conclude that $\mathfrak{G}$ is an A_1-group. This contradiction shows that $\mathfrak{G}$ must be simple. Thus (i) holds.

Next suppose $m(\mathfrak{S}) \leqslant 2$. First of all, $\mathfrak{S}$ must be abelian of type $(2^a, 2^a)$; for if this were not the case, then $\mathfrak{S} \subseteq \mathbf{Z}(\mathbf{N}(\mathfrak{S}))$, and Burnside's Transfer Theorem would imply that $\mathfrak{G}$ possesses a normal 2-complement, contrary to the simplicity of $\mathfrak{G}$. But then $a = 1$ by Theorem B and so $\mathfrak{G}$ satisfies the conclusion of Theorem 1 by Theorem A, a contradiction. Thus $m(\mathfrak{S}) \geqslant 3$. Furthermore, if $\mathfrak{S}$ is elementary and $m(\mathfrak{S}) = 3$, then $|\mathbf{N}(\mathfrak{S})/\mathbf{C}(\mathfrak{S})| = 7$ by Theorem C. Finally, conditions (iv) and (v) follow from Theorems D and E respectively.

Thus a minimal counter-example to Theorem 1 satisfies the hypotheses of Theorem 3, and hence Theorem 1 will follow once Theorem 3 is established. Thus the balance of the paper is devoted to the proof of Theorem 3.

We now introduce a distinguished subset σ of $\pi(\mathfrak{G})$, which will play a fundamental role throughout the paper.

Definition. Let $\mathfrak{S}$ be an S_2-subgroup of $\mathfrak{G}$. Denote by σ the set of those odd primes p in $\pi(\mathfrak{G})$ such that $\mathfrak{S}$ does not centralize some maximal element of $\mathsf{И}(\mathfrak{S}; p)$.

Clearly σ is determined independently of the choice of $\mathfrak{S}$. Furthermore, for any odd prime p not in σ, $\mathfrak{S}$ centralizes every element of $\mathsf{И}(\mathfrak{S}; p)$.

We now prove

PROPOSITION 3. σ is non-empty.

Proof. Suppose that σ is empty. We shall show that any two distinct S_2-subgroups $\mathfrak{S}$ and $\mathfrak{S}_1$ of $\mathfrak{G}$ have a trivial intersection, contrary to Proposition 2(iv). Assume then, by way of contradiction, that $\mathfrak{S} \cap \mathfrak{S}_1 \neq 1$. Then $\mathfrak{C} = \mathbf{C}(\mathfrak{S} \cap \mathfrak{S}_1)$ contains both $\mathfrak{S}$ and $\mathfrak{S}_1$ and $\mathfrak{C}$ is solvable. By Lemma 2.4(i), $\mathfrak{C} = \mathbf{O}(\mathfrak{C})\mathbf{N}_{\mathfrak{C}}(\mathfrak{S})$. But since σ is empty, $\mathfrak{S}$ centralizes every subgroup of odd order which it normalizes, and hence $\mathbf{O}(\mathfrak{C}) \subseteq \mathbf{C}(\mathfrak{S})$. Thus $\mathfrak{C} = \mathbf{N}_{\mathfrak{C}}(\mathfrak{S})$ and $\mathfrak{S} \lhd \mathfrak{C}$, whence $\mathfrak{S} = \mathfrak{S}_1$, a contradiction. Therefore $\mathfrak{S} \cap \mathfrak{S}_1 = 1$, as asserted.

Remark. Proposition 2, parts (iv), (v), and the second assertion of (ii) (and hence Theorems D, E, and C respectively) are each used only once in the proof of Theorem 3 — the first in the preceding proposition, the second in Section 5 at the end of Proposition 5, and the third in Section 7 in Proposition 8. Furthermore, Theorems A and B are not used again in the paper.

4. A transitivity theorem and some consequences. The following proposition is basic for all our work:

PROPOSITION 4. *Let $\mathfrak{T}$ be a 2-subgroup of $\mathfrak{G}$.*

(i) *If $m(\mathfrak{T}) = 2$ and $\mathbf{N}(\mathfrak{T}) \supset \mathbf{C}(\mathfrak{T})$, then $\mathbf{N}(\mathfrak{T})$ acts transitively on the maximal elements of $\mathsf{И}(\mathfrak{T}; p)$ for all odd p.*

(ii) *If $m(\mathfrak{T}) \geqslant 3$, then $\mathbf{C}(\mathfrak{T})$ acts transitively on the maximal elements of $\mathsf{И}(\mathfrak{T}; p)$ for all odd p.*

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Proof. The proof of (i) is essentially identical with that of **(6**, Lemma 6.1**)**. Let T_1, T_2, T_3 be the involutions of $\mathfrak{T}$, let $\mathfrak{P}$ be a maximal element of $\mathsf{M}_{\mathfrak{N}}(\mathfrak{T}; p)$, where $\mathfrak{N} = \mathbf{C}(T_1)$, and let $\mathfrak{P}_1$ be a maximal element of $\mathsf{M}(\mathfrak{T}; p)$ containing $\mathfrak{P}$. Suppose that (i) is false, and choose $\mathfrak{P}_2$ to be a maximal element of $\mathsf{M}(\mathfrak{T}; p)$ which is not conjugate to $\mathfrak{P}_1$ under the action of $\mathbf{N}(\mathfrak{T})$ in such a way that $\mathfrak{D} = \mathfrak{P}_1 \cap \mathfrak{P}_2$ has maximal order. Then clearly $\mathfrak{P}_i \neq 1$, $i = 1, 2$. Since $\mathbf{N}(\mathfrak{T}) \supset \mathbf{C}(\mathfrak{T})$, $|\mathbf{N}(\mathfrak{T})/\mathbf{C}(\mathfrak{T})| = 3$ and $\mathbf{N}(\mathfrak{T})$ contains a 3-element R which cyclically permutes the involutions T_1, T_2, T_3. Since $\mathbf{C}_{\mathfrak{P}_2}(T_i) \neq 1$ for some $i = 1, 2, 3$, $\mathfrak{P}_0 = \mathfrak{P}_2^{R^j} \cap \mathfrak{N} \neq 1$ for some $j = 0, 1, 2$. Now $\mathfrak{N}$ is solvable and $\mathfrak{P}_0 \in \mathsf{M}_{\mathfrak{N}}(\mathfrak{T}; p)$. Hence $\mathfrak{P}_0^E \subseteq \mathfrak{P} \subseteq \mathfrak{P}_1$, where $E \in \mathbf{C}_{\mathfrak{N}}(\mathfrak{T})$ by Lemma 2.4(ii). Setting $Y = R^jE$, we conclude that $\mathfrak{P}_2^Y \cap \mathfrak{P}_1 \neq 1$ and that $Y \in \mathbf{N}(\mathfrak{T})$. But then $\mathfrak{P}_2^Y$ is not conjugate to $\mathfrak{P}_1$ by an element of $\mathbf{N}(\mathfrak{T})$, and it follows from our maximal choice of $\mathfrak{D}$ that $\mathfrak{D} \neq 1$.

Now set $\mathfrak{H} = \mathbf{N}(\mathfrak{D})$ and $\mathfrak{P}_1' = \mathfrak{P}_i \cap \mathfrak{H}$. Clearly $\mathfrak{D} \subset \mathfrak{P}_i$, $i = 1, 2$, and hence $\mathfrak{D} \subset \mathfrak{P}_i'$, $i = 1, 2$. Let $\mathfrak{P}_3^*$ be a maximal element of $\mathsf{M}_{\mathfrak{H}}(\mathfrak{T}; p)$ containing $\mathfrak{P}_1'$. If $\mathfrak{H}$ is an A_0-group or an A_1-group of characteristic 2^n, then by Lemmas 2.4(ii) and 2.5(ii), $\mathfrak{P}_2'^X \subseteq \mathfrak{P}_3^*$, where $X \in \mathbf{C}_{\mathfrak{H}}(\mathfrak{T}) \subseteq \mathbf{N}_{\mathfrak{H}}(\mathfrak{T})$. On the other hand, if $\mathfrak{H}$ is an A_1-group of odd characteristic, then $\mathfrak{H}$ is a D-group and it follows from **(6**, Lemma 3.6(i)**)** that $\mathfrak{P}_2'^X \subseteq \mathfrak{P}_3^*$ for some X in $\mathbf{N}_{\mathfrak{H}}(\mathfrak{T})$. In either case, this leads to a contradiction as in the proof of **(6**, Lemma 6.1**)**.

To prove (ii), let $\mathfrak{P}_1$ and $\mathfrak{P}_2$ be maximal elements of $\mathsf{M}(\mathfrak{T}; p)$ which are not conjugate by an element of $\mathbf{C}(\mathfrak{T})$ and chosen so that $\mathfrak{D} = \mathfrak{P}_1 \cap \mathfrak{P}_2$ has maximal order. If $\mathfrak{D} \neq 1$, we proceed as above, setting $\mathfrak{H} = \mathbf{N}(\mathfrak{D})$, $\mathfrak{P}_i' = \mathfrak{P}_i \cap \mathfrak{H}$, $i = 1, 2$, and denoting by $\mathfrak{P}_3^*$ a maximal element of $\mathsf{M}_{\mathfrak{H}}(\mathfrak{T}; p)$ containing $\mathfrak{P}_1'$. Since $m(\mathfrak{T}) \geqslant 3$, $\mathfrak{H}$ is either an A_0-group or an A_1-group of characteristic 2^n, whence $\mathfrak{P}_2'^X \subseteq \mathfrak{P}_3^*$ for some X in $\mathbf{C}_{\mathfrak{H}}(\mathfrak{T})$ by Lemmas 2.4(ii) and 2.5(ii), which again leads to a contradiction as in the proof of **(6**, Lemma 6.1**)**. Thus if $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are any two maximal elements of $\mathsf{M}(\mathfrak{T}; p)$ which are not conjugate by an element of $\mathbf{C}(\mathfrak{T})$, then we must have $\mathfrak{P}_1 \cap \mathfrak{P}_2 = 1$.

On the other hand, since $m(\mathfrak{T}) \geqslant 3$, there exists an involution T in $\mathfrak{T}$ such that $\mathbf{C}_{\mathfrak{P}_i}(\mathfrak{T}) \neq 1$, $i = 1$ and 2. Set $\mathfrak{C} = \mathbf{C}(T)$, let $\mathfrak{P}_i^*$ be a maximal element of $\mathsf{M}_{\mathfrak{C}}(\mathfrak{T}; p)$ containing $\mathbf{C}_{\mathfrak{P}_i}(T)$, and let $\mathfrak{Q}_i$ be a maximal element of $\mathsf{M}(\mathfrak{T}; p)$ containing $\mathfrak{P}_i^*$. Since $\mathfrak{C}$ is solvable, $\mathfrak{P}_2^{*E} = \mathfrak{P}_1^*$ for some E in $\mathbf{C}_{\mathfrak{C}}(\mathfrak{T})$ by Lemma 2.4(ii). Since $\mathfrak{Q}_2^E \cap \mathfrak{Q}_1 \supseteq \mathfrak{P}_1^*$ and $E \in \mathbf{C}(\mathfrak{T})$, it follows that $\mathfrak{Q}_1 = \mathfrak{Q}_2^F$ for some F in $\mathbf{C}(\mathfrak{T})$. Since $\mathfrak{Q}_i \cap \mathfrak{P}_i \supseteq \mathbf{C}_{\mathfrak{P}_i}(T) \neq 1$, we have also $\mathfrak{Q}_i = \mathfrak{P}_i^{F_i}$, where $F_i \in \mathbf{C}(\mathfrak{T})$, $i = 1, 2$. Thus $\mathfrak{P}_1 = \mathfrak{P}_2^Z$ where $Z = F_2 F F_1^{-1} \in \mathbf{C}(\mathfrak{T})$, a contradiction. This completes the proof of the proposition.

As a corollary we have

LEMMA 4.1. *Let $\mathfrak{S}$ be an S_2-subgroup of $\mathfrak{G}$ and let $\mathfrak{P}$ be a maximal element of $\mathsf{M}(\mathfrak{S}; p)$. Then:*

(i) $\mathbf{N}(\mathfrak{S}) = [\mathbf{N}(\mathfrak{S}) \cap \mathbf{N}(\mathfrak{P})]\mathbf{C}(\mathfrak{S})$.

(ii) $\mathfrak{S} \subseteq [\mathbf{N}(\mathfrak{P}), \mathbf{N}(\mathfrak{P})]$.

(iii) *If $p \in \sigma$, then $\mathfrak{S}/\mathbf{C}_{\mathfrak{S}}(\mathfrak{P})$ is non-cyclic.*

Proof. Let $X \in \mathbf{N}(\mathfrak{S})$. Then $\mathfrak{P}^X$ is a maximal element of $\mathsf{M}(\mathfrak{S}; p)$. Since $m(\mathfrak{S}) \geqslant 3$, the preceding proposition implies that $\mathfrak{P}^X = \mathfrak{P}^Y$ for some Y in $\mathbf{C}(\mathfrak{S})$. Thus $XY^{-1} = Z \in \mathbf{N}(\mathfrak{S}) \cap \mathbf{N}(\mathfrak{P})$, whence

$$X \in [\mathbf{N}(\mathfrak{S}) \cap \mathbf{N}(\mathfrak{P})]\mathbf{C}(\mathfrak{S}),$$

and (i) follows.

Set $\mathfrak{K} = \mathbf{N}_{\mathbf{N}(\mathfrak{P})}(\mathfrak{S})$. If $\mathfrak{S} \not\subseteq [\mathfrak{K}, \mathfrak{K}]$, then $\mathbf{N}(\mathfrak{P})$ possesses a normal subgroup of index 2 by Grün's Theorem (**7**, p. 215, Theorem 14.4.5). But then $\mathbf{N}(\mathfrak{S})$ possesses a normal subgroup of index 2 by (i), and another application of Grün's Theorem yields that $\mathfrak{G}$ possesses a normal subgroup of index 2, a contradiction. Thus $\mathfrak{S} \subseteq [\mathfrak{K}, \mathfrak{K}] \subseteq [\mathbf{N}(\mathfrak{P}), \mathbf{N}(\mathfrak{P})]$ and (ii) holds.

Finally if $p \in \sigma$, $\mathfrak{S}$ does not centralize some maximal element $\mathfrak{Q}$ of $\mathsf{M}(\mathfrak{S}; p)$. Since $\mathfrak{Q}$ and $\mathfrak{P}$ are conjugate by an element of $\mathbf{C}(\mathfrak{S})$, $\mathfrak{S}$ does not centralize $\mathfrak{P}$. If $\mathbf{N}(\mathfrak{P})$ is an A_0-group, (iii) follows now from Lemma 2.4(iv). On the other hand, if $\mathbf{N}(\mathfrak{P})$ is an A_1-group, $\mathbf{C}_{\mathfrak{S}}(\mathfrak{P}) = 1$ by Lemma 2.5(i), and again (iii) follows.

LEMMA 4.2. *Let $\mathfrak{S}$ be an S_2-subgroup of $\mathfrak{G}$ and let $\mathfrak{P}$ be a maximal element of $\mathsf{M}(\mathfrak{S}; p)$, where $p \in \sigma$. Then $\mathbf{C}_{\mathfrak{P}}(X)$ contains a subgroup of type (p, p, p) for any element X of $\mathfrak{P}$ of order p. In particular, $\mathscr{SCN}_3(\mathfrak{P})$ is non-empty.*

Proof. Let $\mathfrak{C}$ be a characteristic subgroup of $\mathfrak{P}$ chosen in accordance with (**4**, Lemma 8.2), and set $\mathfrak{D} = \Omega_1(\mathfrak{C})$. Then $\mathfrak{D}$ is of exponent p and of class at most 2. Since $\mathfrak{S}$ does not centralize $\mathfrak{P}$, $\mathfrak{S}$ does not centralize $\mathfrak{C}$ and hence does not centralize $\mathfrak{D}$. But by Lemma 4.1(ii) $\mathfrak{S} \subseteq [\mathbf{N}(\mathfrak{P}), \mathbf{N}(\mathfrak{P})]$, and consequently $\mathscr{SCN}_3(\mathfrak{D})$ is non-empty by Lemma 2.7. In particular, $\mathscr{SCN}_3(\mathfrak{P})$ is non-empty.

Now let X be an element of $\mathfrak{P}$ of order p. If $X \in \mathbf{Z}(\mathfrak{D})$, $\mathfrak{D} \subseteq \mathbf{C}_{\mathfrak{P}}(X)$, so $\mathbf{C}_{\mathfrak{P}}(X)$ contains a subgroup of type (p, p, p). If $X \in \mathfrak{D} - \mathbf{Z}(\mathfrak{D})$, and $|\mathbf{Z}(\mathfrak{D})| \geqslant p^2$, $\langle \mathbf{Z}(\mathfrak{D}), X\rangle \subseteq \mathbf{C}_{\mathfrak{P}}(X)$ and again $\mathbf{C}_{\mathfrak{P}}(X)$ contains a subgroup of type (p, p, p). If $X \in \mathfrak{D} - \mathbf{Z}(\mathfrak{D})$, and $|\mathbf{Z}(\mathfrak{D})| = p$, then $\mathfrak{D}$ is extra special, and $\mathbf{C}_{\mathfrak{D}}(X)$ contains a subgroup of type (p, p, p). We may therefore assume that $X \in \mathfrak{P} - \mathfrak{D}$. If $\mathbf{C}_{\mathfrak{D}}(X)$ contains a subgroup of type (p, p), then $\mathbf{C}_{\mathfrak{P}}(X)$ contains a subgroup of type (p, p, p). Finally consider that $|\mathbf{C}_{\mathfrak{D}}(X)| = p$, in which case $|\mathbf{C}_{\mathfrak{D}^*}(X)| = p^2$, where $\mathfrak{D}^* = \langle \mathfrak{D}, X\rangle$. Thus $\mathfrak{D}^*$ is of maximal class. Hence if we set $\mathfrak{D}_0 = \mathfrak{D}$ and $\mathfrak{D}_{i+1} = [\mathfrak{D}_i, \mathfrak{P}]$, we have $|\mathfrak{D}_i : \mathfrak{D}_{i+1}| = p$, $i = 0, 1, \ldots, n-1$, where $|\mathfrak{D}| = p^n$. But then each $\mathfrak{D}_i \lhd \mathfrak{N} = \mathbf{N}(\mathfrak{P})$ and consequently $[\mathfrak{N}/\mathfrak{D}_{i+1}, \mathfrak{N}/\mathfrak{D}_{i+1}]$ centralizes $\mathfrak{D}_i/\mathfrak{D}_{i+1}$. Since $\mathfrak{S} \subseteq [\mathfrak{N}, \mathfrak{N}]$, it follows that $\mathfrak{S}$ stabilizes the chain $\mathfrak{D} = \mathfrak{D}_0 \supset \mathfrak{D}_1 \supset \ldots \supset \mathfrak{D}_n = 1$, whence $\mathfrak{S}$ centralizes $\mathfrak{D}$, a contradiction. The lemma follows.

LEMMA 4.3. *Let $\mathfrak{S}$ be an S_2-subgroup of $\mathfrak{G}$ and let $\mathfrak{P}$ be a maximal element of $\mathsf{M}(\mathfrak{S}; p)$ for some prime p in σ. Then $\mathfrak{S}$ possesses a four-subgroup $\mathfrak{T}$ such that $\mathbf{C}_{\mathfrak{P}}(T)$ contains a subgroup of type (p, p, p) for each involution T in $\mathfrak{T}$.*

Proof. Let $\mathfrak{D}$ be as in the preceding lemma, so that $\mathfrak{D}$ is of exponent p, class at most 2, and $\mathscr{SCN}_3(\mathfrak{D})$ is non-empty. We first show that $\mathfrak{S}_1 = \Omega_1(\mathfrak{S})$ possesses a subgroup $\mathfrak{S}_0$ such that $m(\mathfrak{S}_1/\mathfrak{S}_0) \leqslant 3$ and $\mathbf{C}_{\mathfrak{P}}(\mathfrak{S}_0)$ contains a subgroup of type (p, p, p). Suppose first that $\mathfrak{Z} = \mathbf{Z}(\mathfrak{D})$ has order at least p^3. Let $\mathfrak{S}_0$ be a subgroup of $\mathfrak{S}_1$ of maximal order such that $\mathfrak{Z}_0 = \mathbf{C}_{\mathfrak{Z}}(\mathfrak{S}_0)$ has order at least p^3, and let $\mathfrak{T}_0$ be a complement to $\mathfrak{S}_0$ in $\mathfrak{S}_1$. Since $\mathfrak{S}$ is abelian, $\mathfrak{T}_0$ normalizes $\mathfrak{Z}_0$. If $m(\mathfrak{T}_0) \geqslant 4$, then $|\mathbf{C}_{\mathfrak{Z}_0}(T)| \geqslant p^3$ for some T in $\mathfrak{T}_0^{\#}$ and $\langle\mathfrak{S}_0, \mathfrak{T}\rangle$ centralizes a subgroup of $\mathfrak{Z}$ of order at least p^3, contrary to our maximal choice of $\mathfrak{S}_0$. Thus $m(\mathfrak{T}_0) = m(\mathfrak{S}_1/\mathfrak{S}_0) \leqslant 3$ and the desired assertion holds in this case.

Hence we may suppose that $|\mathfrak{Z}| \leqslant p^2$. Since $\mathfrak{S} \subseteq [\mathbf{N}(\mathfrak{P}), \mathbf{N}(\mathfrak{P})]$ by Lemma 4.1(ii), it follows from Lemma 2.7 that $\mathfrak{S}$ centralizes $\mathfrak{Z}$. Now set $\bar{\mathfrak{D}} = \mathfrak{D}/\mathfrak{Z}$. Since $\mathfrak{S}$ does not centralize $\mathfrak{D}$, $\mathfrak{S}$ does not centralize $\bar{\mathfrak{D}}$; hence applying Lemma 2.7 once again, we conclude that $|\bar{\mathfrak{D}}| \geqslant p^3$. Arguing now as in the preceding paragraph, it follows that if $\mathfrak{S}_0$ is a maximal subgroup of $\mathfrak{S}_1$ such that $\mathbf{C}_{\bar{\mathfrak{D}}}(\mathfrak{S}_0) = \bar{\mathfrak{D}}_0$ has order at least p^3, then $m(\mathfrak{S}_1/\mathfrak{S}_0) \leqslant 3$. Setting $\mathfrak{D}_0$ equal to the inverse image of $\bar{\mathfrak{D}}_0$ in $\mathfrak{D}$, we see that $\mathfrak{S}_0$ stabilizes the chain $1 \subset \mathfrak{Z} \subset \mathfrak{D}_0$, and hence $\mathfrak{S}_0$ centralizes $\mathfrak{D}_0$. Since $\mathfrak{D}_0$ contains a subgroup of type (p, p, p), the desired assertion follows in this case as well.

If $m(\mathfrak{S}_0) \geqslant 2$, we can choose $\mathfrak{T}$ to be any four-subgroup of $\mathfrak{S}_0$, and $\mathfrak{T}$ will have the required properties. We can therefore assume that $m(\mathfrak{S}_0) \leqslant 1$ and hence that $m(\mathfrak{S}) = m(\mathfrak{S}_1) \leqslant 4$. Suppose first that $m(\mathfrak{S}) = 3$. Since

$$\mathfrak{S} \subseteq [\mathbf{N}(\mathfrak{P}), \mathbf{N}(\mathfrak{P})]$$

and $\mathfrak{S}$ is abelian, $\mathbf{N}(\mathfrak{P})$ contains a 7-element Y which acts transitively on the elements of $\mathfrak{S}_1^{\#}$. Suppose first that $\mathfrak{S}_1$ does not centralize $\mathfrak{Z}$. Then some irreducible constituent $\mathfrak{V}$ of the representation of $\langle\mathfrak{S}_1, Y\rangle$ on $\mathfrak{Z}$ does not have $\mathfrak{S}_1$ in its kernel. But then Clifford's theorem implies that $|\mathfrak{V}| = p^{7k}$, $k \geqslant 1$. Let $\mathfrak{T}$ be any four subgroup of $\mathfrak{S}_1$, let T_1, T_2, T_3 be the involutions of $\mathfrak{T}$, and let $\mathfrak{V} = \mathfrak{V}_1 \mathfrak{V}_2 \mathfrak{V}_3$ be the $\mathfrak{T}$-decomposition of $\mathfrak{V}$. Since $|\mathfrak{V}| \geqslant p^7$, it follows that $|\mathfrak{V}_i| \geqslant p^3$ for some $i = 1, 2, 3$. Thus $\mathbf{C}_{\mathfrak{Z}}(T_i)$ contains a subgroup of type (p, p, p) for some $i = 1, 2, 3$. But as Y acts transitively on the involutions of $\mathfrak{S}_1$, $\mathbf{C}_{\mathfrak{Z}}(T_i)$ contains a subgroup of type (p, p, p) for each $i = 1, 2, 3$. Hence the lemma holds in this case. On the other hand, if $\mathfrak{S}_1$ centralizes $\mathfrak{Z}$, the lemma follows by essentially the same argument applied to $\bar{\mathfrak{D}} = \mathfrak{D}/\mathfrak{Z}$.

We may therefore assume that $m(\mathfrak{S}) = 4$. Consider first the case that $|\mathbf{N}(\mathfrak{S})/\mathbf{C}(\mathfrak{S})|$ is divisible by 5, in which case $\mathbf{N}(\mathfrak{P})$ possesses a 5-element X which normalizes, but does not centralize, $\mathfrak{S}_1$. As above, suppose first that $\mathfrak{S}_1$ does not centralize $\mathfrak{Z}$. Then by Clifford's Theorem an irreducible constituent $\mathfrak{W}$ of the representation of $\langle\mathfrak{S}_1, X\rangle$ on $\mathfrak{Z}$ not containing S_1 in its kernel has order p^{5k}, $k \geqslant 1$. Let $\mathfrak{W}_1$ be a subgroup of $\mathfrak{W}$ on which $\mathfrak{S}_1$ is represented irreducibly, and let $\mathfrak{K}_1$ be the kernel of this representation. Then $|\mathfrak{W}_1| = p$ and $|\mathfrak{K}_1| = 8$. Furthermore, $\mathfrak{W}^* = \mathfrak{W}_1 \mathfrak{W}_1{}^X \mathfrak{W}_1{}^{X^2}$ has order p^3 and

$$\mathfrak{K}^* = \mathfrak{K}_1 \cap \mathfrak{K}_1{}^X \cap \mathfrak{K}_1{}^{X^2}$$

has order 2. Hence if $\mathfrak{K}^* = \langle T_1 \rangle$, T_1 centralizes $\mathfrak{W}^*$. But then $T_2 = T_1{}^X$ centralizes $\mathfrak{W}^{*X}$, and consequently $\mathfrak{T} = \langle T_1, T_2 \rangle$ centralizes $\mathfrak{W}^* \cap \mathfrak{W}^{*X}$, which has order p^2. Thus $|\mathbf{C}_{\mathfrak{W}}(\mathfrak{T})| \geqslant p^2$.

We shall show that $\mathfrak{T}$ satisfies the requirements of the lemma. Set $T_3 = T_1 T_2$. If $|\mathbf{C}_{\mathfrak{W}}(\mathfrak{T})| \geqslant p^3$, then $\mathbf{C}_{\mathfrak{P}}(\mathfrak{T})$ contains a subgroup of type (p, p, p) and hence so does $\mathbf{C}_{\mathfrak{P}}(T_i)$ for all i. Hence consider that $|\mathbf{C}_{\mathfrak{W}}(\mathfrak{T})| = p^2$, and let

$$\mathfrak{W} = \mathfrak{W}_0 \times \mathfrak{W}_1{}' \times \mathfrak{W}_2{}' \times \mathfrak{W}_3{}'$$

be the $\mathfrak{T}$-decomposition of $\mathfrak{W}$. Here $\mathfrak{W}_0 = \mathbf{C}_{\mathfrak{W}}(\mathfrak{T})$ and $W_i{}'$ is the subset of $\mathbf{C}_{\mathfrak{W}}(T_i)$ inverted by T_j, $j \neq i$. Now $|\mathfrak{W}_1{}'| = |\mathfrak{W}_2{}'|$ since $T_2 = T_1{}^X$. But then $|\mathfrak{W}| = p^2 |W_1{}'|^2 |W_3{}'|$. Since $|\mathfrak{W}| = p^{5k}$ and 5^k is odd, it follows that $\mathfrak{W}_3{}' \neq 1$. Hence $|\mathbf{C}_{\mathfrak{W}}(T_3)| = |\mathfrak{W}_0 \ \mathfrak{W}_3{}'| \geqslant p^3$. Since T_1 and T_2 centralize $\mathfrak{W}^*$, and $|\mathfrak{W}^*| = p^3$, we conclude that $|\mathbf{C}_{\mathfrak{W}}(T_i)| \geqslant p^3$ for all $i = 1, 2, 3$. Thus $\mathbf{C}_{\mathfrak{P}}(T_i)$ contains a subgroup of type (p, p, p) for each $i = 1, 2, 3$, as required. Therefore the lemma holds if $\mathfrak{S}_1$ does not centralize $\mathfrak{Z}$. On the other hand, if $\mathfrak{S}_1$ centralizes $\mathfrak{Z}$, we apply the same argument to $\bar{\mathfrak{D}} = \mathfrak{D}/\mathfrak{Z}$, and the lemma follows in this case as well.

It thus remains to treat the case $m(\mathfrak{S}) = 4$ and $\mathbf{N}(\mathfrak{S})/\mathbf{C}(\mathfrak{S})$ is a 3-group. As above, we assume first that $\mathfrak{S}_1$ does not centralize $\mathfrak{Z}$. Let $\mathfrak{R}$ be an S_3-subgroup of $\mathbf{N}_{\mathbf{N}(\mathfrak{P})}(\mathfrak{S})$. Since $\mathbf{N}(\mathfrak{S})/\mathbf{C}(\mathfrak{S})$ is a 3-group, Lemma 4.1 (i) and (ii) implies that $\mathfrak{S} \subseteq [\mathfrak{S}\mathfrak{R}, \mathfrak{S}\mathfrak{R}]$; and hence that $\mathfrak{S}_1 \subseteq [\mathfrak{S}_1\mathfrak{R}, \mathfrak{S}_1\mathfrak{R}]$. If $\mathfrak{U}$ denotes an irreducible constituent of the representation of $\mathfrak{S}_1\mathfrak{R}$ on $\mathfrak{Z}$, not containing $\mathfrak{S}_1$ in its kernel, it follows once again from Clifford's Theorem that $|\mathfrak{U}| = p^{3k}$, $k \geqslant 1$. Choose R in $\mathfrak{R}$ with R not in $\mathbf{C}(S_1)$. Then R normalizes, but does not centralize, some four-subgroup $\mathfrak{T}_1$ of $\mathfrak{S}_1$. If T_1, T_2, T_3 are the involutions of $\mathfrak{T}_1$, this implies that $|\mathfrak{U}_1| = |\mathfrak{U}_2| = |\mathfrak{U}_3|$, where $\mathfrak{U}_i = \mathbf{C}_{\mathfrak{U}}(T_i)$, $i = 1, 2, 3$. But $\mathfrak{U} = \mathfrak{U}_1 \mathfrak{U}_2 \mathfrak{U}_3$. Hence if $k > 1$, it follows that $|\mathfrak{U}_i| \geqslant p^3$, $i = 1, 2, 3$; whence $\mathbf{C}_{\mathfrak{P}}(T_i)$ contains a subgroup of type (p, p, p) for all i, and T_1 can be taken as the required subgroup T.

Consider next the case $k = 1$. Let $\mathfrak{K}$ be the kernel of the representation of $\mathfrak{S}_1 \mathfrak{R}$ on $\mathfrak{U}$. Since $m(\mathfrak{S}_1) = 4$ and $m(\mathfrak{U}) = 3$, $\mathfrak{S}_1$ cannot be faithfully represented on $\mathfrak{U}$, and consequently $\mathfrak{S}_1 \cap \mathfrak{K} \neq 1$. If $|\mathfrak{S}_1 \cap \mathfrak{K}| = 2$, then $\mathfrak{S}_1 \mathfrak{K}/\mathfrak{K}$ is elementary of order 8. Since $\mathfrak{R}$ is a 3-group, this implies that $\mathfrak{S}_1 \mathfrak{R}/\mathfrak{K}$ possesses a normal subgroup of index 2. But then $\mathfrak{S}_1 \mathfrak{R}$ has a normal subgroup of index 2, contrary to $\mathfrak{S}_1 \subseteq [\mathfrak{S}_1\mathfrak{R}, \mathfrak{S}_1\mathfrak{R}]$. Thus $|\mathfrak{S}_1 \cap \mathfrak{K}| \geqslant 4$ and $\mathfrak{S}_1$ possesses a four-subgroup $\mathfrak{T}$ which acts trivially on $\mathfrak{U}$. Since $\mathfrak{U}$ is elementary of type (p, p, p), we conclude that $\mathfrak{T}$ satisfies the required conditions.

Finally if $\mathfrak{S}_1$ centralizes $\mathfrak{Z}$, the lemma follows once again by the same argument applied to $\bar{\mathfrak{D}}$. This completes the proof.

LEMMA 4.4. *Let $\mathfrak{S}$ be an S_2-subgroup of $\mathfrak{G}$ and $\mathfrak{T}$ a four-subgroup of $\mathfrak{S}$. Let $\mathfrak{Q}$ be a maximal element of $\mathsf{M}(\mathfrak{S}; p)$ and $\mathfrak{P}$ a maximal element of $\mathsf{M}(\mathfrak{T}; p)$ containing $\mathfrak{Q}$, where $p \in \sigma$. Then:*

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(i) $\mathfrak{Q}$ *contains a $\mathfrak{T}$-invariant S_p-subgroup of* $\mathbf{O}(\mathbf{C}(T))$ *for each involution T in $\mathfrak{T}$.*

(ii) $\mathfrak{P} = \mathfrak{Q}\mathbf{C}_{\mathfrak{P}}(\mathfrak{T})$.

Proof. Let T_1, T_2, T_3 be the involutions of $\mathfrak{T}$. Let $\mathfrak{R}_i$ be an $\mathfrak{S}$-invariant S_p-subgroup of $\mathbf{O}(\mathbf{C}(T_i))$, and let $\mathfrak{Q}_i$ be a maximal element of $\mathsf{M}(\mathfrak{S}; p)$ containing $\mathfrak{R}_i$, $i = 1, 2, 3$. By Proposition 4(ii), we have $\mathfrak{Q} = \mathfrak{Q}_i^{X_i}$ for some X_i in $\mathbf{C}(\mathfrak{S})$, $i = 1, 2, 3$, and consequently $\mathfrak{R}_i^{X_i} \subseteq \mathfrak{Q}$ for each i. But since $\mathbf{C}(\mathfrak{S}) \subseteq \mathbf{C}(T_i)$, $\mathfrak{R}_i^{X_i}$ is a $\mathfrak{T}$-invariant S_p-subgroup of $\mathbf{O}(\mathbf{C}(T_i))$, $i = 1, 2, 3$, and (i) holds.

Now let $\mathfrak{P} = \mathfrak{P}_0\mathfrak{P}_1'\mathfrak{P}_2'\mathfrak{P}_3'$ and $\mathfrak{Q} = \mathfrak{Q}_0\mathfrak{Q}_1'\mathfrak{Q}_2'\mathfrak{Q}_3'$ be the $\mathfrak{T}$-decompositions of $\mathfrak{P}$ and $\mathfrak{Q}$, respectively. Since $\mathfrak{Q} \subseteq \mathfrak{P}$, $\mathfrak{Q}_i' \subseteq \mathfrak{P}_i'$, $i = 1, 2, 3$. But each $\mathfrak{P}_i' \subseteq \gamma\mathfrak{P}\mathfrak{T}$. Since $\mathbf{C}(T_i)$ is solvable, it follows therefore from Lemma 2.4(iii) that $\mathfrak{P}_i' \subseteq \mathbf{O}(\mathbf{C}(T_i))$ for each i. Hence $\mathfrak{P}_i'^{Y_i} \subseteq \mathfrak{R}_i$ for some Y_i in $\mathbf{C}(\mathfrak{T})$. Setting $Z_i = Y_i X_i$, we have $Z_i \in \mathbf{C}(\mathfrak{T})$ and $P_i'^{Z_i} \subseteq \mathfrak{Q}$, whence $P_i'^{Z_i} \subseteq Q_i'$. Thus $|\mathfrak{Q}_i'| \geqslant |\mathfrak{P}_i'|$, and consequently $\mathfrak{Q}_i' = \mathfrak{P}_i'$, $i = 1, 2, 3$. Hence $\mathfrak{P} = \mathfrak{P}_0\mathfrak{Q} = \mathbf{C}_{\mathfrak{P}}(\mathfrak{T})\mathfrak{Q}$, and (ii) also holds.

The next two lemmas are needed to study the problem of p-constraint for the primes in σ; compare Section 6 below.

LEMMA 4.5. *Let $\mathfrak{T}$ be a 2-subgroup of $\mathfrak{G}$, and let $\mathfrak{P}$ be a maximal element of $\mathsf{M}(\mathfrak{T}; p)$, where $p \in \sigma$. Assume that the following conditions hold:*

(a) $\mathfrak{P} \not\subseteq \mathbf{C}(\mathfrak{T})$.

(b) *For some non-trivial $\mathfrak{T}$-invariant subgroup $\mathfrak{D}$ of $\mathfrak{P}$, $\mathbf{C}(\mathfrak{D})$ is an A_1-group of which $\mathfrak{T}$ is an S_2-subgroup.*

Then if $\mathfrak{D}^*$ is a $\mathfrak{T}$-invariant S_p-subgroup of $\mathbf{O}_{p',p}(\mathbf{C}(\mathfrak{D}))$, $\mathfrak{T}$ does not centralize $\mathfrak{D}^*$.

Proof. Our conditions imply that $\mathfrak{T}$ is elementary of order at least 4 and that $\mathbf{N}(\mathfrak{T}) \supset \mathbf{C}(\mathfrak{T})$. Let $\mathfrak{S}$ be an S_2-subgroup of $\mathfrak{G}$ containing $\mathfrak{T}$. Obviously every maximal element of $\mathsf{M}(\mathfrak{S}; p)$ is contained in some maximal element of $\mathsf{M}(\mathfrak{T}; p)$. Since any two maximal elements of $\mathsf{M}(\mathfrak{T}; p)$ are conjugate by an element of $\mathbf{N}(\mathfrak{T})$ by Proposition 4, $\mathfrak{P}$ contains a maximal element $\mathfrak{Q}$ of $\mathsf{M}(\mathfrak{S}^X; p)$ for some element X in $\mathbf{N}(\mathfrak{T})$. Since $\mathfrak{T} \subseteq \mathfrak{S}^X$, we can assume without loss that $\mathfrak{Q}$ is a maximal element of $\mathsf{M}(\mathfrak{S}; p)$.

By hypothesis, $\mathfrak{P} \not\subseteq \mathbf{C}(\mathfrak{T})$; hence $\mathfrak{P} \not\subseteq \mathbf{C}(\mathfrak{T}^*)$ for some four-subgroup $\mathfrak{T}^*$ of $\mathfrak{T}$. Let T_1, T_2, T_3 be the involutions of $\mathfrak{T}^*$ and let

$$\mathfrak{P} = \mathfrak{P}_1\mathfrak{P}_2\mathfrak{P}_3 = \mathfrak{P}_0\mathfrak{P}_1'\mathfrak{P}_2'\mathfrak{P}_3'$$

be the $\mathfrak{T}^*$-decompositions of $\mathfrak{P}$. Then $\mathfrak{P}_i' \neq 1$ for some $i = 1, 2, 3$, say, $i = 1$. Thus $\mathfrak{P}_0 \subset \mathfrak{P}$ and $\mathbf{N}(\mathfrak{P}_0) \cap \mathfrak{P}_1' \neq 1$. But $\mathbf{N}(\mathfrak{P}_0) \cap \mathfrak{P}_1'$ centralizes $\mathfrak{P}_0$ by **(6**, Lemma 1.1**)**, whence $\mathbf{C}_{\mathfrak{P}_1'}(\mathfrak{P}_0) \neq 1$. But $\mathfrak{D}$ centralizes $\mathfrak{T}^*$ and hence $\mathfrak{D} \subseteq \mathfrak{P}_0$. Thus $\mathbf{C}_{\mathfrak{P}_1'}(\mathfrak{D}) = \mathfrak{D}_1' \neq 1$. Furthermore, since $\mathfrak{T}$ is abelian, $\mathfrak{P}_1'$, and hence also $\mathfrak{D}_1'$, is $\mathfrak{T}$-invariant. Setting $\mathfrak{C} = \mathbf{C}(\mathfrak{D})$, we conclude that $\mathfrak{D}_1' \in \mathsf{M}_{\mathfrak{C}}(\mathfrak{T}; p)$ and that $\mathfrak{D}_1' \not\subseteq \mathbf{C}(\mathfrak{T})$.

Suppose first that $m(\mathfrak{T}) \geqslant 3$, in which case $\mathfrak{C}$ has characteristic $2^{m(\mathfrak{T})}$. But then $\mathfrak{D}_1' \subseteq \gamma\mathfrak{D}_1'\mathfrak{T} \subseteq \mathbf{O}(\mathfrak{C})$ by Lemma 2.5(iii). Now $\mathfrak{D}_1'$ normalizes some

$\mathfrak{T}$-invariant S_p-subgroup $\mathfrak{F}^*$ of $\mathbf{O}_{p',p}(\mathfrak{C})$. If $\mathfrak{F}^* \subseteq \mathbf{C}(\mathfrak{T})$, then $\mathfrak{D}_1'$ would centralize $\mathfrak{F}^*$ by **(6**, Lemma 1.1**)**, whence $\mathfrak{D}_1' \subseteq \mathfrak{F}^*$ by **(9**, Lemma 1.2.3**)**, contrary to the fact that $\mathfrak{D}_1'$ does not centralize $\mathfrak{T}$. Thus $\mathfrak{F}^* \not\subseteq \mathbf{C}(\mathfrak{T})$. But $\mathfrak{F}^*$ and $\mathfrak{D}^*$, being $\mathfrak{T}$-invariant S_p-subgroups of $\mathbf{O}_{p',p}$ $(\mathfrak{C})$, are conjugate by an element of $\mathbf{C}(\mathfrak{T})$. Hence $\mathfrak{D}^* \not\subseteq \mathbf{C}(\mathfrak{T})$, proving the lemma in this case.

Consider then that $m(\mathfrak{T}) = 2$, in which case $\mathfrak{T} = \mathfrak{T}^*$ and $\mathfrak{C}$ is of characteristic $q \geqslant 5$, q odd. It will suffice to prove that $\mathfrak{P}_i' \neq 1$, $i = 1, 2$, and 3, for then it will follow as above that $\mathfrak{D}_i' = \mathbf{C}_{\mathfrak{P}_i'}(\mathfrak{D}) \neq 1$ for all $i = 1, 2, 3$, and **(6**, Lemma 3.6(vi)**)** will imply that $\mathfrak{D}_i' \subseteq \mathbf{O}(\mathfrak{C})$ for at least two values of i. But then the argument of the preceding paragraph will apply without change to give the desired conclusion.

Now $\mathfrak{P}_1' \subseteq \mathbf{O}(\mathbf{C}(T_1))$ by Lemma 2.4(iii), and $\mathfrak{P}_1' \neq 1$ by assumption. This implies that $\mathfrak{T}$ does not centralize any $\mathfrak{T}$-invariant S_p-subgroup of $\mathbf{O}(\mathbf{C}(T_1))$. But $\mathfrak{C}$ contains an element which cyclically permutes the involutions T_1, T_2, T_3, and hence $\mathfrak{T}$ does not centralize any $\mathfrak{T}$-invariant S_p-subgroup of $\mathbf{O}(\mathbf{C}(T_i))$ for each $i = 1, 2, 3$. On the other hand, by Lemma 4.4(i) $\mathfrak{Q}$ contains an S_p-subgroup of $\mathbf{O}(\mathbf{C}(T_i))$ for each $i = 1, 2, 3$. Thus if

$$\mathfrak{Q} = \mathfrak{Q}_1\mathfrak{Q}_2\mathfrak{Q}_3 = \mathfrak{Q}_0\mathfrak{Q}_1'\mathfrak{Q}_2'\mathfrak{Q}_3'$$

denotes the $\mathfrak{T}$-decompositions of $\mathfrak{Q}$, we conclude that $\mathfrak{Q}_i \supset \mathfrak{Q}_0$ and $\mathfrak{Q}_i' \neq 1$ for each $i = 1, 2, 3$. Since $\mathfrak{Q} \subseteq \mathfrak{P}$, $\mathfrak{Q}_i' \subseteq \mathfrak{P}_i'$ and therefore $\mathfrak{P}_i' \neq 1$, $i = 1, 2, 3$, as required.

LEMMA 4.6. *Let $\mathfrak{T}$ be a four-subgroup of $\mathfrak{G}$, let $\mathfrak{S}$ be an S_2-subgroup of $\mathfrak{G}$ containing $\mathfrak{T}$, let $\mathfrak{Q}$ be a maximal element of $\mathsf{V}\!\!\mathsf{I}(\mathfrak{S}; p)$, where $p \in \sigma$, and let $\mathfrak{P}$ be a maximal element of $\mathsf{V}\!\!\mathsf{I}(\mathfrak{T}; p)$ containing $\mathfrak{Q}$. Assume that the following conditions hold:*

(a) $\mathfrak{P} \subseteq \mathbf{C}(\mathfrak{T})$.

(b) *For some non-trivial subgroup $\mathfrak{D}$ of $\mathfrak{P}$, $\mathbf{C}(\mathfrak{D})$ is an A_1-group of which $\mathfrak{T}$ is an S_2-subgroup.*

(c) *$\mathbf{C}(\mathfrak{Q})$ is solvable.*

Then one of the following two statements holds:

(i) *$\mathfrak{Q}$ contains an $\mathfrak{S}$-invariant subgroup $\mathfrak{R}$ with $\mathscr{SCN}_3(\mathfrak{R})$ non-empty such that $\mathbf{C}(\mathfrak{R}^*)$ is an A_1-group of characteristic 3^t, $t \geqslant p$, for some maximal subgroup $\mathfrak{R}^*$ of $\mathfrak{R}$.*

(ii) *$|\mathfrak{D}| = p$, $\mathfrak{N} = \mathbf{N}(\mathfrak{D})$ is an A_1-group of characteristic 3^p, $\mathfrak{D}$ is an S_p-subgroup of $\mathbf{O}(\mathfrak{N})$, and a maximal element of $\mathsf{V}\!\!\mathsf{I}_{\mathfrak{N}}(\mathfrak{T}; p)$ has order p^2.*

Proof. Let $\mathfrak{H}$ be any A_1-subgroup of $\mathfrak{G}$ containing $\mathfrak{T}$ and let $\mathfrak{X}$ be a maximal element of $\mathsf{V}\!\!\mathsf{I}_{\mathfrak{H}}(\mathfrak{T}; \mathfrak{X})$. Since $\mathbf{N}(\mathfrak{T}) \supset \mathbf{C}(\mathfrak{T})$ by Condition (b), we have $\mathfrak{X}^Y \subseteq \mathfrak{P}$ for some Y in $\mathbf{N}(\mathfrak{T})$ by Proposition 4(i), and consequently $\mathfrak{T}$ centralizes $\mathfrak{X}$. But then we can apply Lemma 2.8 to conclude that either $\mathbf{C}_{\mathfrak{H}}(\mathfrak{X})$ is an A_1-group or that $\mathfrak{H}$ has characteristic 3^{p^n}, $n \geqslant 1$, and $\mathfrak{X}$ possesses a maximal subgroup $\mathfrak{X}^*$ containing $\mathfrak{X} \cap \mathbf{O}(\mathfrak{H})$ such that $\mathbf{C}_{\mathfrak{H}}(\mathfrak{X}^*)$ is an A_1-group of characteristic 3^p. This result will be used repeatedly in the proof.

Suppose first that $\mathfrak{G}$ possesses an A_1-subgroup $\mathfrak{K}$ containing $\mathfrak{T}$ and also containing a maximal element of $\mathsf{M}(\mathfrak{T}; p)$. Then by Proposition 4(i), we can assume without loss that $\mathfrak{P} \subseteq \mathfrak{K}$. Now by the preceding assertion, either $\mathbf{C}_{\mathfrak{K}}(\mathfrak{P})$ is an A_1-group or else $\mathfrak{K}$ has characteristic 3^{p^n} and $\mathfrak{P}$ possesses a maximal subgroup $\mathfrak{P}^*$ such that $\mathbf{C}_{\mathfrak{K}}(\mathfrak{P}^*)$ is an A_1-group of characteristic 3^p. However, in the first case, $\mathbf{C}(\mathfrak{Q})$ is an A_1-group since $\mathfrak{Q} \subseteq \mathfrak{P}$, in contradiction to (c). The same contradiction occurs in the second case if $\mathfrak{Q} \subseteq \mathfrak{P}^*$. Hence $\mathfrak{P}^* \cap \mathfrak{Q}$ is a maximal subgroup of $\mathfrak{Q}$. We shall show that (i) holds with $\mathfrak{Q} = \mathfrak{R}$ and $\mathfrak{P}^* \cap \mathfrak{Q} = \mathfrak{R}^*$.

First of all, $\mathscr{SCN}_3(\mathfrak{Q})$ is non-empty by Lemma 4.2. Furthermore, since $\mathbf{C}_{\mathfrak{K}}(\mathfrak{P}^*) \subseteq \mathbf{C}(\mathfrak{R}^*)$ and $\mathbf{C}_{\mathfrak{K}}(\mathfrak{P}^*)$ is of characteristic 3^p, Lemma 2.6(i) implies that $\mathbf{C}(\mathfrak{R}^*)$ is an A_1-group of characteristic 3^t for some $t \geqslant p$. Thus (i) holds.

We may therefore assume that no A_1-subgroup of $\mathfrak{G}$ containing $\mathfrak{T}$ contains a maximal element of $\mathsf{M}(\mathfrak{T}; p)$. Suppose next that $\mathfrak{P}$ possesses a subgroup $\mathfrak{D}_1$ with $|\mathfrak{D}_1| > p$ such that $\mathbf{C}(\mathfrak{D}_1)$ is an A_1-group. Since $\mathfrak{Q} \subseteq \mathfrak{P}$, certainly $\mathfrak{P}$ is non-cyclic. Hence there exists an element $\mathfrak{B}$ in $\mathscr{U}(\mathfrak{P})$. Since $|\mathfrak{D}_1| > p$, $\mathfrak{D}_0 = \mathbf{C}_{\mathfrak{D}_1}(\mathfrak{B}) \neq 1$. Since $\mathbf{C}(\mathfrak{D}_1) \subseteq \mathbf{C}(\mathfrak{D}_0) = \mathfrak{C}$, it follows that $\mathfrak{C}$ is an A_1-subgroup of $\mathfrak{G}$ containing $\mathfrak{T}$ and $\mathfrak{B}$. Let $\mathfrak{X}$ be a maximal element of $\mathsf{M}_{\mathfrak{C}}(\mathfrak{T}; p)$ containing $\mathfrak{B}$. Then $\mathbf{C}_{\mathfrak{C}}(\mathfrak{X}^*)$ is an A_1-group for some subgroup $\mathfrak{X}^*$ of index at most p in $\mathfrak{X}$. But then $\mathfrak{B}^* = \mathfrak{B} \cap \mathfrak{X}^* \neq 1$, and consequently $\mathfrak{K} = \mathbf{C}(\mathfrak{B}^*)$ is an A_1-subgroup of $\mathfrak{G}$ containing $\mathfrak{T}$. Furthermore, $\mathfrak{P} \cap \mathfrak{K}$ has index 1 or p in $\mathfrak{P}$, and $\mathfrak{P} \subseteq \mathfrak{K}$ if $\mathfrak{B}^* \subseteq \mathbf{Z}(\mathfrak{P})$. Hence under our present assumptions $|\mathfrak{P} : \mathfrak{P} \cap \mathfrak{K}| = p$, $\mathfrak{P} \cap \mathfrak{K}$ is a maximal element of $\mathsf{M}_{\mathfrak{K}}(\mathfrak{T}; p)$, and $\mathfrak{B}^* \not\subseteq \mathbf{Z}(\mathfrak{P})$. In particular, $\mathbf{Z}(\mathfrak{P})$ is cyclic. Furthermore, $\mathbf{C}_{\mathfrak{K}}(\mathfrak{P} \cap \mathfrak{K})$ is not an A_1-group; otherwise $\mathbf{N}(\mathfrak{P} \cap \mathfrak{K})$ would be an A_1-group containing $\mathfrak{T}$ and $\mathfrak{P}$, contrary to our present assumption. Thus by the first assertion of the proof, $\mathfrak{K}$ has characteristic 3^{p^n}, $n \geqslant 1$, and $\mathfrak{P} \cap \mathfrak{K}$ contains a maximal subgroup $\mathfrak{P}^*$ such that $\mathbf{C}_{\mathfrak{K}}(\mathfrak{P}^*)$ is an A_1-group of characteristic 3^p.

Let $\mathfrak{P}_0 = [\mathfrak{P} \cap \mathfrak{K}, \mathfrak{P} \cap \mathfrak{K}]$. Then $\mathfrak{P}_0 \subseteq \mathfrak{P}^*$ and hence $\mathbf{C}_{\mathfrak{K}}(\mathfrak{P}_0)$ is an A_1-group. But $\mathfrak{P}_0$ char $\mathfrak{P} \cap \mathfrak{K} \lhd \mathfrak{P}$. Hence if $\mathfrak{P}_0 \neq 1$, $\mathbf{N}(\mathfrak{P}_0)$ is an A_1-subgroup of $\mathfrak{G}$ containing $\mathfrak{T}$ and $\mathfrak{P}$, contrary to assumption. Thus $\mathfrak{P}_0 = 1$ and $\mathfrak{P} \cap \mathfrak{K}$ is abelian. Since $\mathfrak{P}_1 = \mho^1(\mathfrak{P} \cap \mathfrak{K}) \subseteq \mathfrak{P}^*$ and since $\mathfrak{P}_1$ char $\mathfrak{P} \cap \mathfrak{K}$, we reach the same contradiction if $\mathfrak{P}_1 \neq 1$. Hence $\mathfrak{P}_0 = 1$ and it follows that $\mathfrak{P} \cap \mathfrak{K}$ is elementary. Now $\mathbf{Z}(\mathfrak{P}) \subseteq \mathfrak{P} \cap \mathfrak{K}$ since $\mathbf{Z}(\mathfrak{P}) \subseteq \mathbf{C}(\mathfrak{B}^*) = \mathfrak{K}$. Since $\mathbf{Z}(\mathfrak{P})$ is cyclic, it follows that $|\mathbf{Z}(\mathfrak{P})| = p$. If $\mathbf{Z}(\mathfrak{P}) \subseteq \mathfrak{P}^*$, then $\mathbf{N}(\mathbf{Z}(\mathfrak{P}))$ is an A_1-subgroup of $\mathfrak{G}$ containing $\mathfrak{T}$ and $\mathfrak{P}$, once again yielding a contradiction. Thus $\mathbf{Z}(\mathfrak{P}) \cap \mathfrak{P}^* = 1$ and we conclude that $\mathfrak{P} \cap \mathfrak{K} = \mathbf{Z}(\mathfrak{P}) \times \mathfrak{P}^*$. Since $|\mathbf{Z}(\mathfrak{P})| = p$ and $\mathfrak{P} \cap \mathfrak{K}$ is an elementary normal abelian subgroup of $\mathfrak{P}$ of index p, $\mathfrak{P}$ is of maximal class and $\mathrm{cl}(\mathfrak{P}) = a$, where $p^a = |\mathfrak{P} \cap \mathfrak{K}|$. Since $\mathscr{SCN}_3(\mathfrak{P})$ is non-empty, we also have $a \geqslant 3$.

Now $\mathfrak{Q} \subseteq \mathfrak{P}$. If $\mathfrak{Q} \subseteq \mathfrak{P} \cap \mathfrak{K}$, then we put $\mathfrak{Q} = \mathfrak{R}$ and $\mathfrak{P}^* \cap \mathfrak{Q} = \mathfrak{R}^*$. Then $\mathscr{SCN}_3(\mathfrak{R})$ is non-empty, $|\mathfrak{R} : \mathfrak{R}^*| = p$, and by Lemma 2.6(i), $\mathbf{C}(\mathfrak{R}^*)$ is an A_1-subgroup of $\mathfrak{G}$ of characteristic 3^t, $t \geqslant p$, since $\mathbf{C}_{\mathfrak{K}}(\mathfrak{P}^*) \subseteq \mathbf{C}(\mathfrak{R}^*)$. Thus (i) holds in this case. Assume finally that $\mathfrak{Q} \not\subseteq \mathfrak{P} \cap \mathfrak{K}$. In this case we set $\mathfrak{R} = \mathfrak{Q} \cap \mathfrak{P} \cap \mathfrak{K}$. Then $|\mathfrak{Q} : \mathfrak{R}| = p$ and hence $\mathfrak{R} \lhd \mathfrak{Q}$. If $Y \in \mathfrak{Q} - \mathfrak{R}$, then $Y \in \mathfrak{P} - (\mathfrak{P} \cap \mathfrak{K})$, and $|\mathbf{C}_{\mathfrak{P}}(Y)| = p^2$, whence $|\mathbf{C}_{\mathfrak{Q}}(Y)| = p^2$. Thus also $\mathfrak{Q}$ is of maximal class. Since $\mathscr{SCN}_3(\mathfrak{Q})$ is non-empty, it follows that $\mathrm{cl}(\mathfrak{Q}) = b \geqslant 3$, where $|\mathfrak{R}| = p^b$. In particular, $\mathscr{SCN}_3(\mathfrak{R})$ is non-empty. Furthermore, $\mathfrak{R} = \mathbf{C}_{\mathfrak{Q}}(\mathbf{Z}_2(\mathfrak{Q}))$ char $\mathfrak{Q}$ and hence $\mathfrak{R}$ is $\mathfrak{S}$-invariant. Finally $\mathfrak{P}^* \cap \mathfrak{R}$ contains a maximal subgroup $\mathfrak{R}^*$ of $\mathfrak{R}$. As above, $\mathbf{C}(\mathfrak{R}^*)$ is an A_1-subgroup of $\mathfrak{G}$ of characteristic 3^t, $t \geqslant p$. Thus all parts of (i) hold in this case as well.

It remains therefore to consider the case that for any non-trivial subgroup $\mathfrak{D}_1$ of $\mathfrak{P}$ such that $\mathbf{C}(\mathfrak{D}_1)$ is an A_1-group, we have $|\mathfrak{D}_1| = p$. Now let $\mathfrak{D}$ be a non-trivial subgroup of $\mathfrak{P}$ satisfying (b). We shall show that the conditions of (ii) hold. First of all, our assumptions force $|\mathfrak{D}| = p$. Set $\mathfrak{N} = \mathbf{N}(\mathfrak{D})$, and let $\mathfrak{X}$ be a maximal element of $\mathsf{M}_{\mathfrak{N}}(\mathfrak{T}; p)$ containing $\mathfrak{P} \cap \mathfrak{N}$. Since $\mathfrak{P}$ is non-cyclic, $|\mathfrak{P} \cap \mathfrak{N}| \geqslant p^2$ and hence $|\mathfrak{X}| \geqslant p^2$. Our conditions force $\mathbf{C}(\mathfrak{X})$ to be solvable. Hence by the first statement of the proof, $\mathfrak{N}$ is an A_1-group of characteristic 3^{p^n}, $n \geqslant 1$, and $\mathfrak{X}$ possesses a maximal subgroup $\mathfrak{X}^*$ containing $\mathfrak{X} \cap \mathbf{O}(\mathfrak{N})$ such that $\mathbf{C}_{\mathfrak{N}}(\mathfrak{X}^*)$ is an A_1-group of characteristic 3^p. Now our assumptions force $|\mathfrak{X}^*| = p$, whence $|\mathfrak{X}| = p^2$. Furthermore, since $\mathfrak{D} \subseteq \mathfrak{X} \cap \mathbf{O}(\mathfrak{N}) \subseteq \mathfrak{X}^*$, it follows that $\mathfrak{D} = \mathfrak{X}^*$ is an S_p-subgroup of $\mathbf{O}(\mathfrak{N})$ and also that $\mathbf{C}_{\mathfrak{N}}(\mathfrak{D})$ is an A_1-group of characteristic 3^p. But $\mathbf{C}_{\mathfrak{N}}(\mathfrak{D}) \lhd \mathfrak{N}$, and consequently $\mathfrak{N}$ is an A_1-group of characteristic 3^p. Thus all parts of (ii) are established. This completes the proof of the lemma.

5. The uniqueness condition. Using the results of the preceding section, we shall now investigate the consequences of the assumption that the uniqueness condition holds for some prime p in σ. The essential lines of the proof are based upon arguments of J. G. Thompson which were communicated to the author and which are used in **(12)**. Our main result is the following:

PROPOSITION 5. *Assume that $\mathfrak{G}$ satisfies the uniqueness condition for some prime p in σ. Let $\mathfrak{S}$ be an S_2-subgroup of $\mathfrak{G}$, let $\mathfrak{P}$ be a maximal element of $\mathsf{M}(\mathfrak{S}; p)$, let $\bar{\mathfrak{P}}$ be an S_p-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$, and let $\mathfrak{M}$ be the unique subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$ which is maximal subject to the conditions that $\mathfrak{M}$ contain an element of $\mathscr{A}_4(\bar{\mathfrak{P}})$ and that $p \in \pi_s(\mathfrak{M})$. Then the following hold:*

(i) $\mathfrak{S} \subset \mathfrak{M}$ and $\mathfrak{S}$ *centralizes* $\mathbf{O}_p(\mathfrak{M})$.

(ii) *The characteristic of every A_1-subgroup of $\mathfrak{G}$ is a power of* 2.

Remark. In Sections 6 and 7, we shall argue that if σ is non-empty, then, in fact, $\mathfrak{G}$ satisfies the uniqueness condition for some prime p in σ for which the corresponding subgroup $\mathfrak{M}$ of $\mathfrak{G}$ violates the conclusion of Proposition 5. This will force σ to be empty, in contradiction to Proposition 3, and will thus establish Theorem 3.

We carry out the proof of Proposition 5 in a sequence of lemmas, which are of some independent interest.

LEMMA 5.1. *The following conditions hold:*
(i) $\mathfrak{S} \subset \mathfrak{M}$.
(ii) $\mathbf{C}(T) \subseteq \mathfrak{M}$ *for each involution* T *in* $\mathfrak{S}$.
(iii) $\mathbf{N}(\mathfrak{T}) \subseteq \mathfrak{M}$ *for every non-trivial subgroup* $\mathfrak{T}$ *of* $\mathfrak{S}$.
(iv) $\mathfrak{P} \subseteq \mathbf{O}(\mathfrak{M})$.

Proof. By Lemma 4.3, $\mathfrak{S}$ possesses a four-subgroup $\mathfrak{T}$ such that $\mathbf{C}_{\mathfrak{P}}(T)$ contains a subgroup of type (p, p, p) for each T in $\mathfrak{T}^{\#}$. But then $\mathbf{C}_{\mathfrak{P}}(T) \in \mathscr{A}_3(\mathfrak{P})$. Since $\mathbf{C}(T)$ is solvable, it follows that $\mathbf{C}(T) \subseteq \mathfrak{M}$ for each T in $\mathfrak{T}^{\#}$. Since $\mathbf{C}(\mathfrak{S}) \subseteq \mathbf{C}(\mathfrak{T})$, $\mathbf{C}(\mathfrak{S}) \subseteq \mathfrak{M}$; and, in particular, (i) holds. Furthermore, $\mathbf{N}(\mathfrak{P}) \subseteq \mathfrak{M}$ since $\mathfrak{P} \in \mathscr{A}_3(\bar{\mathfrak{P}})$. Since

$$\mathbf{N}(\mathfrak{S}) = [\mathbf{N}(\mathfrak{S}) \cap \mathbf{N}(\mathfrak{P})]\mathbf{C}(\mathfrak{S})$$

by Lemma 4.1(i), it also follows that $\mathbf{N}(\mathfrak{S}) \subseteq \mathfrak{M}$.

Now let $\mathfrak{T}_0$ be an arbitrary subgroup of $\mathfrak{S}$ and set $\mathfrak{K} = \mathbf{N}(\mathfrak{T}_0)$, $\mathfrak{L} = \mathbf{C}(\mathfrak{T}_0)$. Then $\mathfrak{L}$ is solvable and $\mathfrak{S} \subseteq \mathfrak{L}$. Hence $|\mathfrak{K}/\mathfrak{L}|$ is odd, whence $\mathfrak{K}/\mathfrak{L}$, and consequently $\mathfrak{K}$, is solvable. Thus $\mathfrak{K} = \mathbf{O}(\mathfrak{K})\mathbf{N}_{\mathfrak{K}}(\mathfrak{S})$ by Lemma 2.4(i). But

$$\mathbf{O}(\mathfrak{K}) = \langle \mathbf{C}_{\mathbf{O}(\mathfrak{K})}(T) | T \in \mathfrak{T}^{\#} \rangle,$$

and hence $\mathbf{O}(\mathfrak{K}) \subseteq M$. Since $\mathbf{N}_{\mathfrak{K}}(\mathfrak{S}) \subseteq \mathbf{N}(\mathfrak{S}) \subseteq \mathfrak{M}$, it follows that $\mathfrak{K} \subseteq \mathfrak{M}$. Thus (iii) holds. Since (ii) is the special case of (iii) corresponding to $\mathfrak{T}_0 = \langle T \rangle$, (ii) also holds.

Finally (iv) follows from Lemmas 2.4(iv) and 2.5(iv).

The second statement of the proposition is an immediate consequence of this lemma. For let $\mathfrak{H}$ be an A_1-subgroup of $\mathfrak{G}$ which is not of characteristic 2^n for any n. Then $\mathfrak{H}$ is of odd characteristic $q > 5$. By replacing $\mathfrak{H}$ by a conjugate, if necessary, we may assume that $\mathfrak{T} = \mathfrak{S} \cap \mathfrak{H}$ is an S_2-subgroup of $\mathfrak{H}$. But then $\mathfrak{H} = \langle \mathbf{C}_{\mathfrak{H}}(T) | T \in \mathfrak{T}^{\#} \rangle$ by Lemma 2.6(ii) and consequently $\mathfrak{H} \subseteq \mathfrak{M}$ by Lemma 5.1(ii). Since $\mathfrak{H}$ is an A_1-group, also $\mathfrak{M}$ is an A_1-group. But $\mathfrak{S} \subset \mathfrak{M}$ and $m(\mathfrak{S}) \geqslant 3$, whence $\mathfrak{M}$ has characteristic $2^{m(\mathfrak{S})}$ by Lemma 2.5(i). However, this is impossible by Lemma 2.6(i), since $\mathfrak{H} \subseteq \mathfrak{M}$ and $\mathfrak{H}$ has odd characteristic $q > 5$.

To prove the first statement of the proposition, we need several additional lemmas.

LEMMA 5.2. $\mathbf{N}(\mathfrak{S})$ *has only one class of involutions.*

Proof. We first argue that $\mathfrak{G}$ has only one class of involutions. Let T be a fixed involution in $\mathfrak{S}$ and suppose $\mathfrak{G}$ possesses a conjugate class $\mathfrak{W}$ of involutions with $T \notin \mathfrak{W}$. Choose T_0 in $\mathfrak{W}$. Since T and T_0 are not conjugate in $\mathfrak{G}$, $\langle T, T_0 \rangle$ contains an involution T_1 which commutes with both T and T_0. But $\mathbf{C}(T) \subseteq \mathfrak{M}$ by Lemma 5.1(ii), and hence $T_1 \in \mathfrak{M}$. Thus $T_1 = T_2{}^M$ for some involution T_2 in $\mathfrak{S}$ and some M in $\mathfrak{M}$. Since $\mathbf{C}(T_2) \subseteq \mathfrak{M}$, we have $\mathbf{C}(T_1) \subseteq \mathfrak{M}$ and hence $T_0 \in \mathfrak{M}$. Thus $\mathfrak{W} \subseteq \mathfrak{M}$. But then $\langle \mathfrak{W} \rangle$ is a normal subgroup of $\mathfrak{G}$ contained in $\mathfrak{M}$, contrary to the simplicity of $\mathfrak{G}$. Thus all involutions of $\mathfrak{G}$ are conjugate.

Since $\mathfrak{S}$ is abelian, two involutions of $\mathfrak{S}$ which are conjugate in $\mathfrak{G}$ are already conjugate in $\mathbf{N}(\mathfrak{S})$. Hence $\mathbf{N}(\mathfrak{S})$ possesses only one class of involutions.

In the following lemmas, T will denote a fixed involution of $\mathfrak{S}$ and we set $\mathfrak{H} = \mathbf{C}(T)\mathbf{O}_p(\mathfrak{M})$. Since $\mathbf{C}(T) \subseteq \mathfrak{M}$, $\mathfrak{H}$ is a subgroup of $\mathfrak{M}$.

LEMMA 5.3. *Assume* $\Omega_1(\mathfrak{S})$ *does not centralize* $\mathfrak{P}$. *Then if* X *is an element of order* p *in* $\mathfrak{H}$, $\mathbf{C}_{\mathfrak{M}}(X)$ *contains a subgroup of type* (p, p, p).

Proof. By Lemma 5.1(iv), $\mathfrak{P} \subseteq \mathbf{O}(\mathfrak{M})$. Since $\mathfrak{P}$ is a maximal element of $\mathscr{M}(\mathfrak{S}; p)$, this implies that $\mathfrak{P}$ is an S_p-subgroup of $\mathbf{O}(\mathfrak{M})$. But then $\mathfrak{M} = \mathbf{O}(\mathfrak{M})\mathfrak{N}$ by Sylow's Theorem, where $\mathfrak{N} = \mathbf{N}_{\mathfrak{M}}(\mathfrak{P})$. It follows at once that $\mathfrak{K} = \mathbf{C}_{\mathfrak{N}}(T)\mathfrak{P}$ contains an S_p-subgroup of $\mathfrak{H}$. Hence if $\mathfrak{P}^*$ is an S_p-subgroup of $\mathfrak{K}$, it will suffice to show that $\mathbf{C}_{\mathfrak{P}^*}(X)$ contains a subgroup of type (p, p, p) for every element X of order p in $\mathfrak{P}^*$. If $X \in \mathfrak{P}$, this is a consequence of Lemma 4.2. We may therefore assume that $X \in \mathfrak{P}^* - \mathfrak{P}$ and, in particular, that $\mathfrak{P}$ is not an S_p-subgroup of $\mathfrak{K}$.

Let $\mathfrak{C}$ be a characteristic subgroup of $\mathfrak{P}$ chosen in accordance with (**4**, Lemma 8.2), and set $\mathfrak{D} = \Omega_1(\mathfrak{C})$, so that $\mathfrak{D}$ is of class at most 2 and of exponent p. The lemma follows if $|\mathbf{C}_{\mathfrak{D}}(X)| > p$; therefore we may assume that $|\mathfrak{C}_{\mathfrak{D}}(X)| = p$. Thus $\mathfrak{D}^* = \langle \mathfrak{D}, X \rangle$ is of maximal class. To derive a contradiction from this assumption, it will suffice to show that $\bar{\mathfrak{D}} = \mathfrak{D}/\mathbf{D}(\mathfrak{D})$ has order at least p^{p+1}. For assume this to be the case. Since every homomorphic image of a p-group of maximal class is easily seen to be of maximal class, $\bar{\mathfrak{D}}^* = \mathfrak{D}^*/\mathbf{D}(\mathfrak{D})$ is necessarily of maximal class. Since $\bar{\mathfrak{D}}$ is a maximal subgroup of $\bar{\mathfrak{D}}^*$ of order at least p^{p+1}, $\mathrm{cl}(\bar{\mathfrak{D}}^*) \geqslant p + 1$. On the other hand, since $\bar{\mathfrak{D}}$ is elementary, we clearly have $\mathrm{cl}(\bar{\mathfrak{D}}^*) \leqslant p$, thus yielding a contradiction.

Now since $\mathscr{SCN}_3(\mathfrak{P})$ is non-empty, $\mathbf{N}(\mathfrak{P}) \subseteq \mathfrak{M}$, and consequently $\mathfrak{N} = \mathbf{N}(\mathfrak{P})$. But then $\mathbf{N}(\mathfrak{S}) = \mathbf{N}_{\mathfrak{N}}(\mathfrak{S})\mathbf{C}(\mathfrak{S})$ by Lemma 4.1(i), and it follows from the preceding lemma that $\mathfrak{L} = \mathbf{N}_{\mathfrak{N}}(\mathfrak{S})$ has only one class of involutions, whence $\bar{\mathfrak{L}} = \mathfrak{L}/\mathbf{O}(\mathfrak{L})$ has only one class of involutions. Furthermore, $\mathbf{C}_{\mathfrak{N}}(T) \subseteq \mathbf{O}(\mathfrak{N})\mathbf{C}_{\mathfrak{L}}(T)$ by Lemma 2.4(v). Since $\mathfrak{K} = \mathbf{C}_{\mathfrak{N}}(T)\mathfrak{P}$ and $\mathfrak{P}$ is not an S_p-subgroup of $\mathfrak{K}$, this implies that $\mathfrak{P} \cap \mathbf{C}_{\mathfrak{L}}(T)$ is not an S_p-subgroup of $\mathbf{C}_{\mathfrak{L}}(T)$. Let $\bar{T}$ be the image of T in $\bar{\mathfrak{L}}$, in which case $\mathbf{C}_{\bar{\mathfrak{L}}}(\bar{T})$ is the image of $\mathbf{C}_{\mathfrak{L}}(T)$ in $\bar{\mathfrak{L}}$. Since $\mathfrak{P}$ contains an S_p-subgroup of $\mathbf{O}(\mathfrak{L})$ and $\mathfrak{P} \cap \mathbf{C}_{\mathfrak{L}}(T)$ is not an S_p-subgroup of $\mathbf{C}_{\mathfrak{L}}(T)$, we conclude that p divides $|\mathbf{C}_{\bar{\mathfrak{L}}}(\bar{T})|$.

On the other hand, $\bar{\mathfrak{L}}$, being solvable, possesses a 2-complement $\mathfrak{Y}$. Thus $\bar{\mathfrak{L}} = \bar{\mathfrak{S}}\mathfrak{Y}$, where $\bar{\mathfrak{S}}$ denotes the image of $\mathfrak{S}$ in $\bar{\mathfrak{L}}$. Set $\bar{\mathfrak{S}}_1 = \Omega_1(\bar{\mathfrak{S}})$ and $\bar{\mathfrak{L}}_1 = \bar{\mathfrak{S}}_1 \mathfrak{Y}$. Since any element of $\mathfrak{Y}$ which centralizes $\bar{\mathfrak{S}}_1$ necessarily centralizes $\bar{\mathfrak{S}}$, we conclude from (**9**, Lemma 1.2.3) that $\mathbf{O}(\bar{\mathfrak{L}}_1) \subseteq \mathbf{O}(\bar{\mathfrak{L}})$. But $\bar{\mathfrak{L}} = \mathfrak{L}/\mathbf{O}(\mathfrak{L})$, and consequently $\mathbf{O}(\bar{\mathfrak{L}}_1) = \mathbf{O}(\bar{\mathfrak{L}}) = 1$. Furthermore, since all involutions of $\bar{\mathfrak{L}}$ are conjugate, $\bar{\mathfrak{L}}_1$ also has only one class of involutions. It follows therefore from Lemma 2.3(iii) that $|\mathbf{C}_{\bar{\mathfrak{L}}_1}(\bar{T})| = 2^{m(\bar{\mathfrak{S}}_1)}w$, where $w | m(\bar{\mathfrak{S}}_1)$. Since p divides $|\mathbf{C}_{\bar{\mathfrak{L}}}(\bar{T})|$, p also divides $|\mathbf{C}_{\bar{\mathfrak{L}}_1}(\bar{T})|$, and it follows that p divides $m(\bar{\mathfrak{S}}_1) = m(\mathfrak{S})$.

By Lemma 2.3(ii), we also have that $\mathfrak{Y}$ possesses a cyclic subgroup $\mathfrak{R}$ of order $r \geqslant (2^{m(\mathfrak{S})} - 1)/d$, where $d = (2^{m(\mathfrak{S})} - 1, m(\mathfrak{S}))$, which acts regularly on $\bar{\mathfrak{S}}_1$. Let $\mathfrak{R}$ be its inverse image in $\mathfrak{L}$ and set $\mathfrak{S}_1 = \Omega_1(\mathfrak{S})$. Then $\mathfrak{R}/\mathbf{C}_{\mathfrak{R}}(\mathfrak{S}_1)$ is cyclic of order r. In turn, $\mathfrak{R}\mathfrak{S}_1$ is represented on $\bar{\mathfrak{D}} = \mathfrak{D}/\mathbf{D}(\mathfrak{D})$. Furthermore, $\mathfrak{S}_1$ is not represented trivially on $\bar{\mathfrak{D}}$, for otherwise $\mathfrak{S}_1 \subseteq \mathbf{C}(\mathfrak{D})$, whence $\mathfrak{S}_1 \subseteq \mathbf{C}(\mathfrak{C})$. But then $\mathfrak{S}_1 \subseteq \mathbf{C}(\mathfrak{P})$ by **(4**, Lemma 8.2**)**, contrary to hypothesis. Hence there exists a subgroup $\bar{\mathfrak{E}}$ of $\bar{\mathfrak{D}}$ on which $\mathfrak{R}\mathfrak{S}_1$ is represented irreducibly and on which $\mathfrak{S}_1$ is represented non-trivially. It follows therefore from Clifford's Theorem that $|\bar{\mathfrak{E}}| = p^a$, where $a \geqslant r$. But then $a \geqslant (2^{m(\mathfrak{S})} - 1)/d$. Since $d = (2^{m(\mathfrak{S})} - 1, m(\mathfrak{S}))$, we conclude at once from this inequality that $a \geqslant m(\mathfrak{S}) + 1$. But $p | m(\mathfrak{S})$, and consequently $a \geqslant p + 1$ whence

$$|\bar{\mathfrak{D}}| \geqslant |\bar{\mathfrak{E}}| \geqslant p^{p+1},$$

and the lemma is proved.

LEMMA 5.4. *Assume $\Omega_1(\mathfrak{S})$ does not centralize $\mathfrak{P}$. Then if X is an element of prime order in $\mathfrak{H}$ which is strongly real*† *in $\mathfrak{G}$, we have $\mathbf{C}^*(X) \subseteq \mathfrak{M}$.*

Proof. Let X have order q. If $q = 2$, $X^M \in \mathfrak{S}$ for some M in $\mathfrak{M}$, and it follows from Lemma 5.1 (ii) that $\mathbf{C}^*(X) = \mathbf{C}(X) \subseteq \mathfrak{M}$. If $q = p$, then $\mathbf{C}^*_{\mathfrak{M}}(X)$ contains a subgroup of type (p, p, p) by the preceding lemma. But then $\mathbf{C}^*_{\mathfrak{M}}(X^M)$ contains an element of $\mathscr{A}_3(\mathfrak{P})$ for some M in $\mathfrak{M}$. Since

$$p \in \pi_s(\mathbf{C}^*(X^M)),$$

it follows at once that $\mathbf{C}^*(X^M) \subseteq \mathfrak{M}$ and hence also that $\mathbf{C}^*(X) \subseteq \mathfrak{M}$. We may therefore assume that q is odd and that $q \neq p$.

Since $\mathfrak{H} = \mathbf{C}(T)\mathbf{O}_p(\mathfrak{M})$ and $\mathbf{O}_p(\mathfrak{M}) \lhd \mathfrak{H}$, it follows that $Y = X^C \in \mathbf{C}(T)$ for some C in $\mathfrak{H}$. Furthermore, since X is strongly real in $\mathfrak{G}$, so also is Y, and consequently $\mathfrak{C}^* = \mathbf{C}^*(Y)$ contains an involution not in $\mathfrak{C} = \mathbf{C}(Y)$. Since $T \in \mathfrak{C}$, it follows that an S_2-subgroup $\mathfrak{S}^*$ of $\mathfrak{C}^*$ containing T is non-cyclic. Now $\mathfrak{S}^* \subseteq \mathbf{C}(T) \subseteq \mathfrak{M}$ and hence $\mathbf{N}_{\mathfrak{C}^*}(\mathfrak{S}^*) \subseteq \mathfrak{M}$ by Lemma 5.1 (iii). Furthermore, since $\mathfrak{C}$ is a normal subgroup of index 2 in $\mathfrak{C}^*$, $\mathfrak{C}^*$ is not an A_1-group by Lemma 2.6 (iii). Thus $\mathfrak{C}^*$ is solvable and consequently $\mathfrak{C}^* = \mathbf{O}(\mathfrak{C}^*)\mathbf{N}_{\mathfrak{C}^*}(\mathfrak{S}^*)$ by Lemma 2.4 (i). It therefore suffices to show that $\mathbf{O}(\mathfrak{C}^*) \subseteq \mathfrak{M}$. But

$$\mathbf{O}(\mathfrak{C}^*) = \langle \mathbf{C}_{\mathbf{O}(\mathfrak{C}^*)}(T^*) | T^* \in \mathfrak{S}^{*\#} \rangle.$$

Since $\mathfrak{S}^{*M} \subseteq \mathfrak{S}$ for some M in $\mathfrak{M}$, $\mathbf{C}(T^*) \subseteq \mathfrak{M}$ for each T^* in $\mathfrak{S}^{*\#}$ by Lemma 5.1 (ii), and it follows that $\mathbf{O}(\mathfrak{C}^*) \subseteq \mathfrak{M}$. The lemma is proved.

LEMMA 5.5. *Assume $\Omega_1(\mathfrak{S})$ does not centralize $\mathfrak{P}$. Then each coset of $\mathfrak{H}$ which does not lie in $\mathfrak{M}$ contains at most one involution.*

Proof. Let $\mathfrak{H}Y$ be a coset of $\mathfrak{H}$ in $\mathfrak{G}$ with $Y \in \mathfrak{G} - \mathfrak{M}$, and suppose $\mathfrak{H}Y$ contains two distinct involutions T_1, T_2. Then $T_1 T_2 = X_0 \in \mathfrak{H}^\#$ and T_1 inverts X_0. Let X be a power of X_0 having prime order. Then $X \in \mathfrak{H}^\#$ and T_1

†An element of a group $\mathfrak{G}$ is called *strongly real* if it is inverted by some involution of $\mathfrak{G}$.

inverts X. But then $\mathbf{C}^*(X) \subseteq \mathfrak{M}$ by the preceding lemma, whence $T_1 \in \mathfrak{M}$ and consequently $Y \in \mathfrak{M}$, a contradiction.

LEMMA 5.6. $\Omega_1(\mathfrak{S})$ *centralizes* $\mathbf{O}_p(\mathfrak{M})$.

Proof. Assume by way of contradiction that $\Omega_1(\mathfrak{S})$ does not centralize $\mathbf{O}_p(\mathfrak{M})$. Then certainly $\Omega_1(\mathfrak{S})$ does not centralize $\mathfrak{P}$, and hence we can apply the preceding lemma. First of all, since $\mathbf{N}(\mathfrak{S}) \subseteq \mathfrak{M}$ and $\mathbf{N}(\mathfrak{S})$ has only one class of involutions, it follows that $\mathbf{O}_p(\mathfrak{M}) \not\subseteq \mathbf{C}(T)$. Thus $\mathbf{C}(T) \subset \mathfrak{H}$. Furthermore, $\mathfrak{M}$ and $\mathfrak{G}$ each have only class of involutions.

Now set $c = |\mathbf{C}(T)|$, $ch = |\mathfrak{H}|$, $chm = |\mathfrak{M}|$, and $chmg_0 = |\mathfrak{G}|$. Since $\mathbf{C}(T) \subset \mathfrak{H}$, we have $h \geqslant 3$. Now the number of involutions in $\mathfrak{M}$ is hm. Hence, by the preceding lemma, the number of involutions in $\mathfrak{G}$ is at most $hm + (mg_0 - m)$. On the other hand, the number of involutions in $\mathfrak{G}$ is hmg_0. Thus we have

$$hmg_0 \leqslant hm + mg_0 - m,$$

which yields

$$(h - 1)(g_0 - 1) \leqslant 0.$$

Since $h \geqslant 3$, this forces $g_0 = 1$ and $\mathfrak{M} = \mathfrak{G}$, a contradiction.

We are now in a position to prove part (i) of the proposition. Suppose, by way of contradiction, that $\mathfrak{S}$ does not centralize $\mathbf{O}_p(\mathfrak{M})$. Since $\Omega_1(\mathfrak{S})$ centralizes $\mathbf{O}_p(\mathfrak{M})$, $\mathfrak{S}$ is not elementary. As $\mathbf{N}(\mathfrak{S})$ possesses only one class of involutions, it follows that $\mathfrak{S}$ is homocyclic of type $(2^a, 2^a, \ldots, 2^a)$ with $a > 1$. Since an S_2-subgroup of an A_1-subgroup is elementary, we conclude in particular that $\mathfrak{M}$ is solvable.

To obtain a contradiction, we shall now argue that the normalizer of every non-identity subgroup of $\mathfrak{G}$ is solvable, contrary to Proposition 2(v). Suppose this is false, and let $\mathfrak{K}_0$ be a non-trivial solvable subgroup of $\mathfrak{G}$ such that $\mathfrak{K} = \mathbf{N}(\mathfrak{K}_0)$ is non-solvable. If $\mathfrak{Q}_0$ is an S_q-subgroup of $\mathfrak{K}_0$ for any q in $\pi(\mathfrak{K}_0)$, then $\mathfrak{K} = \mathfrak{K}_0 \mathbf{N}_{\mathfrak{K}}(\mathfrak{Q}_0)$ by Sylow's Theorem, and consequently $\mathbf{N}(\mathfrak{Q}_0)$ is non-solvable. Hence if $\mathfrak{Q}$ is a q-group of maximal order such that $\mathfrak{H} = \mathbf{N}(\mathfrak{Q})$ is non-solvable, we have $\mathfrak{Q} \neq 1$. If $\mathfrak{T}$ is an S_2-subgroup of $\mathfrak{H}$, we can assume without loss that $\mathfrak{T} \subseteq \mathfrak{S}$. Now $\mathfrak{H}$, being non-solvable, is an A_1-group, and hence $\mathfrak{H}$ has characteristic 2^n, $n \geqslant 2$, by part (ii) of the proposition, which has been established above. Furthermore, Sylow's Theorem and the maximality of $\mathfrak{Q}$ imply that $\mathfrak{Q}$ is an S_q-subgroup of $\mathbf{O}(\mathfrak{H})$. It follows therefore from Lemma 2.5(iv), if $n \geqslant 3$, that $\mathfrak{Q}$ is a maximal element of $\mathsf{M}_{\mathfrak{H}}(\mathfrak{T}; q)$. On the other hand, if $n = 2$, the same conclusion clearly holds, since then $\mathfrak{H}/\mathbf{O}(\mathfrak{H})$ is isomorphic to the alternating group A_5. Since $\mathfrak{H} = \mathbf{N}(\mathfrak{Q})$, it follows that $\mathfrak{Q}$ is a maximal element of $\mathsf{M}(\mathfrak{T}; q)$. But $\mathfrak{Q} \subseteq \mathfrak{M}$ since $\mathfrak{Q} = \langle \mathbf{C}_{\mathfrak{Q}}(T) | \; T \in \mathfrak{T}^{\#} \rangle$ and $\mathbf{C}(T) \subseteq \mathfrak{M}$ for each T in $\mathfrak{T}^{\#}$. We conclude therefore that $\mathfrak{Q}$ is a maximal element of $\mathsf{M}_{\mathfrak{M}}(\mathfrak{T}; q)$. Now $\mathfrak{M}$ is solvable and has only one class of involutions. But then Lemma 2.4(vi) implies that an S_2-subgroup $\mathfrak{T}^*$ of $\mathbf{N}_{\mathfrak{M}}(\mathfrak{Q})$ is homo-

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cyclic of type $(2^a, 2^a, \ldots, 2^a)$. Since $\mathfrak{T} \subseteq \mathfrak{M}$, we can assume that $\mathfrak{T} \subseteq \mathfrak{T}^*$. But then $\mathfrak{T} = \mathfrak{T}^*$, since $\mathfrak{T}$ is an S_2-subgroup of $\mathfrak{H} = \mathbf{N}(\mathfrak{Q})$. Since $a > 1$, we conclude that $\mathfrak{T}$ is not elementary, contrary to the fact that $\mathfrak{T}$ is an S_2-subgroup of the A_1-group $\mathfrak{H}$. This completes the proof of the proposition.

6. The set of tame primes. In the present section we shall determine for the primes in σ the precise conditions under which the modified definitions of weakly p-tame, p-tame, and τ-tame, as discussed in Section 1, are satisfied. We first study the concept of weak p-constraint. Our results are contained in the following lemma:

LEMMA 6.1. *If $p \in \sigma$ and $p \geqslant 5$, then $\mathfrak{G}$ is weakly p-constrained. If $3 \in \sigma$, then either $\mathfrak{G}$ is weakly 3-constrained or else $\mathfrak{G}$ possesses an A_1-subgroup of characteristic 3^t, $t \geqslant 3$.*

Proof. Let $p \in \sigma$, and suppose $\mathfrak{G}$ is not weakly p-constrained. Then $\mathfrak{G}$ possesses a p-subgroup $\mathfrak{D} \neq 1$ with the following properties: (a) if $\mathfrak{C} = \mathfrak{D}\mathbf{C}(\mathfrak{D})$ and if $\mathfrak{D}^*$ denotes an S_p-subgroup of $\mathbf{O}_{p',p}(\mathfrak{C})$, then $\mathbf{C}(\mathfrak{D}^*)$ is non-solvable; and (b) $\mathbf{N}(\mathfrak{D})$ contains an element of $\mathscr{A}_4(\mathfrak{P})$ for some S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$. Since $\mathfrak{D} \subseteq \mathfrak{D}^*$, $\mathbf{C}(\mathfrak{D}^*) \subseteq \mathfrak{C}$, and therefore $\mathfrak{C}$ is an A_1-group. Let $\mathfrak{T}$, $\mathfrak{S}$, $\mathfrak{Q}$, and $\mathfrak{P}$ be respectively an S_2-subgroup of $\mathfrak{C}$, an S_2-subgroup of $\mathfrak{G}$ containing $\mathfrak{T}$, a maximal element of И$(\mathfrak{S}; p)$, and a maximal element of И$(\mathfrak{T}, p)$ containing $\mathfrak{Q}$. Since $\mathbf{N}(\mathfrak{T}) \supset \mathbf{C}(\mathfrak{T})$, we may assume in view of Proposition 4 that $\mathfrak{D} \subseteq \mathfrak{P}$.

Suppose first that $\mathfrak{P} \not\subseteq \mathbf{C}(\mathfrak{T})$. Then a $\mathfrak{T}$-invariant S_p-subgroup $\mathfrak{P}^*$ of $\mathbf{O}_{p',p}(\mathbf{C}(\mathfrak{D}))$ does not centralize $\mathfrak{T}$ by Lemma 4.5. Now $\mathfrak{D}\mathfrak{P}^*$ is a $\mathfrak{T}$-invariant S_p-subgroup of $\mathbf{O}_{p',p}(\mathfrak{C})$ and hence is conjugate to $\mathfrak{D}^*$ by an element of $\mathbf{C}(\mathfrak{T})$. Since $\mathfrak{D}\mathfrak{P}^* \not\subseteq \mathbf{C}(\mathfrak{T})$, it follows that $\mathfrak{D}^* \not\subseteq \mathbf{C}(\mathfrak{T})$. Since $\mathfrak{C}$ is an A_1-group and $\mathfrak{T}$ is an S_2-subgroup of $\mathfrak{C}$, this implies that $\mathbf{C}_{\mathfrak{C}}(\mathfrak{D}^*) \subseteq \mathbf{O}(\mathfrak{C})$ and hence that $\mathbf{C}_{\mathfrak{C}}(\mathfrak{D}^*)$ is solvable. Since $\mathbf{C}(\mathfrak{D}^*) \subseteq \mathfrak{C}$, we conclude that $\mathbf{C}(\mathfrak{D}^*)$ is solvable, contrary to assumption. Therefore $\mathfrak{P} \subseteq \mathbf{C}(\mathfrak{T})$.

Now let $\mathfrak{D}_0$ be a maximal $\mathfrak{T}$-invariant subgroup of $\mathfrak{P}$ containing $\mathfrak{D}^*$ such that $\mathfrak{N} = \mathbf{N}(\mathfrak{D}_0)$ is an A_1-group. Let $\mathfrak{D}_1$ be a maximal element of И$_{\mathfrak{N}}(\mathfrak{T}, p)$. Applying Proposition 4 once again, we can assume with no loss of generality that $\mathfrak{D}_1 \subseteq \mathfrak{P}$. Maximality of $\mathfrak{D}_0$ implies that $\mathfrak{D}_0$ is an S_p-subgroup of $\mathbf{O}(\mathfrak{N})$. Let $\mathfrak{S}_1$ be an S_2-subgroup of $\mathfrak{N}$ containing $\mathfrak{T}$ and suppose first that $m(\mathfrak{S}_1) \geqslant 3$, in which case $\mathfrak{N}$ has characteristic $2^{m(\mathfrak{S}_1)}$. Since $m(\mathfrak{T}) \geqslant 2$, and $\mathfrak{D}_1 \subseteq \mathfrak{P} \subseteq \mathbf{C}(\mathfrak{T})$, Lemma 2.8 implies that $\mathbf{C}_{\mathfrak{N}}(\mathfrak{D}_1)$ is an A_1-group. Hence $\mathfrak{D}_1 = \mathfrak{D}_0$ by the maximality of $\mathfrak{D}_0$. Thus $\mathfrak{D}_0$ is a maximal element of И$_{\mathbf{N}(\mathfrak{D}_0)}(\mathfrak{T}; p)$, and so is a maximal element of И$(\mathfrak{T}; p)$. We conclude that $\mathfrak{D}_0 = \mathfrak{P}$ and that $\mathbf{C}_{\mathfrak{N}}(\mathfrak{P})$ is an A_1-group. But then as $\mathfrak{Q} \subseteq \mathfrak{P}$, also $\mathbf{C}(\mathfrak{Q})$ is an A_1-group. But $\mathfrak{S} \subseteq \mathbf{N}(\mathfrak{Q})$, and hence $\mathfrak{S}$ centralizes $\mathfrak{Q}$ by Lemma 2.6(v), contrary to the fact that $\mathfrak{Q}$ is a maximal element of И$(\mathfrak{S}; p)$ and $p \in \sigma$. Thus $m(\mathfrak{S}_1) = 2$ and $\mathfrak{T} = \mathfrak{S}_1$ is an S_2-subgroup of $\mathfrak{N}$.

To treat this case, observe first of all, that $\mathbf{C}(\mathfrak{Q})$ is solvable; otherwise $\mathbf{C}(\mathfrak{Q})$ would be an A_1-group, and Lemma 2.6(v) would yield the same con-

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tradiction as in the preceding paragraph. Thus the hypotheses of Lemma 4.6 are satisfied. Suppose first that the conditions of Lemma 4.6(i) hold. Then $\mathfrak{Q}$ contains an $\mathfrak{S}$-invariant subgroup $\mathfrak{R}$ such that $\mathscr{SCN}_3(\mathfrak{R})$ is non-empty and such that $\mathbf{C}(\mathfrak{R}^*)$ is an A_1-group of characteristic 3^t, $t \geqslant p$, where $\mathfrak{R}^*$ is a suitable maximal subgroup of $\mathfrak{R}$. We shall show that this case is impossible. We first argue that $m(\mathfrak{S}) \geqslant 4$; so assume the contrary. Then $m(\mathfrak{S}) = 3$, and hence both $\mathfrak{G}$ and $\mathbf{N}(\mathfrak{S})$ possess only one class of involutions. But then $\mathbf{N}(\mathfrak{Q})$ has only one class of involutions by Lemma 4.1(i). Since $\mathfrak{P} \subseteq \mathbf{C}(\mathfrak{T})$, $\mathfrak{Q} \subseteq \mathbf{C}(\mathfrak{T})$, and consequently $\mathfrak{Q}$ centralizes some involution of $\mathfrak{S}$. Since $\mathbf{N}(\mathfrak{Q})$ has only one class of involutions, it follows that $\Omega_1(\mathfrak{S})$ centralizes $\mathfrak{Q}$ and hence centralizes $\mathfrak{R}^*$. But then $\Omega_1(\mathfrak{S}) \subseteq \mathbf{C}(\mathfrak{R}^*)$. Since $m(\Omega_1(\mathfrak{S})) = 3$ and $\mathbf{C}(\mathfrak{R}^*)$ is an A_1-group, Lemma 2.5(i) implies that $\mathbf{C}(\mathfrak{R}^*)$ has characteristic 2^3, contrary to the fact that $\mathbf{C}(\mathfrak{R}^*)$ has characteristic 3^t. Therefore $m(\mathfrak{S}) \geqslant 4$, as asserted.

Thus $\mathfrak{S}$ possesses a four-subgroup $\mathfrak{T}^*$ disjoint from $\mathfrak{T}$. Let $\mathfrak{T}_i^*$, $i = 1, 2, 3$, be the involutions of $\mathfrak{T}^*$. To reach a contradiction, it will suffice to show that $\mathfrak{R}$ contains a non-cyclic subgroup $\mathfrak{B}$ which is either centralized or inverted by one of the involutions T_i^*. Indeed, if this is the case, $\mathfrak{B} \cap \mathfrak{R}^* = \mathfrak{B}^* \neq 1$, and $\mathbf{N}(\mathfrak{B}^*)$ contains the 2-group $\langle \mathfrak{T}, T_i^* \rangle$ of order 8. Hence $\mathbf{N}(\mathfrak{B}^*)$ is either solvable or is an A_1-group of characteristic 2^m by Lemma 2.5(i). But $\mathbf{C}(\mathfrak{R}^*) \subseteq \mathbf{C}(\mathfrak{B}^*) \subseteq \mathbf{N}(\mathfrak{B}^*)$, and consequently $\mathbf{N}(\mathfrak{B}^*)$ is an A_1-group of characteristic 3^n by Lemma 2.6(i), thus yielding the desired contradiction.

Now let $\mathfrak{R} = \mathfrak{R}_1 \mathfrak{R}_2 \mathfrak{R}_3 = \mathfrak{R}_0 \mathfrak{R}_1' \mathfrak{R}_2' \mathfrak{R}_3'$ be the $\mathfrak{T}^*$-decompositions of $\mathfrak{R}$. If $\mathfrak{R}_i$ is non-cyclic for any i, then we can take $\mathfrak{R}_i$ as $\mathfrak{B}$ since T_i^* centralizes $\mathfrak{R}_i$. Hence we may suppose that $\mathfrak{R}_i$ is cyclic for each $i = 1, 2, 3$. Since $\mathscr{SCN}_3(\mathfrak{R})$ is non-empty, this implies that $\mathfrak{R}_0 = 1$ and that $\mathfrak{R}_i'$ is cyclic for each i. Since $\mathbf{Z}(\mathfrak{R})$ is $\mathfrak{T}^*$-invariant, $\mathfrak{R}_i' \cap \mathbf{Z}(\mathfrak{R}) \neq 1$ for some i, say $i = 1$. Then $\Omega_1(\mathfrak{R}_1') \subseteq \mathbf{Z}(\mathfrak{R})$ and consequently $\mathfrak{B} = \langle \Omega_1(\mathfrak{R}_1'), \Omega_1(\mathfrak{R}_2') \rangle$ is abelian of type (p, p). Furthermore, since T_3^* inverts $\mathfrak{R}_1'$ and $\mathfrak{R}_2'$, T_3^* inverts $\mathfrak{B}$, and thus $\mathfrak{B}$ has the required properties. We conclude that the conditions of Lemma 4.6(ii) must hold.

Now $\mathfrak{D}_0$ is a maximal $\mathfrak{T}$-invariant p-subgroup of $\mathfrak{G}$ such that $\mathfrak{N} = \mathbf{N}(\mathfrak{D}_0)$ is an A_1-group and $\mathfrak{P} \cap \mathfrak{N} = \mathfrak{D}_1$ is a maximal element of $\mathsf{N}_{\mathfrak{N}}(\mathfrak{T}; p)$. It follows therefore from Lemma 4.6(ii) that $|\mathfrak{P} \cap \mathfrak{N}| = p^2$, that $|\mathfrak{P} \cap \mathbf{O}(\mathfrak{N})| = p$, and that $\mathfrak{N}$ has characteristic 3^p. In particular, $\mathfrak{D} = \mathfrak{D}^* = \mathfrak{D}_0$, whence $\mathfrak{N} = \mathbf{N}(\mathfrak{D})$. Suppose now that $p \geqslant 5$. Then by Lemma 2.8(ii), $\mathfrak{P} \cap \mathfrak{N}$ is an S_p-subgroup of $\mathfrak{N}$. Now by assumption, $\mathfrak{N}$ contains an element $\mathfrak{X}$ of $\mathscr{A}_4(\tilde{\mathfrak{P}})$ for some S_p-subgroup $\tilde{\mathfrak{P}}$ of $\mathfrak{G}$. Since an S_p-subgroup of $\mathfrak{N}$ has order p^2 and since $|\mathfrak{X}| \geqslant p^2$, it follows that $\mathfrak{X}$ is an S_p-subgroup of $\mathfrak{N}$. In particular, $\mathfrak{D} \subseteq \mathfrak{X}$ and $\mathbf{N}_{\tilde{\mathfrak{P}}}(\mathfrak{D}) = \mathfrak{X}$. But now it follows at once from the definition of $\mathscr{A}_i(\tilde{\mathfrak{P}})$, $i = 1, 2, 3, 4$, that $\mathfrak{X} \in \mathscr{A}_1(\tilde{\mathfrak{P}})$, which is clearly impossible since every element of $\mathscr{A}_1(\tilde{\mathfrak{P}})$ possesses a subgroup of type (p, p, p). This contradiction shows that $\mathfrak{G}$ is weakly p-constrained for all $p \geqslant 5$. Finally if $p = 3$, our argument shows that either $\mathfrak{G}$ is weakly 3-constrained or else $\mathbf{N}(\mathfrak{D})$ is an A_1-group of characteristic 3^3; and all parts of the lemma are proved.

Proposition 6. *If $p \in \sigma$ and $p \geqslant 5$, then $\mathfrak{G}$ is p-tame. If $3 \in \sigma$, then either $\mathfrak{G}$ is p-tame or else $\mathfrak{G}$ possesses an A_1-subgroup of characteristic 3^t, $t \geqslant 3$.*

Proof. If $\mathfrak{G}$ is not weakly p-constrained for some p in σ, then by the preceding lemma, $p = 3$ and $\mathfrak{G}$ possesses an A_1-subgroup of characteristic 3^t, $t \geqslant 3$. In this case the final alternative of the proposition holds. Hence we need only prove the proposition for those primes p in σ for which $\mathfrak{G}$ is weakly p-constrained.

We first argue that $\mathfrak{G}$ is weakly p-tame for each p in σ. First of all, $\mathscr{SCN}_3(p)$ is non-empty by Lemma 4.2. Furthermore, every proper subgroup of $\mathfrak{G}$ is either an A_0-group, an A_1-group of characteristic 2^n, or else is a D-group. It follows therefore from Proposition 1(i) and from **(6**, Proposition 6**)** that every proper subgroup of $\mathfrak{G}$ is p-stable, whence $\mathfrak{G}$ itself is p-stable. Suppose next that $\mathfrak{H}$ is a proper subgroup of $\mathfrak{G}$ such that $p \notin \pi_s(\mathfrak{H})$ and such that $\mathfrak{A} \subseteq \mathfrak{H}$ for some element $\mathfrak{A}$ of $\mathscr{SCN}_3(p)$. Clearly this implies that $\mathfrak{H}$ is non-solvable and hence that $\mathfrak{H}$ is an A_1-group. Since $\mathfrak{A}$ is mapped isomorphically into $\bar{\mathfrak{H}} = \mathfrak{H}/\mathbf{O}(\mathfrak{H})$, $\bar{\mathfrak{H}}$ contains a subgroup of type (p, p, p). It follows therefore from Lemma 2.2(iv) that $\bar{\mathfrak{H}}$, and hence also $\mathfrak{H}$, is of odd characteristic. Thus $\mathfrak{H}$ is a non-solvable D-group. But now the proof of **(6**, Lemma 7.4**)** implies that every element of $\text{И}_{\mathfrak{H}}(\mathfrak{A})$ lies in $\mathbf{O}(\mathfrak{H})$. We conclude from the definition that $\mathfrak{G}$ is weakly p-tame.

To complete the proof of the proposition, we must show, in addition, that if $\tilde{\mathfrak{P}}$ is an S_p-subgroup of $\mathfrak{G}$ and if $\mathfrak{Q}$ is a non-trivial element of $\text{И}(\tilde{\mathfrak{P}}; q)$ for any prime q, then $\tilde{\mathfrak{P}} \cap \mathbf{S}(\mathbf{N}(\mathfrak{Q})) \neq 1$. We shall follow the argument of **(6**, Proposition 10**)**. We let $\mathfrak{A} \in \mathscr{SCN}_3(\tilde{\mathfrak{P}})$ and set $\mathfrak{B} = \mathbf{V}(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{A}); \tilde{\mathfrak{P}})$. We assume the desired conclusion is false, and choose $\mathfrak{Q}$ of maximum order in $\text{И}(\tilde{\mathfrak{P}}; q)$ so that $\tilde{\mathfrak{P}} \cap \mathbf{S}(\tilde{\mathfrak{N}}(\mathfrak{Q})) = 1$. Then certainly $\mathfrak{N} = \mathbf{N}(\mathfrak{Q})$ is non-solvable and, in particular, $\mathbf{S}(\mathfrak{N}) = \mathbf{O}(\mathfrak{N})$. But now the argument in the first part of the proof of **(6**, Proposition 10**)** applies without change to show that $\mathfrak{Q}$ is, in fact, a maximal element of $\text{И}(\tilde{\mathfrak{P}}, q)$. Furthermore, since $\tilde{\mathfrak{P}} \cap \mathbf{O}(\mathfrak{N}) = 1$, $\bar{\mathfrak{N}} = \mathfrak{N}/\mathbf{O}(\mathfrak{N})$ contains a subgroup of type (p, p, p) and we conclude once again from Lemma 2.2(iv) that $\mathfrak{N}$ is a non-solvable D-group. But now the proof of **(6**, Proposition 10**)** applies to yield that $\mathfrak{N}$ has characteristic p^m for some m, that $\mathfrak{A}$ is the unique element of $\mathscr{SCN}_3(\tilde{\mathfrak{P}})$, and that $\mathfrak{A}$ is characteristic in every subgroup of $\tilde{\mathfrak{P}}$ containing $\mathfrak{A}$. In particular, $\mathfrak{B} = \mathfrak{A}$ and $\mathfrak{N}_1 = \mathbf{N}(\mathbf{Z}(\mathfrak{B})) = \mathbf{N}(\mathfrak{A})$.

To derive a contradiction from these conditions, it will suffice to show that there exists a subgroup $\mathfrak{P}$ of $\tilde{\mathfrak{P}}$ containing $\mathfrak{A}$ with the property that an S_2-subgroup $\mathfrak{T}$ of $\mathbf{N}(\mathfrak{P})$ is non-cyclic. For assume this to be the case. Since $\mathfrak{A}$ char $\mathfrak{P}$, $\mathbf{N}(\mathfrak{P}) \subseteq \mathbf{N}(\mathfrak{A})$ and consequently $\mathfrak{T} \subseteq \mathbf{N}(\mathfrak{A}) = \mathfrak{N}_1$. On the other hand, since $\mathfrak{G}$ is weakly p-tame and $\mathfrak{Q}$ is a maximal element of $\text{И}(\tilde{\mathfrak{P}}; q)$, it follows from **(5**, Lemma 4.2**)** that $\mathfrak{N}_1 = \mathbf{O}_{p'}(\mathfrak{N}_1)(\mathfrak{N}_1 \cap \mathfrak{N})$. Thus $\mathfrak{T}^X = \mathfrak{T}' \in \mathfrak{N}_1 \cap \mathfrak{N}$ for some X in $\mathbf{O}_p{}'(\mathfrak{N}_1)$. Since $\mathfrak{N}_1 = \mathbf{N}(\mathfrak{A})$, X centralizes $\mathfrak{A}$, and consequently $\mathfrak{T}'$ normalizes $\mathfrak{A}$. But then the image $\bar{\mathfrak{T}}'$ of $\mathfrak{T}'$ normalizes the image $\bar{\mathfrak{A}}$ of $\mathfrak{A}$ in

$\bar{\mathfrak{N}} = \mathfrak{N}/\mathbf{O}(\mathfrak{N})$. But $\bar{\mathfrak{N}}$ is isomorphic to a subgroup of $\mathrm{P\Gamma L}(2, p^m)$ containing $\mathrm{PSL}(2, p^m)$ and $\bar{\mathfrak{A}}$ contains a subgroup of type (p, p, p). Since $\bar{\mathfrak{A}} \in \mathsf{M}_{\bar{\mathfrak{N}}}(\bar{\mathfrak{T}}'; p)$, this contradicts **(6**, Lemma 3.1 (vii)**)**.

Let then $\mathfrak{S}$ be an S_2-subgroup of $\mathfrak{G}$ and let $\mathfrak{P}^*$ be a maximal element of $\mathsf{M}(\mathfrak{S}; p)$. By Lemma 4.1 (iii), $\mathfrak{S}/\mathbf{C}_{\mathfrak{S}}(\mathfrak{P}^*)$ is non-cyclic and hence an S_2-subgroup of $\mathbf{N}(\mathfrak{P}^*)/\mathbf{C}(\mathfrak{P}^*)$ is non-cyclic. Let $\mathfrak{P}$ be a p-subgroup of $\mathfrak{G}$ of maximal order containing $\mathfrak{P}^*$ such that $\mathbf{N}(\mathfrak{P})/\mathbf{C}(\mathfrak{P})$ has a non-cyclic S_2-subgroup. By replacing $\mathfrak{P}$ by a suitable conjugate, we can assume without loss that $\bar{\mathfrak{P}} \cap \mathfrak{N}_0$ is an S_p-subgroup of $\mathfrak{N}_0$, where $\mathfrak{N}_0 = \mathbf{N}(\mathfrak{P})$. Let $\mathfrak{P}_1$ be an S_p-subgroup of $\mathbf{O}_{p',p}(\mathfrak{N}_0)$ and set $\mathfrak{C}_1 = \mathbf{C}_{\mathfrak{N}_0}(\mathfrak{P}_1)$ and $\mathfrak{N}_1 = \mathbf{N}_{\mathfrak{N}_0}(\mathfrak{P}_1)$. Since $\mathfrak{N}_0 = \mathbf{O}_{p'}(\mathfrak{N}_0)\mathfrak{N}_1$ by Sylow's Theorem, $\mathbf{O}_{p'}(\mathfrak{N}_0)\mathfrak{C}_1 \lhd \mathfrak{N}_1$ and $\mathfrak{N}_0/\mathbf{O}_{p'}(\mathfrak{N}_0)\mathfrak{C}_1$ is isomorphic to $\mathfrak{N}_1/\mathfrak{C}_1$. On the other hand, $\mathfrak{C}_1 \subseteq \mathbf{C}(\mathfrak{P})$, since $\mathfrak{P} \subseteq \mathfrak{P}_1$. Since $\mathbf{O}_{p'}(\mathfrak{N}_0)$ clearly centralizes $\mathfrak{P}$, it follows that $\mathbf{O}_{p'}(\mathfrak{N}_0)\mathfrak{C}_1 \subseteq \mathbf{C}(\mathfrak{P})$. But now our conditions imply that an S_2-subgroup of $\mathfrak{N}_0/\mathbf{O}_{p'}(\mathfrak{N}_0)\mathfrak{C}_1$ is non-cyclic and consequently that an S_2-subgroup of $\mathfrak{N}_1/\mathfrak{C}_1$ is non-cyclic. But then certainly an S_2-subgroup of $\mathbf{N}(\mathfrak{P}_1)/\mathbf{C}(\mathfrak{P}_1)$ is non-cyclic. Hence $\mathfrak{P}_1 = \mathfrak{P}$ by our maximal choice of $\mathfrak{P}$, and so $\mathfrak{P}$ is an S_p-subgroup of $\mathbf{O}_{p',p}(\mathbf{N}(\mathfrak{P}))$. Furthermore, since $\mathfrak{P}^* \subseteq \mathfrak{P}$, Lemma 4.2 implies that $\mathfrak{P} \in \mathscr{A}_3(\bar{\mathfrak{P}})$. Since $\mathfrak{G}$ is weakly p-tame, we can therefore apply **(5**, Lemma 3.4**)** to conclude that $\mathfrak{P}$ contains every element of $\mathscr{SCN}_3(\bar{\mathfrak{P}})$. Thus $\mathfrak{A} \subseteq \mathfrak{P} \subseteq \bar{\mathfrak{P}}$. Since $\mathbf{N}(\mathfrak{P})$ possesses a non-cyclic S_2-subgroup, $\mathfrak{P}$ has the required properties. This completes the proof of the proposition.

In view of the alternatives of Proposition 6, we now define for convenience a subset σ^* of σ as follows. We set $\sigma^* = \sigma$ if either $3 \notin \sigma$ or if $3 \in \sigma$ and $\mathfrak{G}$ does not possess an A_1-subgroup of characteristic 3^t, $t \geqslant 3$, and we set $\sigma^* = \sigma - \{3\}$ in the contrary case. Thus, by the proposition, $\mathfrak{G}$ is p-tame for each p in σ^*.

PROPOSITION 7. *$\mathfrak{G}$ is σ^*-tame.*

Proof. We have just remarked that $\mathfrak{G}$ is p-tame for each p in σ^*. Hence to show that $\mathfrak{G}$ is σ^*-tame, we need only prove that $p \sim q$ for any two primes p and q in σ^*. Let $\mathfrak{S}$ be an S_2-subgroup of $\mathfrak{G}$, let $\mathfrak{P}$ be a maximal element of $\mathsf{M}(\mathfrak{S}; p)$, and let $\mathfrak{Q}$ be a maximal element of $\mathsf{M}(\mathfrak{S}; q)$. For definiteness, assume that $p > q$. We apply Lemma 4.3 (ii) and conclude that $\mathfrak{S}$ possesses a four-subgroup $\mathfrak{T}$ such that $\mathbf{C}_{\mathfrak{P}}(T)$ contains a subgroup of type (p, p, p) for each T in $\mathfrak{T}^{\#}$. Let T_1, T_2, T_3 be the involutions of $\mathfrak{T}$, and suppose that for some $i = 1, 2, 3$, $\mathfrak{Q}_i = \mathbf{C}_{\mathfrak{Q}}(T_i)$ contains a subgroup of type (q, q, q). Then $\mathbf{C}(T_i)$ contains a subgroup of both type (p, p, p) and (q, q, q). Since $\mathbf{C}(T_i)$ is solvable, it follows at once that $p \sim q$. We may therefore assume that $\mathscr{SCN}_3(\mathfrak{Q}_i)$ is empty for each $i = 1, 2, 3$. In particular, this implies that $\mathfrak{T}$ does not centralize at least one $\mathfrak{Q}_i$; for otherwise $\mathfrak{T}$ centralizes $\mathfrak{Q} = \langle \mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3 \rangle$, whence $\mathfrak{Q} = \mathfrak{Q}_1 = \mathfrak{Q}_2 = \mathfrak{Q}_3$, contrary to the fact that $\mathscr{SCN}_3(\mathfrak{Q})$ is non-empty by Lemma 4.2.

Assume for definiteness that $\mathfrak{T}$ does not centralize $\mathfrak{Q}_1$, and set $\mathfrak{C} = \mathbf{C}(T_1)$. Now $\mathfrak{Q}_1$ and $\mathfrak{P}_1 = \mathbf{C}_{\mathfrak{P}}(T_1)$ are each $\mathfrak{S}$-invariant and $\mathfrak{S} \subseteq \mathfrak{C}$. It follows therefore from Lemma 2.4(iv) that $\mathfrak{P}_1$ and $\mathfrak{Q}_1$ each lie in $\mathbf{O}(\mathfrak{C})$. Let $\mathfrak{K}$ be an $\mathfrak{S}$-invariant $S_{p,q}$-subgroup of $\mathbf{O}(\mathfrak{C})$ containing $\mathfrak{P}_1$, and set $\mathfrak{K} = \mathfrak{K}_p \mathfrak{K}_q$, where $\mathfrak{K}_p$ and $\mathfrak{K}_q$ are respectively $\mathfrak{S}$-invariant S_p- and S_q-subgroups of $\mathfrak{K}$. Since $\mathscr{SCN}_3(\mathfrak{K}_p)$ is non-empty, we may assume that $\mathscr{SCN}_3(\mathfrak{K}_q)$ is empty, But then as $p > q$, it follows from **(4**, Lemma 8.5**)** that $\mathfrak{K}_p \lhd \mathfrak{K}$. Furthermore, since $\mathfrak{Q}_1$ is contained in an $\mathfrak{S}$-invariant S_q-subgroup of $\mathbf{O}(\mathfrak{K})$ and since any two such are conjugate by an element of $\mathbf{C}_{\mathbf{O}(\mathfrak{C})}(\mathfrak{S})$, we see that $\mathfrak{T}$ does not centralize $\mathfrak{K}_q$. We have thus proved the existence of an $\mathfrak{S}$-invariant $\{p, q\}$-subgroup $\mathfrak{K} = \mathfrak{K}_p \mathfrak{K}_q$ with the following properties: (a) $\mathscr{SCN}_3(\mathfrak{K}_p)$ is non-empty, (b) $\mathfrak{K}_p \lhd \mathfrak{K}$, and (c) $\mathfrak{S}$ does not centralize $\mathfrak{K}_q$.

Among all such $\{p, q\}$-subgroups of $\mathfrak{G}$, choose $\mathfrak{K}^* = \mathfrak{K}_p^*\mathfrak{K}_q^*$ of maximal order. In virtue of Proposition 4, we may assume without loss that $\mathfrak{K}_p^* \subseteq \mathfrak{P}$. Suppose $\mathfrak{K}_p^* \subset \mathfrak{P}$; and set $\mathfrak{N}^* = \mathbf{N}(\mathfrak{K}_p^*)$. Then $\mathfrak{P} \cap \mathfrak{N}^* \supset \mathfrak{K}_p^*$ and $\mathfrak{P} \cap \mathfrak{N}^* \subseteq \mathbf{O}(\mathfrak{N}^*)$ by Lemma 2.4(iv). Furthermore, by the same lemma, $\mathfrak{K}_q^* \subseteq \mathbf{O}(\mathfrak{N}^*)$. Arguing therefore as in the preceding paragraph, we conclude that either $p \sim q$ or that an $\mathfrak{S}$-invariant $\mathfrak{S}_{p,q}$-subgroup $\mathfrak{K}' = \mathfrak{K}_p'\mathfrak{K}_q'$ satisfies conditions (a), (b), and (c). But in the latter case, $|\mathfrak{K}'| > |\mathfrak{K}^*|$, since $|\mathfrak{K}_p'| \geqslant |\mathfrak{P} \cap \mathbf{O}(\mathfrak{N}^*)| > |\mathfrak{K}_p^*|$ and $|\mathfrak{K}_q'| \geqslant |\mathfrak{K}_q^*|$, contrary to our maximal choice of $\mathfrak{K}^*$. Hence either $p \sim q$ or $\mathfrak{K}_p^* = \mathfrak{P}$.

So finally consider the possibility $\mathfrak{K}_p^* = \mathfrak{P}$ and set $\mathfrak{N} = \mathbf{N}(\mathfrak{P})$. Then, as above, $\mathfrak{K}_q^* \subseteq \mathbf{O}(\mathfrak{N})$. Hence if $\mathfrak{Q}^*$ is a maximal element of $\mathsf{M}_{\mathfrak{N}}(\mathfrak{S}; q)$ containing $\mathfrak{K}_p^*$, $\mathfrak{S}$ does not centralize $\mathfrak{Q}^*$. But $\mathfrak{S} \subseteq [\mathfrak{N}, \mathfrak{N}]$ by Lemma 4.1(ii) and consequently $\mathscr{SCN}_3(\mathfrak{Q}^*)$ is non-empty by Lemma 2.4(iv). Thus $p \sim q$, and the proposition is proved.

7. Elimination of the tame primes. We are now in a position to apply the main results of **(5)** (as modified in Section 1). We begin with an analysis of the set $\sigma^* \cap \pi_4$. (As in **(4, 5,** and **6)**, $\pi_3 = \pi_3(\mathfrak{G})$ and $\pi_4 = \pi_4(\mathfrak{G})$ denote the set of primes p such that $\mathscr{SCN}_3(p)$ is non-empty and $\mathsf{M}(\tilde{\mathfrak{P}})$ is respectively non-trivial or trivial, $\tilde{\mathfrak{P}}$ an S_p-subgroup of $\mathfrak{G}$.)

LEMMA 7.1. *If $p \in \sigma^* \cap \pi_4$ and $\mathfrak{G}$ does not satisfy the uniqueness condition for p, then $\mathfrak{G}$ possesses a subgroup $\mathfrak{X}$ which satisfies the following conditions:*

(i) $\mathbf{O}_{p'}(\mathfrak{X}) = 1$, $\mathbf{O}_p(\mathfrak{X}) \neq 1$, *and $\mathfrak{X}$ contains a subgroup of type (p, p, p).*

(ii) *$\mathbf{O}_p(\mathfrak{X})$ is an S_p-subgroup of $\mathbf{O}(\mathfrak{X})$.*

(iii) *An S_p-subgroup of $\mathbf{N}(\mathbf{O}_p(\mathfrak{X}))$ is an S_p-subgroup of $\mathfrak{X}$.*

(iv) *Either $\mathfrak{X}$ has characteristic p^n with $n > 1$ or $p = 3$ and $\mathfrak{X}/\mathbf{O}(\mathfrak{X})$ is isomorphic to* $\mathrm{P\Gamma L}(2, 8)$.

(v) *If $\mathfrak{T}$ is a four-subgroup of $\mathfrak{X}$, then a maximal element of $\mathsf{M}_{\mathfrak{X}}(\mathfrak{T}; p)$ is a maximal element of $\mathsf{M}(\mathfrak{T}; p)$.*

(vi) *If $\mathfrak{X}$ has characteristic* 8, *then so does any subgroup $\mathfrak{Y}$ of $\mathfrak{G}$ containing $\mathfrak{X}$ such that $p \in \pi_s(\mathfrak{Y})$.*

Proof. Since $\mathfrak{G}$ does not satisfy the uniqueness condition for p, it follows from **(5**, Theorems D and E**)** that $\mathfrak{G}$ is not strongly p-tame. But $\mathfrak{G}$ is p-tame by Proposition 6 and the definition of σ^*; thus either $\mathfrak{G}$ is not p-restricted or not p-reductive. We conclude therefore from the definition of these concepts that there exists a subgroup $\mathfrak{H}$ of $\mathfrak{G}$ satisfying the following conditions:

(a) $\mathbf{O}_{p'}(\mathfrak{H}) = 1$, $\mathbf{O}_p(\mathfrak{H}) \neq 1$, and $\mathfrak{H}$ contains a subgroup of type (p, p, p).

(b) An S_p-subgroup of $\mathfrak{H}$ is an S_p-subgroup of $\mathbf{N}(\mathbf{O}_p(\mathfrak{H}))$.

(c) $\mathfrak{H}$ is either not p-restricted or not p-reductive.

Now by Proposition 1(ii), a subgroup $\mathfrak{H}$ of $\mathfrak{G}$ satisfying condition (a) is p-restricted and p-reductive if either $\mathfrak{H}$ is an A_0-group, $\mathfrak{H}$ is an A_1-group of characteristic 2^n, $n > 3$, or $\mathfrak{H}$ is an A_1-group of characteristic 8 and either $p \neq 3$ or $\mathfrak{H}/\mathbf{O}(\mathfrak{H})$ is not isomorphic to $\mathrm{P\Gamma L}(2, 8)$. Furthermore, by **(6**, Propositions 7 and 8**)**, $\mathfrak{H}$ will also be p-restricted and p-reductive if $\mathfrak{H}$ is an A_1-group of odd characteristic q and $q \neq p^n$ with $n > 1$. Thus one of the following holds:

(d_1) $\mathfrak{H}$ is an A_1-group of characteristic p^n with $n > 1$.

(d_2) $p = 3$ and $\mathfrak{H}/\mathbf{O}(\mathfrak{H})$ is isomorphic to $\mathrm{P\Gamma L}(2, 8)$.

Furthermore, since $\mathfrak{G}$ is p-tame and $\mathfrak{G}$ does not satisfy the uniqueness condition for p, it follows that $\mathfrak{G}$ satisfies Hypotheses E of **(5**, Section 11**)**. Hence by **(5**, Lemma 11.4**)**, if $\tilde{\mathfrak{P}}$ is an S_p-subgroup of $\mathfrak{G}$, then $\mathscr{U}(\tilde{\mathfrak{P}})$ contains an element $\mathfrak{B}$ which is weakly embedded in $\mathfrak{G}$—that is, an element $\mathfrak{B}$ such that $\gamma^2\mathbf{C}(B)\mathfrak{B}^2 = 1$ for each B in $B^\#$.

We first treat the case that $\mathfrak{H}$ satisfies conditions (a), (b), and (d_1). Among all subgroups $\mathfrak{H}$ of $\mathfrak{G}$ satisfying these conditions, choose $\mathfrak{H}$ so that if $\mathfrak{T}$ is an S_2-subgroup of $\mathfrak{H}$, then a maximal element of $\mathsf{M}_{\mathfrak{H}}(\mathfrak{T}; p)$ has maximal order. Let $\mathfrak{P}$ be a maximal element of $\mathsf{M}(\mathfrak{T}; p)$ such that $\mathfrak{P} \cap \mathfrak{H}$ is a maximal element of $\mathsf{M}_{\mathfrak{H}}(\mathfrak{T}; p)$. We shall argue that $\mathfrak{P} \subseteq \mathfrak{H}$.

Since $\mathfrak{G}$ is p-tame and $\mathfrak{H}$ contains a subgroup of type (p, p, p), **(5**, Lemma 3.4**)** implies that $\mathbf{O}_p(\mathfrak{H})$ contains every element of $\mathscr{SCN}_3(\tilde{\mathfrak{P}})$, where $\tilde{\mathfrak{P}}$ is an S_p-subgroup of $\mathfrak{G}$ such that $\tilde{\mathfrak{P}} \cap \mathfrak{H}$ is an S_p-subgroup of $\mathfrak{H}$. Hence if $\mathfrak{N} = \mathbf{N}(\mathbf{O}_p(\mathfrak{H}))$, we have $\mathbf{O}_{p'}(\mathfrak{N}) = 1$ by **(5**, Corollary 4.3**)** since $p \in \pi_4$. But $\mathfrak{H} \subseteq \mathfrak{N}$, whence $\mathfrak{N}$ is an A_1-group of characteristic p^m with $m \geqslant n$ by Lemma 2.6(i). Thus $\mathfrak{N}$ also satisfies conditions (a), (b), and (d_1), so that by our maximal choice of $\mathfrak{H}$, we have that $\mathfrak{P} \cap \mathfrak{H}$ is a maximal element of $\mathsf{M}_{\mathfrak{N}}(\mathfrak{T}; p)$. Since $\mathbf{O}_p(\mathfrak{H}) \subseteq \mathfrak{P} \cap \mathfrak{H}$, it also follows that $\mathbf{Z}(\mathfrak{P}) \subseteq \mathfrak{P} \cap \mathfrak{H}$. Furthermore, $\mathbf{C}_{\mathfrak{H}}(\mathbf{O}_p(\mathfrak{H}))$ is solvable since $\mathfrak{G}$ is weakly p-constrained, and consequently $\mathbf{C}_{\mathfrak{H}}(\mathbf{O}_p(\mathfrak{H})) \subseteq \mathbf{O}_p(\mathfrak{H})$ by **(9**, Lemma 1.2.3**)**. In particular, $\mathbf{Z}(\mathfrak{P}) \subseteq \mathbf{O}_p(\mathfrak{H})$.

We may assume that $\tilde{\mathfrak{P}}$ is chosen to contain $\mathfrak{P} \cap \mathfrak{H}$. Let $\mathfrak{B}$ be an element of $\mathscr{U}(\tilde{\mathfrak{P}})$ which is weakly embedded in $\mathfrak{G}$. Then $\mathfrak{B}$ is contained in some element of $\mathscr{SCN}_3(\tilde{\mathfrak{P}})$ by **(4**, Lemma 8.9**)**, and hence $\mathfrak{B} \subseteq \mathbf{O}_p(\mathfrak{H})$ since $\mathbf{O}_p(\mathfrak{H})$ contains every element of $\mathscr{SCN}_3(\tilde{\mathfrak{P}})$. Thus $\mathfrak{B} = \mathbf{V}(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{B}); \mathfrak{P} \cap \mathfrak{H}) \neq 1$. We now apply **(6**, Lemma 8.3**)**. The argument of the preceding paragraph shows that the hypotheses of that Lemma are satisfied with $\mathfrak{H}$, $\mathfrak{T}$, $\mathfrak{P} \cap \mathfrak{H}$, $\mathbf{Z}(\mathfrak{P})$, and $\mathfrak{B}$ respectively in the roles of $\mathfrak{K}$, $\mathfrak{T}_i$, $\mathfrak{P}$, $\mathfrak{Z}$, and $\mathfrak{D}$; we conclude therefore that

$\mathfrak{H} = \mathfrak{L}\mathbf{N}_{\mathfrak{H}}(\mathfrak{B})$, where $\mathfrak{L}$ is the largest normal subgroup of $\mathfrak{H}$ which centralizes $\mathbf{Z}(\mathfrak{P})$.

Suppose first that $\mathfrak{L} \not\subseteq \mathbf{O}(\mathfrak{H})$. Then $\mathfrak{L}$ is an A_1-group of the same characteristic p^n as $\mathfrak{H}$. Since $\mathfrak{L} \subseteq \mathfrak{C} = \mathbf{C}(\mathbf{Z}(\mathfrak{P}))$, Lemma 2.6(i) implies once again that $\mathfrak{C}$ is an A_1-group of characteristic p^r with $r \geqslant n$. For the same reason $\mathfrak{H}_1 = \mathbf{N}(\mathbf{O}_p(\mathfrak{C}))$ is an A_1-group of characteristic p^t with $t \geqslant r$. Furthermore, $\mathfrak{P} \subseteq \mathfrak{C} \subseteq \mathfrak{H}_1$ and $\mathfrak{T} \subseteq \mathfrak{L} \subseteq \mathfrak{H}_1$. Since $\mathfrak{P}$ contains every element of $\mathscr{SCN}_3(\tilde{\mathfrak{P}})$, $\mathbf{O}_{p'}(\mathfrak{H}_1) = 1$ by **(5**, Corollary 4.3**)**, and it follows that $\mathfrak{H}_1$ satisfies conditions (a), (b), and (d_1). Since $\mathfrak{T}$ is an S_2-subgroup of $\mathfrak{H}_1$ and $\mathfrak{P}$ is a maximal element of $\mathsf{V}\!\!\mathsf{I}(\mathfrak{T}; p)$ contained in $\mathfrak{H}_1$, the maximality of $\mathfrak{H}$ implies that $\mathfrak{P} \subseteq \mathfrak{H}$.

On the other hand, if $\mathfrak{L} \subseteq \mathbf{O}(\mathfrak{H})$, then $\mathfrak{B} \subseteq \mathbf{O}(\mathfrak{H})$ and $\mathbf{N}_{\mathfrak{H}}(\mathfrak{B})$ is an A_1-group of the same characteristic p^n as $\mathfrak{H}$, and $\mathbf{N}_{\mathfrak{H}}(\mathfrak{B})$ contains $\mathfrak{T}$. Reasoning now as in the preceding paragraph we conclude that $\mathfrak{H}_0 = \mathbf{N}(\mathbf{O}_p(\mathbf{N}(\mathfrak{B})))$ satisfies conditions (a), (b), and (d_1). But then if $\mathfrak{P} \not\subseteq \mathfrak{H}$, we have $\mathfrak{P} \cap \mathbf{N}(\mathfrak{B}) \supset \mathfrak{P} \cap \mathfrak{H}$, whence $\mathfrak{P} \cap \mathfrak{H}_0 \supset \mathfrak{P} \cap \mathfrak{H}$ and $\mathfrak{H}_0$ is greater in our ordering than $\mathfrak{H}$. This contradiction shows that $\mathfrak{P} \subseteq \mathfrak{H}$ in this case as well.

Finally we set $\mathfrak{X} = \mathbf{N}(\mathfrak{P} \cap \mathbf{O}(\mathfrak{H}))$. Then $\mathfrak{T} \subseteq \mathfrak{X}$ and, as above, $\mathfrak{X}$ contains every element of $\mathscr{SCN}_3(\tilde{\mathfrak{P}})$, $\mathbf{O}_{p'}(\mathfrak{X}) = 1$, and $\mathfrak{X}$ is an A_1-group of characteristic p^s with $s > 1$. Furthermore, since $\mathfrak{P}$ is a maximal element of $\mathsf{V}\!\!\mathsf{I}(\mathfrak{T}; p)$, $\mathfrak{P} \cap \mathbf{O}(\mathfrak{X})$ is an S_p-subgroup of $\mathbf{O}(\mathfrak{X})$. But since $\mathfrak{H} = \mathbf{O}(\mathfrak{H})(\mathfrak{H} \cap \mathfrak{X})$ by Sylow's Theorem, $\mathfrak{P} \cap \mathbf{O}(\mathfrak{X}) \subseteq \mathbf{O}(\mathfrak{H})$, and consequently

$$\mathfrak{P} \cap \mathbf{O}(\mathfrak{X}) = \mathfrak{P} \cap \mathbf{O}(\mathfrak{H}) \subseteq \mathbf{O}_p(\mathfrak{X}).$$

Thus $\mathbf{O}_p(\mathfrak{X})$ is an S_p-subgroup of $\mathbf{O}(\mathfrak{X})$, and we conclude that $\mathfrak{X}$ satisfies all the conditions of the lemma.

Thus in the balance of the proof we may assume that $\mathfrak{G}$ does not possess a subgroup satisfying conditions (a), (b), and (d_1). Hence every subgroup of $\mathfrak{G}$ satisfying conditions (a) and (b) is either p-restricted and p-reductive, or else satisfies condition (d_2). Consider first the case that $\mathfrak{G}$ possesses a unique subgroup $\mathfrak{M}$ which is maximal subject to containing $\tilde{\mathfrak{P}}$ and such that $p \in \pi_s(\mathfrak{M})$. Set $\mathscr{A}_1^*(\tilde{\mathfrak{P}}) = \{\mathfrak{P}_0 | \mathfrak{P}_0 \subseteq \tilde{\mathfrak{P}}$ and $\mathfrak{A}^M \subseteq \mathfrak{P}_0$ for some $\mathfrak{A}$ in $\mathscr{SCN}_3(\tilde{\mathfrak{P}})$ and suitable M in $\mathfrak{M}\}$. Then clearly $\mathscr{A}_1(\tilde{\mathfrak{P}}) \subseteq \mathscr{A}_1^*(\tilde{\mathfrak{P}})$. Since $\mathfrak{G}$ does not satisfy the uniqueness condition for p, it follows therefore from **(5**, Lemma 11.1**)** that there exists a subgroup $\mathfrak{H}$ of $\mathfrak{G}$ with $\mathfrak{H} \not\subseteq \mathfrak{M}$ such that $\mathfrak{P}_0 \subseteq \mathfrak{H}$ for some element $\mathfrak{P}_0$ of $\mathscr{A}_1^*(\tilde{\mathfrak{P}})$ and $p \in \pi_s(\mathfrak{H})$. Among all such subgroups, choose $\mathfrak{H}$ so that $|\mathfrak{P}_0|$ is maximal. Then as in the proof of **(5**, Lemma 11.1**)**, $\mathfrak{P}_0$ is an S_p-subgroup of $\mathfrak{H}$. Furthermore, since $p \in \pi_4$ and $\mathfrak{P}_0$ contains an element of $\mathscr{SCN}_3(p)$, it follows from **(5**, Corollary 4.3**)** that $\mathbf{O}_{p'}(\mathfrak{H}) = 1$. Since $\mathfrak{H} \subseteq \mathbf{N}(\mathbf{O}_p(\mathfrak{H}))$ and $\mathfrak{H} \not\subseteq \mathfrak{M}$, we have $\mathbf{N}(\mathbf{O}_p(\mathfrak{H})) \not\subseteq \mathfrak{M}$. We conclude at once from our maximal choice of $\mathfrak{H}$ that $\mathfrak{P}_0$ is an S_p-subgroup of $\mathbf{N}(\mathbf{O}_p(\mathfrak{H}))$. But then by **(5**, Lemma 3.4**)**, $\mathbf{O}_p(\mathfrak{H})$ contains every element of $\mathscr{SCN}_3(\tilde{\mathfrak{P}})$ and, in particular, contains $\mathbf{Z}(\tilde{\mathfrak{P}})$. Finally $\mathfrak{P}_0 \subset \tilde{\mathfrak{P}}$ since $\mathfrak{H} \not\subseteq \mathfrak{M}$.

Now set $\mathfrak{K} = \mathbf{O}(\mathfrak{H})\mathfrak{P}_0$ and $\mathfrak{X} = \mathbf{N}(\mathfrak{P})$, where $\mathfrak{P} = \mathfrak{P}_0 \cap \mathbf{O}(\mathfrak{H})$. Then $\mathbf{O}_{p'}(\mathfrak{K}) = 1$ and $\mathbf{Z}(\tilde{\mathfrak{P}}) \subseteq \mathbf{O}_p(\mathfrak{K})$. Since $\mathfrak{K}$ is p-restricted, we can therefore apply **(5**, Lemma 10.3**)** to conclude that $\mathfrak{K} = \mathfrak{L}\mathbf{N}_{\mathfrak{K}}(\mathfrak{V}_0)$, where $\mathfrak{L}$ is the largest normal subgroup of $\mathfrak{K}$ which centralizes $\mathbf{Z}(\tilde{\mathfrak{P}})$ and $\mathfrak{V}_0 = \mathbf{V}(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{V}); \mathfrak{P}_0)$. Now $\mathfrak{L} \subseteq \mathbf{C}(\mathbf{Z}(\tilde{\mathfrak{P}})) \subseteq \mathfrak{M}$. Furthermore, $\mathfrak{P} \cap \mathbf{N}(\mathfrak{V}_0) \supset \mathfrak{P}_0$ and also

$$\mathfrak{P} \cap \mathbf{N}(\mathfrak{V}_0) \in \mathscr{A}_1^*(\tilde{\mathfrak{P}}).$$

But then $\mathbf{N}(\mathfrak{V}_0) \subseteq \mathfrak{M}$ by our maximal choice of $\mathfrak{H}$. Thus $\mathfrak{K} \subseteq \mathfrak{M}$, and consequently $\mathfrak{X} \not\subseteq \mathfrak{M}$. Again by the maximality of $\mathfrak{H}$, we obtain that $\mathfrak{P}_0$ is an S_p-subgroup of $\mathfrak{X}$. Since $\mathfrak{P}_0 \cap \mathbf{O}(\mathfrak{X}) \subseteq \mathbf{O}(\mathfrak{H})$, this implies that

$$\mathfrak{P} = \mathfrak{P}_0 \cap \mathbf{O}(\mathfrak{H}) = \mathfrak{P}_0 \cap \mathbf{O}(\mathfrak{X})$$

is an S_p-subgroup of $\mathbf{O}(\mathfrak{X})$. Furthermore, we have $\mathbf{O}_{p'}(\mathfrak{X}) = 1$ and $\mathbf{Z}(\tilde{\mathfrak{P}}) \subseteq \mathbf{O}_p(\mathfrak{X})$. Hence if $\mathfrak{X}$ were p-restricted, we could apply **(5**, Lemma 10.3**)** to $\mathfrak{X}$ as we did above to $\mathfrak{K}$ to conclude that $\mathfrak{X} \subseteq \mathfrak{M}$, which is not the case. Thus $\mathfrak{X}$ is not p-restricted. But by assumption, $\mathfrak{X}$ is not of characteristic p^n with $n > 1$; and hence $p = 3$ and $\mathfrak{X}/\mathbf{O}(\mathfrak{X})$ is isomorphic to $\mathrm{P\Gamma L}(2, 8)$. Next let $\mathfrak{T}$ be a four-subgroup of $\mathfrak{X}$. By Lemma 2.5(iii) a maximal element of $\mathscr{N}_{\mathfrak{H}}(\mathfrak{T}; p)$ is contained in $\mathbf{O}(\mathfrak{H})$, and consequently $\mathfrak{P}$ is a maximal element of $\mathscr{N}_{\mathfrak{H}}(\mathfrak{T}; p)$. Since $\mathfrak{X} = \mathbf{N}(\mathfrak{P})$, it follows at once that $\mathfrak{P}$ is, in fact, a maximal element of $\mathscr{N}(\mathfrak{T}; p)$. Thus $\mathfrak{X}$ satisfies conditions (i)–(v) of the lemma. We shall show that $\mathfrak{X}$ also satisfies (vi).

Let $\mathfrak{S}_1$ be an S_2-subgroup of $\mathfrak{X}$, which without loss we may assume contains $\mathfrak{T}$. Since $\mathfrak{P}$ is a maximal element of $\mathscr{N}(\mathfrak{T}; p)$, certainly $\mathfrak{P}$ is also a maximal element of $\mathscr{N}(\mathfrak{S}_1; p)$. Now let $\mathfrak{X} \subseteq \mathfrak{Y} \subset \mathfrak{G}$ with $p \in \pi_s(\mathfrak{Y})$ and suppose $\mathfrak{S}_1$ is not an S_2-subgroup of $\mathfrak{Y}$. Then $\mathfrak{Y}$ has characteristic 2^d, $d > 3$, by Lemma 2.5(i), and so $\mathfrak{Y}$ is p-restricted by Proposition 1(ii). But $\mathbf{O}_{p'}(\mathfrak{Y}) = 1$ since $p \in \pi_4$, $\mathfrak{P}_0 \subseteq \mathfrak{Y}$, and $\mathfrak{P}_0$ contains every element of $\mathscr{SCN}_3(\tilde{\mathfrak{P}})$. Furthermore, by our maximal choice of $\mathfrak{H}$, $\mathfrak{P}_0$ is an S_p-subgroup of $\mathbf{N}(\mathbf{O}_p(\mathfrak{Y}))$, whence $\mathbf{Z}(\tilde{\mathfrak{P}}) \subseteq \mathbf{Z}(\mathfrak{P}) \subseteq \mathbf{O}_p(\mathfrak{Y})$. But now, if $\mathfrak{L}$ denotes the largest normal subgroup of $\mathfrak{Y}$ which centralizes $\mathbf{Z}(\tilde{\mathfrak{P}})$, we can apply **(5**, Lemma 10.3**)** once again to conclude that $\mathfrak{Y} = \mathfrak{L}\mathbf{N}_{\mathfrak{Y}}(\mathfrak{V}_0)$. As above, this yields $\mathfrak{X} \subseteq \mathfrak{Y} \subset \mathfrak{M}$. Thus $\mathfrak{S}_1$ is an S_2-subgroup of $\mathfrak{Y}$ and (vi) also holds.

We may therefore assume finally that $\mathfrak{G}$ does not possess a unique subgroup $\mathfrak{M}$ which is maximal subject to containing $\tilde{\mathfrak{P}}$ and $p \in \pi_s(\mathfrak{M})$. Let $\mathfrak{H}$ be a subgroup of $\mathfrak{G}$ containing $\tilde{\mathfrak{P}}$ such that $p \in \pi_s(\mathfrak{H})$. Then $\mathfrak{P} = \tilde{\mathfrak{P}} \cap \mathbf{O}(\mathfrak{H}) \neq 1$. Set $\mathfrak{X} = \mathbf{N}(\mathfrak{P})$, and let $\mathfrak{Y}$ be any subgroup of $\mathfrak{G}$ containing $\mathfrak{X}$ such that $p \in \pi_s(\mathfrak{Y})$. We first consider the case that for each such subgroup $\mathfrak{H}$, there exists such a subgroup $\mathfrak{Y}$ containing $\mathfrak{X}$ which is both p-restricted and p-reductive.

Set $\mathfrak{K} = \mathbf{O}(\mathfrak{H})\tilde{\mathfrak{P}}$. Then $\mathfrak{H} = \mathfrak{K}(\mathfrak{Y} \cap \mathfrak{H})$ by Sylow's Theorem. It follows that for at least one such choice of $\mathfrak{H}$, $\mathbf{Z}(\tilde{\mathfrak{P}})$ is not normal in both $\mathfrak{K}$ and $\mathfrak{Y}$; otherwise $\mathbf{Z}(\tilde{\mathfrak{P}}) \lhd \mathfrak{H}$ for each $\mathfrak{H}$ and $\mathfrak{M} = \mathbf{N}(\mathbf{Z}(\tilde{\mathfrak{P}}))$ would be the unique subgroup of $\mathfrak{G}$ which is maximal subject to $\tilde{\mathfrak{P}} \subseteq \mathfrak{M}$ and $p \in \pi_s(\mathfrak{M})$. But $\mathfrak{K}$, being of odd order, is p-restricted and p-reductive by **(5**, Propositions 6 and 7**)**, and the same is true of $\mathfrak{Y}$ by assumption. Suppose $\mathbf{Z}(\tilde{\mathfrak{P}})$ is not normal in $\mathfrak{K}$. Let $\mathfrak{L}$ be the largest normal subgroup of $\mathfrak{K}$ which centralizes $\mathbf{Z}(\tilde{\mathfrak{P}})$ and set

$\mathfrak{Z} = \mathbf{C}(\mathfrak{L}) \cap \mathbf{Z}(\mathbf{O}_p(\mathfrak{K}))$. Then $\mathfrak{Z} \lhd \mathfrak{K}$, $\mathbf{Z}(\tilde{\mathfrak{P}}) \subseteq \mathfrak{Z}$, and we have $\mathfrak{L} = \mathbf{C}_{\mathfrak{K}}(\mathfrak{Z})$ and $\mathbf{O}_p(\mathfrak{K}/\mathfrak{L}) = 1$. If $\tilde{\mathfrak{P}} \subseteq \mathfrak{L}$, then $\mathfrak{K} = \mathfrak{L}\mathbf{N}_{\mathfrak{K}}(\tilde{\mathfrak{P}})$ by Sylow's Theorem, whence $\mathfrak{K} \subseteq \mathbf{N}(\mathbf{Z}(\tilde{\mathfrak{P}}))$ and $\mathbf{Z}(\tilde{\mathfrak{P}}) \lhd \mathfrak{K}$, a contradiction. Thus the hypotheses of (**5**, Lemma 10.1) are satisfied, and it follows from (**5**, Lemmas 10.1 and 10.2) that $\mathfrak{G}$ possesses an elementary subgroup $\mathfrak{E}$ of order p^3 which is weakly embedded in $\mathfrak{G}$. On the other hand, if $\mathbf{Z}(\tilde{\mathfrak{P}}) \lhd \mathfrak{K}$, then $\mathbf{Z}(\tilde{\mathfrak{P}})$ is not normal in $\mathfrak{Y}$, and we reach the same conclusion by the same argument applied to $\mathfrak{Y}$ in place of $\mathfrak{K}$.

As in (**5**, Section 10), we now set $\mathfrak{W} = \mathbf{V}(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{E}); \tilde{\mathfrak{P}})$ and denote by $\mathfrak{W}^*$ the subgroup of $\tilde{\mathfrak{P}}$ generated by its subgroups $\mathfrak{F}$ which are of index p in $\mathfrak{E}^G$ for suitable G in $\mathfrak{G}$. Since $\mathfrak{K}$ and $\mathfrak{Y}$ are each p-restricted and p-reductive, we can apply (**5**, Lemma 10.4) to each of them to conclude that

$$\mathfrak{K} = (\mathfrak{K} \cap \mathbf{N}(\mathfrak{N}_1))(\mathfrak{K} \cap \mathbf{N}(\mathfrak{N}_2))$$

and

$$\mathfrak{Y} = (\mathfrak{Y} \cap \mathbf{N}(\mathfrak{N}_1)(\mathfrak{Y} \cap \mathbf{N}(\mathfrak{N}_2)),$$

where $\mathfrak{N}_1, \mathfrak{N}_2$ are any one of the three pairs $(\mathbf{Z}(\tilde{\mathfrak{P}}), \mathfrak{W})$, $(\mathbf{Z}(\tilde{\mathfrak{P}}), \mathfrak{W}^*)$, $(\mathbf{Z}(\mathfrak{W}^*), \mathfrak{W})$. Since $\mathfrak{H} = \mathbf{O}(\mathfrak{H})(\mathfrak{Y} \cap \mathfrak{H})$ and $\mathbf{O}(\mathfrak{H}) \subseteq \mathfrak{K}$, it follows that

$$\mathfrak{H} = (\mathfrak{H} \cap \mathbf{N}(\mathfrak{N}_1))(\mathfrak{H} \cap \mathbf{N}(\mathfrak{N}_2))$$

for each of the three pairs $(\mathfrak{N}_1, \mathfrak{N}_2)$ and for each subgroup $\mathfrak{H}$ of $\mathfrak{G}$ containing $\tilde{\mathfrak{P}}$ for which $p \in \pi_s(\mathfrak{H})$. But now the proof of (**5**, Theorem D) applies to show that $\mathfrak{M} = \mathbf{N}(\mathbf{Z}(\tilde{\mathfrak{P}}))\mathbf{N}(\mathfrak{W})$ is a group and that it is the unique subgroup of $\mathfrak{G}$ which is maximal subject to containing $\tilde{\mathfrak{P}}$ and $p \in \pi_s(\mathfrak{M})$, contrary to our present assumption.

There must therefore exist a subgroup $\mathfrak{H}$ such that if any subgroup $\mathfrak{Y}$ contains $\mathfrak{X} = \mathbf{N}(\mathfrak{P})$ and satisfies $p \in \pi_s(\mathfrak{Y})$, then $\mathfrak{Y}$ is either not p-restricted or not p-reductive. Hence under our present assumptions, we conclude that $p = 3$ and that $\mathfrak{Y}/\mathbf{O}(\mathfrak{Y})$ is isomorphic to $\mathrm{P\Gamma L}(2, 8)$ for each choice of $\mathfrak{Y}$. In particular, $\mathfrak{X}/\mathbf{O}(\mathfrak{X})$ is isomorphic to $\mathrm{P\Gamma L}(2, 8)$. Since $\mathfrak{H}$ and $\mathfrak{X}$ contain the S_p-subgroup $\tilde{\mathfrak{P}}$ of $\mathfrak{G}$,

$$\mathfrak{P} = \tilde{\mathfrak{P}} \cap \mathbf{O}(\mathfrak{H}) = \tilde{\mathfrak{P}} \cap \mathbf{O}(\mathfrak{X}),$$

and hence $\mathfrak{P}$ is an S_p-subgroup of $\mathbf{O}(\mathfrak{X})$. Furthermore, $\mathbf{O}_{p'}(\mathfrak{X}) = 1$ since $p \in \pi_4$; and if $\mathfrak{T}$ is any four-subgroup of $\mathfrak{X}$, we conclude, as in the preceding case, that $\mathfrak{P}$ is a maximal element of $\mathsf{V\!\!I}(\mathfrak{T}; p)$. Thus all parts of the lemma hold in this case as well.

PROPOSITION 8. $\mathfrak{G}$ *satisfies the uniqueness condition for all p in $\sigma^* \cap \pi_4$.*

Proof. Let $\mathfrak{X}$ be a subgroup of $\mathfrak{G}$ satisfying the conditions of the preceding lemma. Let $\mathfrak{T}$ be a four-subgroup of $\mathfrak{X}$, let $\mathfrak{T}^*$ be an S_2-subgroup of $\mathfrak{X}$ containing $\mathfrak{T}$, and let $\mathfrak{P}$ be a maximal element of $\mathsf{V\!\!I}_{\mathfrak{X}}(\mathfrak{T}; p)$. Then we know that $\mathfrak{P}$ is a maximal element of $\mathsf{V\!\!I}(\mathfrak{T}; p)$, that $\mathbf{O}_p(\mathfrak{X}) = \mathfrak{P} \cap \mathbf{O}(\mathfrak{X})$, and that either $\mathfrak{X}$ is of characteristic p^n, $n > 1$, or $p = 3$ and $\mathfrak{X}$ is of characteristic 8. If $\mathfrak{X}$ has characteristic p^n, then $\mathfrak{T} = \mathfrak{T}^*$ and hence $\mathfrak{T}^*$ normalizes $\mathfrak{P}$. On the other hand, if $\mathfrak{X}$ has characteristic 8, then $\mathfrak{P} \subseteq \mathbf{O}(\mathfrak{X})$ by Lemma 2.5(iii), whence $\mathfrak{P} = \mathbf{O}_p(\mathfrak{X})$, and so $\mathfrak{T}^*$ normalizes $\mathfrak{P}$ in this case as well. Furthermore, as in Lemma 7.1, we let $\mathfrak{B}$ be an element of $\mathscr{U}(\tilde{\mathfrak{P}})$ which is weakly embedded in $\mathfrak{G}$, where $\tilde{\mathfrak{P}}$ is an S_p-subgroup of $\mathfrak{G}$ such that $\tilde{\mathfrak{P}} \cap \mathfrak{X}$ is an S_p-subgroup of $\mathfrak{X}$. Since $\mathfrak{X}$ contains an S_p-subgroup of $\mathbf{N}(\mathbf{O}_p(\mathfrak{X}))$, $\mathbf{O}_p(\mathfrak{X})$ contains every element of $\mathscr{SCN}_3(\tilde{\mathfrak{P}})$ by **(5**, Lemma 3.4**)**. Since $\mathfrak{B}$ is contained in such an element, $\mathfrak{B} \subset \mathbf{O}_p(\mathfrak{X})$, and hence $\mathfrak{B} = \mathbf{V}(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{B}); \mathfrak{P}) \neq 1$.

For the sake of clarity, we divide the proof into several statements.

(a) If $\mathfrak{X}$ is of characteristic p^n, $n > 1$, then both $\mathbf{N}(\mathbf{Z}(\mathfrak{P}))$ and $\mathbf{N}(\mathfrak{B})$ are A_1-groups. Set $\mathfrak{N} = \mathfrak{P} \cap \mathbf{O}(\mathfrak{X})$, $k = |\mathfrak{P}/\mathfrak{N}|$, and $r = n/k$. Then by Lemma 2.8, r is an integer and either $r = 1$ and $p = 3$ or else $\mathbf{N}_{\mathfrak{X}}(\mathfrak{P})$ is an A_1-group of characteristic p^r. However, since $n > 1$, it follows in the first case that $n = k = 3^t$, $t \geqslant 1$, so that $\mathfrak{X}$ is an A_1-group of characteristic 3^n, $n \geqslant 3$. But then by definition of σ^*, 3 is not in σ^*, contrary to the fact that $p = 3$ is in σ^*. We conclude that $\mathbf{N}(\mathfrak{P})$ is an A_1-group. But as $\mathbf{Z}(\mathfrak{P})$ and $\mathfrak{B}$ are each normal in $\mathfrak{P}$, (a) follows at once.

(b) $\mathfrak{P}$ is permutable with an S_2-subgroup of $\mathfrak{G}$ containing $\mathfrak{T}^*$. Let $\mathfrak{S}$ be an S_2-subgroup of $\mathfrak{G}$ containing $\mathfrak{T}^*$, and let $\mathfrak{Q}$ be a maximal element of $\mathsf{M}(\mathfrak{S}; p)$. If $\mathfrak{T} = \mathfrak{T}^*$, then $\mathbf{N}(\mathfrak{T}) \supset \mathbf{C}(\mathfrak{T})$, and we can assume, in virtue of Proposition 4(i), that $\mathfrak{Q} \subseteq \mathfrak{P}$. On the other hand, if $|\mathfrak{T}^*| = 8$, then clearly $\mathfrak{P}$ is a maximal element of $\mathsf{M}(\mathfrak{T}^*; p)$. In this case, we can assume that $\mathfrak{Q} \subseteq \mathfrak{P}$ by Proposition 4(ii). Now let T_1, T_2, T_3 be the involutions of $\mathfrak{T}$ and let $\mathfrak{P} = \mathfrak{P}_0 \mathfrak{P}_1' \mathfrak{P}_2' \mathfrak{P}_3'$ and $\mathfrak{Q} = \mathfrak{Q}_0 \mathfrak{Q}_1' \mathfrak{Q}_2' \mathfrak{Q}_3'$ be the $\mathfrak{T}$-decompositions of $\mathfrak{P}$ and $\mathfrak{Q}$ respectively. Then by Lemma 4.4(ii), we have $\mathfrak{P}_i' = \mathfrak{Q}_i'$, $i = 1, 2, 3$. Since $\mathfrak{S}$ is abelian, this implies that $\mathfrak{S}$ normalizes

$$\mathfrak{P}^* = \langle \mathfrak{P}_1', \mathfrak{P}_2', \mathfrak{P}_3' \rangle = \langle \mathfrak{Q}_1', \mathfrak{Q}_2', \mathfrak{Q}_3' \rangle.$$

On the other hand, $\mathfrak{P}_0$ normalizes each $\mathfrak{P}_i'$, and hence also normalizes $\mathfrak{P}^*$. Since $\mathfrak{P} = \mathfrak{P}_0 \mathfrak{P}^*$, we conclude that $\mathbf{N}(\mathfrak{P}^*)$ contains both $\mathfrak{S}$ and $\mathfrak{P}$. If $\mathbf{N}(\mathfrak{P}^*)$ is an A_1-group, it follows from Lemma 2.5(iii), since $|\mathfrak{S}| > 4$, that $\mathfrak{P}$ is permutable with an S_2-subgroup $\mathfrak{S}^*$ of $\mathbf{N}(\mathfrak{P}^*)$ containing $\mathfrak{T}$; and the same conclusion follows from Lemma 2.4(iii) if $\mathbf{N}(\mathfrak{P}^*)$ is solvable. But $\mathfrak{S}^*$ is an S_2-subgroup of $\mathfrak{G}$, thus proving (b).

Without loss we may assume that $\mathfrak{S}^* = \mathfrak{S}$. Set $\mathfrak{H} = \mathfrak{S}\mathfrak{P}$. Since $\mathbf{O}_p(\mathfrak{X})$ contains every element of $\mathscr{SCN}_3(\tilde{\mathfrak{P}})$ and $\mathbf{O}_p(\mathfrak{X}) \subseteq \mathfrak{P}$ and since $p \in \pi_4$, it follows from **(5**, Corollary 4.3**)** that $\mathbf{O}_{p'}(\mathfrak{H}) = 1$ and $\mathbf{Z}(\mathfrak{P}) \subseteq \mathbf{O}_p(\mathfrak{H})$. Since $\mathfrak{H}$ is p-restricted by Proposition 1(ii), we can therefore apply **(5**, Lemma 10.3**)** to conclude that $\mathfrak{H} = \mathfrak{L}_0 \mathbf{N}_{\mathfrak{H}}(\mathfrak{B})$, where $\mathfrak{L}_0$ is the largest normal subgroup of $\mathfrak{H}$ which centralizes $\mathbf{Z}(\mathfrak{P})$. We next prove

(c) $\mathfrak{L}_0 \cap \mathfrak{S} \subseteq \mathfrak{T}^*$. Assume by way of contradiction that $\mathfrak{L}_0 \cap \mathfrak{S} \not\subseteq \mathfrak{T}^*$. Since $\mathfrak{L}_0 \mathfrak{T}^* \subseteq \mathfrak{N} = \mathbf{N}(\mathbf{Z}(\mathfrak{P}))$, this implies that $|\mathfrak{S} \cap \mathfrak{N}| > |\mathfrak{T}^*|$. However, if $\mathfrak{X}$ is of characteristic 8, $\mathfrak{T}^*$ is an S_2-subgroup of $\mathfrak{N}$ by Lemma 7.1(vi) since $\mathfrak{X} \subseteq \mathfrak{N}$ and $p \in \pi_s(\mathfrak{N})$. Thus $\mathfrak{X}$ has characteristic p^n, $n > 1$, and $\mathfrak{T} = \mathfrak{T}^*$. Since an S_2-subgroup of $\mathfrak{N}$ has order at least 8, we also have that $\mathfrak{N}$ is of characteristic 2^d by Lemma 2.5(i). Furthermore, since $\mathbf{Z}(\mathfrak{P})$ centralizes $\mathfrak{L}_0 \cap \mathfrak{S} \neq 1$, $\mathfrak{C} = \mathbf{C}(\mathbf{Z}(\mathfrak{P})) \not\subseteq \mathbf{O}(\mathfrak{N})$. Since $\mathfrak{C} \lhd \mathfrak{N}$ and $\mathfrak{N}$ contains no normal subgroup of index 2 by Lemma 2.6(iii), $\mathfrak{C}$ contains an S_2-subgroup of $\mathfrak{N}$ and consequently $\mathfrak{T} \subseteq \mathfrak{C}$. But by Lemma 2.6(vi), $\mathfrak{P} = \mathfrak{R}\mathbf{C}_{\mathfrak{P}}(\mathfrak{T})$, where $\mathfrak{R} = \mathfrak{P} \cap \mathbf{O}(\mathfrak{X})$. It follows therefore from Lemma 2.9 that $\mathfrak{T} \subseteq \mathbf{C}(\mathbf{Z}(\mathfrak{R}))$. Thus $\mathfrak{C}_1 = \mathbf{C}_{\mathfrak{X}}(\mathbf{Z}(\mathfrak{R})) \not\subseteq \mathbf{O}(\mathfrak{X})$. Since $\mathfrak{C}_1 \lhd \mathfrak{X}$, we conclude that $\mathfrak{C}_1$ is an A_1-group of the same characteristic p^n as $\mathfrak{X}$. But since $\mathbf{Z}(\mathfrak{P}) \subseteq \mathbf{Z}(\mathfrak{R})$, $\mathfrak{C}_1 \subseteq \mathbf{C}(\mathbf{Z}(\mathfrak{P})) \subseteq \mathfrak{N}$, and consequently $\mathfrak{N}$ is an A_1-group of characteristic p^r, $r \geqslant n$, by Lemma 2.6(i), contrary to the fact that $\mathfrak{N}$ is of characteristic 2^d.

Using these results, we shall now complete the proof of the proposition. We have $\mathfrak{H} = \mathfrak{L}_0 \mathbf{N}_{\mathfrak{H}}(\mathfrak{V})$, whence $\mathfrak{H}_0 = \mathfrak{L}_0\mathfrak{V} \lhd \mathfrak{H}$. Now $\mathfrak{P} \cap \mathfrak{H}_0$ is an S_p-subgroup of $\mathfrak{H}_0$. Furthermore, since $\mathfrak{L}_0 \cap \mathfrak{S} \subseteq \mathfrak{T}^*$, $\mathfrak{T}^* \cap \mathfrak{H}_0$ is an S_2-subgroup of $\mathfrak{H}_0$, and therefore $\mathfrak{P} \cap \mathfrak{H}_0$ is normalized by an S_2-subgroup of $\mathfrak{H}_0$. But $\mathfrak{H} = \mathfrak{H}_0 \mathfrak{H}_1$ by Sylow's Theorem, where $\mathfrak{H}_1 = \mathbf{N}_{\mathfrak{H}}(\mathfrak{P} \cap \mathfrak{H}_0)$. Thus

$$|\mathfrak{H}| = |\mathfrak{H}_0|\,|\mathfrak{H}_1|/|\mathfrak{H}_0 \cap \mathfrak{H}_1|.$$

Since $\mathfrak{H}_0 \cap \mathfrak{H}_1$ contains an S_2-subgroup of $\mathfrak{H}_0$, it follows that $\mathfrak{H}_1$ contains an S_2-subgroup of $\mathfrak{H}$, and consequently $\mathfrak{P} \cap \mathfrak{H}_0$ is normalized by an S_2-subgroup of $\mathfrak{H}$, which without loss we may assume to be $\mathfrak{S}$ itself. Since $\mathfrak{V} \subseteq \mathfrak{P} \cap \mathfrak{H}_0$, we conclude that $\mathfrak{S} \subseteq \mathfrak{N}^* = \mathbf{N}(\mathfrak{V})$.

Suppose first that $\mathfrak{X}$ is of characteristic 8. Since $\mathfrak{X} \subseteq \mathfrak{N}^*$ and $p \in \pi_s(\mathfrak{N}^*)$, Lemma 7.1(vi) implies that $\mathfrak{N}^*$ is also of characteristic 8. But $\mathfrak{S} \subseteq \mathfrak{N}^*$, and hence $\mathfrak{S}$ is elementary of order 8. Furthermore, $\bar{\mathfrak{X}} = \mathfrak{X}/\mathbf{O}(\mathfrak{X})$ is isomorphic to $P\Gamma L(2, 8)$ by Lemma 7.1(iv), whence $|\mathbf{N}_{\bar{\mathfrak{X}}}(\bar{\mathfrak{S}})/\mathbf{C}_{\bar{\mathfrak{X}}}(\bar{\mathfrak{S}})| = 21$, where $\bar{\mathfrak{S}}$ is the image of $\mathfrak{S}$ is $\bar{\mathfrak{X}}$. We conclude that $|\mathbf{N}(\mathfrak{S})/\mathbf{C}(\mathfrak{S})| = 21$, which contradicts Proposition 2(ii).

Thus $\mathfrak{X}$ has characteristic p^n, $n > 1$. Since $\mathfrak{N}^*$ is an A_1-group by (a), $\mathfrak{N}^*$ is of characteristic $2^{m(\mathfrak{S})}$ by Lemma 2.5(i). But now if $\mathfrak{V} \subseteq \mathfrak{N}$, then $\mathfrak{X} \subseteq \mathfrak{N}^*$, and this is impossible by Lemma 2.6(i). Thus $\mathfrak{V} \not\subseteq \mathfrak{N}$ and hence $\mathfrak{V} \not\subseteq \mathbf{O}(\mathfrak{X})$. But $\mathfrak{X}$ satisfies the hypotheses of **(6**, Lemma 8.3**)**, and consequently $\mathfrak{X} = \mathfrak{L}\mathbf{N}_{\mathfrak{H}}(\mathfrak{V})$, where $\mathfrak{L}$ is the largest normal subgroup of $\mathfrak{X}$ which centralizes $\mathbf{Z}(\mathfrak{P})$. Since $\mathfrak{V} \not\subseteq \mathbf{O}(\mathfrak{X})$, this forces $\mathfrak{L} \not\subseteq \mathbf{O}(\mathfrak{X})$, whence $\mathfrak{L}$ is an A_1-group of characteristic p^n and $\mathfrak{T}$ centralizes $\mathbf{Z}(\mathfrak{P})$. Thus $\mathfrak{L} \subseteq \mathbf{N}(\mathbf{Z}(\mathfrak{P})) = \mathfrak{N}$, and therefore $\mathfrak{N}$ is an A_1-group of characteristic p^m, $m \geqslant n$, by Lemma 2.6(i).

Finally set $\mathfrak{R}^* = \mathfrak{P} \cap \mathbf{O}(\mathfrak{N}^*)$ and $\mathfrak{N}_1 = \mathbf{N}_{\mathfrak{N}^*}(\mathfrak{R}^*)$. Then by Sylow's Theorem $\mathfrak{N}_1$ is an A_1-group of the same characteristic $2^{m(\mathfrak{S})}$ as $\mathfrak{N}^*$. Furthermore, $\mathfrak{T} \subseteq \mathfrak{N}_1$. Also $\mathfrak{P} = \mathfrak{R}^*\mathbf{C}_{\mathfrak{P}}(\mathfrak{T})$ by Lemma 2.5(iii). Since $\mathfrak{T}$ centralizes $\mathbf{Z}(\mathfrak{P})$, Lemma 2.9 implies that $\mathfrak{T}$ centralizes $\mathbf{Z}(\mathfrak{R}^*)$. But then $\mathfrak{C}^* = \mathbf{C}_{\mathfrak{N}_1}(\mathbf{Z}(\mathfrak{R}^*))$, being normal in $\mathfrak{N}_1$, is an A_1-group of characteristic $2^{m(\mathfrak{S})}$. Since $\mathbf{Z}(\mathfrak{P}) \subseteq (\mathbf{Z}(\mathfrak{R}^*))$, $\mathfrak{C}^* \subseteq \mathfrak{N}$, and hence $\mathfrak{N}$ is of characteristic 2^d for some d, contrary to the fact that $\mathfrak{N}$ has characteristic p^m. The proposition is proved.

Proposition 9. *$\sigma^* \cap \pi_4$ is empty.*

Proof. Suppose by way of contradiction that there exists a prime p in

$\sigma^* \cap \pi_4$. Then by the preceding proposition, $\mathfrak{G}$ satisfies the uniqueness condition for p. Let $\mathfrak{S}$ be an S_2-subgroup of $\mathfrak{G}$, let $\mathfrak{P}$ be a maximal element of $\mathsf{M}(\mathfrak{S}; p)$, let $\tilde{\mathfrak{P}}$ be an S_p-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$, and let $\mathfrak{M}$ be the unique subgroup of $\mathfrak{G}$ containing $\tilde{\mathfrak{P}}$ which is maximal subject to the conditions that $\mathfrak{M}$ contains an element of $\mathscr{A}_4(\tilde{\mathfrak{P}})$ and that $p \in \pi_s(\mathfrak{M})$. Then $\mathfrak{S}$ centralizes $\mathbf{O}_p(\mathfrak{M})$ by Proposition 5. Furthermore, $\mathfrak{S} \subseteq \mathfrak{M}$ and $\mathfrak{P} \subseteq \mathbf{O}(\mathfrak{M})$ by Lemma 5.1. Set $\mathfrak{C} = \mathbf{C}_{\mathfrak{M}}(\mathbf{O}_p(\mathfrak{M}))$. Then $\mathfrak{C} \lhd \mathfrak{M}$ and $\mathfrak{S} \subseteq \mathfrak{C}$. It follows therefore that $\gamma\mathfrak{S}\mathfrak{P} \subseteq \mathfrak{P} \cap \mathfrak{C}$. On the other hand, since $p \in \pi_4$, $\mathbf{O}_{p'}(\mathbf{O}(\mathfrak{M})) = 1$. But now as $\mathfrak{P} \subseteq \mathbf{O}(\mathfrak{M})$, (**9**, Lemma 1.2.3) implies that $\mathbf{C}_{\mathfrak{P}}(\mathbf{O}_p(\mathfrak{M})) \subseteq \mathbf{O}_p(\mathfrak{M})$. Thus $\mathfrak{P} \cap \mathfrak{C} \subseteq \mathbf{O}_p(\mathfrak{M})$ and consequently $\gamma\mathfrak{S}\mathfrak{P} \subseteq \mathbf{O}_p(\mathfrak{M})$. Since $\mathfrak{S}$ centralizes $\mathbf{O}_p(\mathfrak{M})$, $\mathfrak{S}$ stabilizes the chain $1 \subset \mathbf{O}_p(\mathfrak{M}) \subseteq \mathfrak{P}$, and we conclude that $\mathfrak{S}$ centralizes $\mathfrak{P}$. But since $p \in \sigma$, $\mathfrak{S}$ does not centralize $\mathfrak{P}$ by Proposition 4 and the definition of σ, and the proposition is proved.

By the preceding proposition, $\sigma^* \subseteq \pi_3$. We shall now show that σ^* is empty. Our argument follows rather closely that of (**6**, Section 9). Assume then that σ^* is non-empty. Since $\mathfrak{G}$ is σ^*-tame by Proposition 7, it follows therefore from (**5**, Theorem A) that $\mathfrak{G}$ satisfies E'_{σ^*}-subgroup and hence that $\mathfrak{G}$ contains an S'_{σ^*}-subgroup. We shall use this result to derive the following lemma.

LEMMA 7.2. *$\mathfrak{G}$ possesses a subgroup $\mathfrak{H}$ with the following properties:*
(i) *$\mathfrak{H}$ contains an S_p-subgroup of $\mathfrak{G}$ for each p in σ^*.*
(ii) *An S_{σ^*}-subgroup of $\mathbf{O}(\mathfrak{H})$ is normal in $\mathfrak{H}$.*
(iii) *$\mathfrak{H}$ possesses a 2-subgroup $\mathfrak{T}$ such that $\mathfrak{T} \subseteq [\mathbf{N}_{\mathfrak{H}}(\mathfrak{T}), \mathbf{N}_{\mathfrak{H}}(\mathfrak{T})]$.*
(iv) *$\mathfrak{T}$ does not centralize $\mathbf{O}_p(\mathfrak{H})$ for some p in σ^*.*

Proof. Let $\mathfrak{L}$ be an S'_{σ^*}-subgroup of $\mathfrak{G}$, and let $\mathfrak{K}$ be an S_{σ^*}-subgroup of $\mathbf{S}(\mathfrak{L})$. Then by definition of an S'_{σ^*}-subgroup, $\mathfrak{K} \lhd \mathfrak{L}$, $\pi(\mathfrak{K}) = \sigma^*$, and $\mathfrak{L}$ contains an S_p-subgroup of $\mathfrak{G}$ for each p in σ^*. In particular, $\mathfrak{K} \subseteq \mathbf{O}(\mathfrak{L})$. Choose p in $\pi(\mathbf{F}(\mathfrak{K}))$. Let $\mathfrak{S}$ be an S_2-subgroup of $\mathfrak{G}$, let $\mathfrak{F}$ be a maximal element of $\mathsf{M}(\mathfrak{S}; p)$, and set $\mathfrak{M} = \mathbf{N}(\mathfrak{F})$. Then by Lemma 4.1, we have $\mathfrak{S} \not\subseteq \mathbf{C}(\mathfrak{F})$, and $\mathfrak{S} \subseteq [\mathbf{N}_{\mathfrak{M}}(\mathfrak{S}), \mathbf{N}_{\mathfrak{M}}(\mathfrak{S})]$. Let $\mathfrak{P}$ be a p-subgroup of $\mathfrak{G}$ of maximal order containing $\mathfrak{F}$ with the property that $\mathfrak{M}_1 = \mathbf{N}(\mathfrak{P})$ possesses a non-trivial 2-subgroup $\mathfrak{T}$ such that $\mathfrak{T} \not\subseteq \mathbf{C}(\mathfrak{P})$ and $\mathfrak{T} \subseteq [\mathbf{N}_{\mathfrak{M}_1}(\mathfrak{T}), \mathbf{N}_{\mathfrak{M}_1}(\mathfrak{T})]$; and assume that $\mathfrak{T}$ is chosen to be of minimal order satisfying these conditions. Finally let $\tilde{\mathfrak{P}}$ be an S_p-subgroup of $\mathfrak{G}$ such that $\tilde{\mathfrak{P}} \cap \mathfrak{M}_1$ is an S_p-subgroup of $\mathfrak{M}_1$. Replacing $\mathfrak{L}$ by a suitable conjugate, if necessary, we can assume without loss that $\tilde{\mathfrak{P}} \subseteq \mathfrak{L}$.

Let $\mathfrak{A} \in \mathscr{SCN}_3(\tilde{\mathfrak{P}})$, let $\mathfrak{B} = \mathbf{V}(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{A}); \tilde{\mathfrak{P}})$, and set $\mathfrak{N}_1 = \mathbf{N}(\mathbf{Z}(\mathfrak{B}))$ and $\mathfrak{D} = \tilde{\mathfrak{P}} \cap \mathbf{O}_{p',p}(\mathfrak{N}_1)$. Then $\mathfrak{B} \subseteq \mathfrak{D}$ by (**5**, Lemma 5.5). Furthermore, if $\mathfrak{P}^*$ is a $\mathfrak{T}$-invariant S_p-subgroup of $\mathbf{O}(\mathfrak{M}_1)$, then certainly $\mathfrak{T} \not\subseteq \mathbf{C}(\mathfrak{P}^*)$, and by Sylow's Theorem, $\mathfrak{T} \subseteq [\mathbf{N}_{\mathfrak{M}^*}(\mathfrak{T}), \mathbf{N}_{\mathfrak{M}^*}(\mathfrak{T})]$, where $\mathfrak{M}^* = \mathbf{N}_{\mathfrak{M}_1}(\mathfrak{P}^*)$. Hence

$$\mathfrak{T} \subseteq [\mathbf{N}_{\mathbf{N}(\mathfrak{P}^*)}(\mathfrak{T}), \mathbf{N}_{\mathbf{N}(\mathfrak{P}^*)}(\mathfrak{T})],$$

and it follows from our maximal choice of $\mathfrak{P}$ that $\mathfrak{P}^* = \mathfrak{P}$. Thus $\mathfrak{P}$ is an S_p-subgroup of $\mathbf{O}(\mathfrak{M}_1) = \mathbf{O}(\mathbf{N}(\mathfrak{P}))$. Since $\mathfrak{P}$ contains a subgroup of type

(p, p, p), and $\mathfrak{G}$ is weakly p-tame, we conclude from **(5**, Lemma 3.4**)** that $\mathfrak{A} \subseteq \mathfrak{P}$. But then $\mathfrak{M}_1 \subseteq \mathfrak{N}_1$ by **(5**, Lemma 5.4**)**. In particular, it follows from the maximality of $\mathfrak{P}$ that $\mathfrak{D} \subseteq \mathfrak{P}$. Furthermore, $\mathbf{Z}(\mathfrak{P}) \subseteq \mathfrak{A}$ since $\mathfrak{A} \subseteq \mathfrak{P}$, and hence $\mathbf{Z}(\mathfrak{P}) \subseteq \mathfrak{B} \subseteq \mathfrak{D}$.

Since $\mathfrak{T} \subseteq [\mathbf{N}_{\mathfrak{M}_1}(\mathfrak{T}), \mathbf{N}_{\mathfrak{M}_1}(\mathfrak{T})]$ and $\mathfrak{T} \not\subseteq \mathbf{C}(\mathfrak{P})$, $\mathbf{N}_{\mathfrak{M}_1}(\mathfrak{T})$ contains a cyclic subgroup $\mathfrak{R}$ of odd order which does not centralize $\mathfrak{T}/\mathfrak{T} \cap \mathbf{C}(\mathfrak{P})$. It follows that $\mathfrak{T}$ possesses an $\mathfrak{R}$-invariant subgroup $\mathfrak{T}_1$ such that $\mathfrak{T}_1 \not\subseteq \mathbf{C}(\mathfrak{P})$ and such that $\mathfrak{R}$ acts irreducibly on $\mathfrak{T}_1/\mathbf{D}(\mathfrak{T}_1)$. But then

$$\mathfrak{T}_1 \subseteq [\mathfrak{T}_1\mathfrak{R}, \mathfrak{T}_1\mathfrak{R}] \subseteq [\mathbf{N}_{\mathfrak{M}_1}(\mathfrak{T}_1), \mathbf{N}_{\mathfrak{M}_1}(\mathfrak{T}_1)].$$

Thus $\mathfrak{T}_1 = \mathfrak{T}$ by our minimal choice of $\mathfrak{T}$, and hence $\mathfrak{R}$ acts irreducibly on $\mathfrak{T}/\mathbf{D}(\mathfrak{T})$.

Now let $\mathfrak{Q}$ be a $\tilde{\mathfrak{P}}$-invariant S_q-subgroup of $\mathbf{O}_{p'}(\mathfrak{K})$ and set $\mathfrak{Z}_q = \mathbf{C}_{\mathbf{Z}(\mathfrak{D})}(\mathfrak{Q})$ and $\mathfrak{Z}^* = \cap_q \mathfrak{Z}_q$, taken over all primes q in $\pi(\mathbf{O}_{p'}(\mathfrak{K}))$. Since $\tilde{\mathfrak{P}}$ normalizes $\mathfrak{Q}$ and $\mathbf{Z}(\mathfrak{D})$, $\mathfrak{Z}_q \lhd \tilde{\mathfrak{P}}$ for each q and hence $\mathfrak{Z}^* \lhd \tilde{\mathfrak{P}}$. Furthermore, $\mathfrak{Z}^*$ centralizes $\mathbf{O}_{p'}(\mathfrak{K})$. On the other hand, since $\tilde{\mathfrak{P}} \cap \mathbf{F}(\mathfrak{K}) \neq 1$ by our choice of p, $\mathfrak{Z}_0 = \mathbf{Z}(\tilde{\mathfrak{P}}) \cap \mathbf{F}(\mathfrak{K}) \neq 1$. But $\mathfrak{Q} \subseteq \mathbf{C}(\mathfrak{Z}_0)$ and $\mathfrak{Z}_0 \subseteq \mathbf{Z}(\tilde{\mathfrak{P}}) \subseteq \mathbf{Z}(\mathfrak{D})$, whence $\mathfrak{Z}_0 \subseteq \mathfrak{Z}_q$ for each q. Thus $\mathfrak{Z}^* \neq 1$.

By **(5**, Lemma 8.1**)**, $\mathbf{O}_{p'}(\mathfrak{K})$ contains a maximal element of $И(\tilde{\mathfrak{P}}; q)$. Then by **(5**, Lemma 4.2**)**, we have $\mathfrak{N}_1 = \mathbf{O}_{p'}(\mathfrak{N}_1)(\mathfrak{N}_1 \cap \mathfrak{N})$, where $\mathfrak{N} = \mathbf{N}(\mathfrak{Q})$. Since $\mathfrak{D} \subseteq \mathfrak{N}_1 \cap \mathfrak{N}$ and $\mathfrak{D}$ is an S_p-subgroup of $\mathbf{O}_{p',p}(\mathfrak{N}_1)$, it follows that $\mathfrak{D}$ is an S_p-subgroup of $\mathbf{O}_{p',p}(\mathfrak{N}_1 \cap \mathfrak{N})$. Hence $\mathfrak{N}_1 = \mathbf{O}_{p'}(\mathfrak{N}_1)\mathfrak{N}_0$, where $\mathfrak{N}_0 = \mathbf{N}_{\mathfrak{N}_1 \cap \mathfrak{N}}(\mathfrak{D})$. But then $\mathfrak{T}^X \subseteq \mathfrak{N}_0$ for some X in $\mathbf{O}_{p'}(\mathfrak{N}_1)$. Clearly $X \in \mathbf{C}(\mathbf{Z}(\mathfrak{B}))$. Since $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{D}$ and $\mathfrak{A}$ is an S_p-subgroup of $\mathbf{C}(\mathfrak{A})$, we have $\mathbf{Z}(\mathfrak{D}) \subseteq \mathbf{Z}(\mathfrak{B})$, whence $X \in \mathbf{C}(\mathbf{Z}(\mathfrak{D}))$. Now $\mathfrak{Q}$ and $\mathfrak{D}$ are each invariant under $\mathfrak{T}^X$, and hence so is $\mathfrak{Z}_q$. But X centralizes $\mathfrak{Z}_q \subseteq \mathbf{Z}(\mathfrak{D})$, and we conclude that $\mathfrak{Z}_q$ is $\mathfrak{T}$-invariant. Furthermore, since $\mathfrak{R} \subseteq \mathfrak{M}_1 \subseteq \mathfrak{N}_1$, we have $\mathfrak{R}^X \subseteq \mathbf{N}_{\mathfrak{N}_1}(\mathfrak{T}^X)$. But since $\mathfrak{N}_1 = \mathbf{O}_{p'}(\mathfrak{N}_1)\mathfrak{N}_0$ and $\mathfrak{T}^X \subseteq \mathfrak{N}_0$,

$$\mathbf{N}_{\mathfrak{N}_1}(\mathfrak{T}^X) \subseteq \mathbf{O}_{p'}(\mathfrak{N}_1)\mathbf{N}_{\mathfrak{N}_0}(\mathfrak{T}^X).$$

Thus if $\mathfrak{R} = \langle R \rangle$, we have $R^X Y \in \mathbf{N}_{\mathfrak{N}_0}(\mathfrak{T}^X)$ for some Y in $\mathbf{O}_{p'}(\mathfrak{N}_1)$. In particular, $R^X Y \in \mathfrak{N}_0$. But now the same argument as above shows that $\mathfrak{Z}_q$ is R-invariant. Since $\mathfrak{R} = \langle R \rangle$ is defined independently of q, we conclude that $\mathfrak{Z}^*$ is invariant under both $\mathfrak{T}$ and $\mathfrak{R}$.

Set $\mathfrak{M}^* = \mathbf{N}(\mathfrak{Z}^*)$. Since $\mathfrak{Z}^* \neq 1$, $\mathfrak{M}^* \subset \mathfrak{G}$. Furthermore,

$$\langle \mathbf{O}_{p'}(\mathfrak{K}), \tilde{\mathfrak{P}}, \mathfrak{T}, \mathfrak{R} \rangle \subseteq \mathfrak{M}^*.$$

Since $\tilde{\mathfrak{P}}$ normalizes $\mathbf{O}_{p'}(\mathfrak{K})$ and $\mathfrak{G}$ is p-tame, $\mathbf{O}_{p'}(\mathfrak{K}) \subseteq \mathbf{O}_{p'}(\mathfrak{M}^*)$ by **(5**, Lemma 4.1**)**. But $\mathbf{O}_{p'}(\mathfrak{K})$ contains a maximal element of $И(\tilde{\mathfrak{P}}; q)$ for each q in $\sigma^* - p$, and hence $\mathbf{O}_{p'}(\mathfrak{K})$ is necessarily an S-subgroup of $\mathbf{O}_{p'}(\mathfrak{M}^*)$. It follows that $\langle \mathfrak{P}, \mathfrak{T}, \mathfrak{R} \rangle = \mathfrak{P}\mathfrak{T}\mathfrak{R}$ normalizes some conjugate of $\mathbf{O}_{p'}(\mathfrak{K})$, which without loss we may take to be $\mathbf{O}_{p'}(\mathfrak{K})$ itself. Moreover, since $\mathfrak{R} \subseteq \mathfrak{M}^*$,

$$\mathfrak{T} \subseteq [\mathbf{N}_{\mathfrak{M}^*}(\mathfrak{T}), \mathbf{N}_{\mathfrak{M}^*}(\mathfrak{T})].$$

As $\mathbf{O}_{p'}(\mathfrak{K})$ is an S-subgroup of $\mathbf{O}_{p'}(\mathfrak{M}^*)$, we also have $\mathfrak{M}^* = \mathbf{O}_{p'}(\mathfrak{M}^*)\mathfrak{M}_0$, where $\mathfrak{M}_0 = \mathbf{N}_{\mathfrak{M}^*}(\mathbf{O}_{p'}(\mathfrak{K}))$. Finally since $\mathfrak{T} \subseteq \mathfrak{M}_0$, it also follows that $\mathfrak{T} \subseteq [\mathbf{N}_{\mathfrak{M}_0}(\mathfrak{T}), \mathbf{N}_{\mathfrak{M}_0}(\mathfrak{T})]$.

First consider the case $\mathbf{O}_{p'}(\mathfrak{K}) \neq 1$, and set $\mathfrak{H}^* = \mathbf{N}(\mathbf{O}_{p'}(\mathfrak{K}))$. Since $\mathfrak{L} \subseteq \mathfrak{H}^*$, $\mathfrak{H}^*$ contains an S_r-subgroup of $\mathfrak{G}$ for each r in σ^*; and in particular, $\mathfrak{H}^*$ contains $\tilde{\mathfrak{P}}$. Furthermore, since $\mathfrak{M}_0 \subseteq \mathfrak{H}^*$, we also have $\mathfrak{T} \subseteq [\mathbf{N}_{\mathfrak{H}^*}(\mathfrak{T}), \mathbf{N}_{\mathfrak{H}^*}(\mathfrak{T})]$. Now let $\mathfrak{E}$ be a $\mathfrak{T}\mathfrak{R}$ invariant S_{σ^*}-subgroup of $\mathbf{O}(\mathfrak{H}^*)$ containing $\mathfrak{P} \cap \mathbf{O}(\mathfrak{H}^*)$ and set $\mathfrak{H} = \mathbf{N}_{\mathfrak{H}^*}(\mathfrak{E})$. Then it follows immediately that $\mathfrak{H}$ satisfies conditions (i), (ii), and (iii) of the lemma. We shall show that condition (iv) also holds. If $\mathfrak{H}$ is solvable or if $\mathfrak{H}$ is an A_1-group of characteristic 2^n, then

$$\mathfrak{P} = (\mathfrak{P} \cap \mathbf{O}(\mathfrak{H}))\mathbf{C}_{\mathfrak{P}}(\mathfrak{T})$$

by Lemmas 2.4(iii) and 2.5(iii). On the other hand, if $\mathfrak{H}$ is an A_1-group of odd characteristic, the same conclusion follows from Lemma 2.6(vi) since $\mathfrak{T}\mathfrak{R} \subseteq \mathbf{N}_{\mathfrak{H}}(\mathfrak{P})$. But $\mathfrak{P} \not\subseteq \mathbf{C}(\mathfrak{T})$, and consequently $\mathfrak{P} \cap \mathbf{O}(\mathfrak{H}) \not\subseteq \mathbf{C}(\mathfrak{T})$. Thus $\mathfrak{T}$ does not centralize a $\mathfrak{T}$-invariant S_p-subgroup $\mathfrak{E}_p$ of $\mathfrak{E}$. Suppose $\mathfrak{T} \subseteq \mathbf{C}(\mathbf{F}(\mathfrak{E}))$. Then

$$\gamma\mathfrak{T}\mathfrak{E}_p \subseteq \mathbf{C}_{\mathfrak{E}}(\mathbf{F}(\mathfrak{E})) \cap \mathfrak{E}_p = \mathbf{F}(\mathfrak{E}) \cap \mathfrak{E}_p,$$

and $\mathfrak{T}$ stabilizes the chain $1 \subseteq \mathbf{F}(\mathfrak{E}) \cap \mathfrak{E}_p \subseteq \mathfrak{E}_p$. Thus $\mathfrak{T}$ centralizes $\mathfrak{E}_p$, a contradiction. Hence $\mathfrak{T} \not\subseteq \mathbf{C}(\mathbf{F}(\mathfrak{E}))$ and we conclude that $\mathfrak{T}$ does not centralize $\mathbf{O}_r(\mathfrak{H})$ for some prime r in σ^*, proving (iv).

Finally consider the case $\mathbf{O}_{p'}(\mathfrak{K}) = 1$. In this case, $\mathfrak{B} \subseteq \mathbf{O}_p(\mathfrak{K})$ by (**5**, Lemma 5.5). Hence $\mathfrak{L} \subseteq \mathfrak{N}_1$ by (**5**, Lemma 5.4) and consequently $\mathfrak{N}_1$ contains an S_r-subgroup of $\mathfrak{G}$ for each r in σ^*. Since $\mathfrak{M}_1 \subseteq \mathfrak{N}_1$, we also have

$$\mathfrak{T} \subseteq [\mathbf{N}_{\mathfrak{N}_1}(\mathfrak{T}), \mathbf{N}_{\mathfrak{N}_1}(\mathfrak{T})]$$

and $\mathfrak{R} \subseteq \mathfrak{N}_1$. But now if we let $\mathfrak{E}$ be a $\mathfrak{T}$-invariant S_{σ^*}-subgroup of $\mathbf{O}(\mathfrak{N}_1)$ containing $\mathfrak{P} \cap \mathbf{O}(\mathfrak{N}_1)$ and set $\mathfrak{H} = \mathbf{N}_{\mathfrak{N}_1}(\mathfrak{E})$, we conclude as in the preceding paragraph that $\mathfrak{H}$ satisfies the requirements of the lemma. This completes the proof.

We now let $\mathfrak{H}$ be a subgroup of $\mathfrak{G}$ satisfying the conditions of the previous lemma and let $\mathfrak{E}$ be an S_{σ^*}-subgroup of $\mathbf{O}(\mathfrak{H})$. Then $\mathfrak{E} \lhd \mathfrak{H}$. Furthermore, $\mathfrak{H}$ contains a 2-subgroup $\mathfrak{T}$ such that $\mathfrak{T} \subseteq [\mathbf{N}_{\mathfrak{H}}(\mathfrak{T}), \mathbf{N}_{\mathfrak{H}}(\mathfrak{T})]$ and such that $\mathfrak{T}$ does not centralize $\mathfrak{P} = \mathbf{O}_p(\mathfrak{E})$ for some prime p. We set $\mathfrak{P}^* = \gamma\mathfrak{P}\mathfrak{T}$. Finally let $\tilde{\mathfrak{P}}$ be an S_p-subgroup of $\mathfrak{H}$ (and hence also of $\mathfrak{G}$) such that $\tilde{\mathfrak{P}} \cap \mathfrak{E}$ is $\mathfrak{T}$-invariant, let $\mathfrak{A} \in \mathscr{SCN}_3(\tilde{\mathfrak{P}})$, and set $\mathfrak{B} = \mathbf{V}(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{A}); \tilde{\mathfrak{P}})$ and $\mathfrak{N}_1 = \mathbf{N}(\mathbf{Z}(\mathfrak{B}))$. Then we have

LEMMA 7.3. *$\mathfrak{P}^*$ centralizes every element of $\mathscr{U}(\tilde{\mathfrak{P}}; q)$ for all primes q, with the possible exception $q = 3$. If $\mathfrak{P}^*$ does not centralize every element of $\mathscr{U}(\tilde{\mathfrak{P}}; 3)$, then $3 \in \sigma - \sigma^*$.*

Proof. We first argue that $\mathfrak{P}^*$ centralizes every element of $\mathscr{U}(\tilde{\mathfrak{P}}; q)$ for q in $\tau = \sigma^* - p$. Since $\mathfrak{P} \lhd \tilde{\mathfrak{P}}$, $\mathfrak{Z} = \mathfrak{P} \cap \mathbf{Z}(\tilde{\mathfrak{P}}) \neq 1$. Set $\mathfrak{K} = \mathbf{N}(\mathfrak{Z})$. Since

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$\mathfrak{P} \subseteq \mathbf{F}(\mathfrak{E})$, $\mathbf{O}_{p'}(\mathfrak{E}) \subseteq \mathfrak{K}$. Furthermore, since $\mathbf{O}_{p'}(\mathfrak{E})$ is $\tilde{\mathfrak{P}}$-invariant and $\tilde{\mathfrak{P}} \subseteq \mathfrak{K}$, **(5**, Lemma 4.1**)** implies that $\mathbf{O}_{p'}(\mathfrak{E}) \subseteq \mathbf{O}_{p'}(\mathfrak{K})$. Hence by **(5**, Lemma 8.1**)**, $\mathbf{O}_{p'}(\mathfrak{E})$ is an S_τ-subgroup of $\mathbf{O}_{p'}(\mathfrak{K})$. Since $\mathfrak{P}^*$ centralizes $\mathbf{O}_{p'}(\mathfrak{E})$ and since any two $\mathfrak{P}^*$-invariant S_τ-subgroups of $\mathbf{O}_{p'}(\mathfrak{K})$ are conjugate by an element of $\mathbf{C}_{\mathfrak{K}}(\mathfrak{P}^*)$, we conclude from $D_{\sigma*}$ in $\mathbf{O}_{p'}(\mathfrak{K})\mathfrak{P}^*$ that $\mathfrak{P}^*$ centralizes every τ-subgroup of $\mathbf{O}_{p'}(\mathfrak{K})$ which it normalizes.

Now let $\mathfrak{Q}$ be a maximal element of $\mathsf{M}(\tilde{\mathfrak{P}}; q)$ with q in τ. Then $\mathbf{O}_{p'}(\mathfrak{E})$ contains a maximal element $\mathfrak{Q}^*$ of $\mathsf{M}(\tilde{\mathfrak{P}}; q)$; and by **(5**, Theorem 1**)**, we have $\mathfrak{Q}^* = \mathfrak{Q}^X$ for some X in $\mathbf{C}(\mathfrak{A})$. Now $\mathbf{C}(\mathfrak{A}) = \mathfrak{A} \times \mathfrak{D}$, where $\mathfrak{D}$ is a p'-group by **(4**, (3.10)**)**. Since $\mathfrak{Q}$ is $\mathfrak{A}$-invariant, we may assume that $X \in \mathfrak{D}$. But $\mathfrak{Z} \subseteq \mathfrak{A}$, whence $\mathfrak{D} \subseteq \mathfrak{K} = \mathbf{N}(\mathfrak{Z})$, and consequently $\mathfrak{D} \subseteq \mathbf{O}_{p'}(\mathfrak{K})$ by **(5**, Lemma 4.1**)**. Since $X \in \mathfrak{D}$ and $\mathfrak{Q}^* \subseteq \mathbf{O}_{p'}(\mathfrak{K})$, it follows that $\mathfrak{Q} \subseteq \mathbf{O}_{p'}(\mathfrak{K})$. But then $\mathfrak{P}^*$ centralizes $\mathfrak{Q}$ by the preceding paragraph. We conclude at once that $\mathfrak{P}^*$ centralizes every element of $\mathsf{M}(\tilde{\mathfrak{P}}; q)$ for q in τ.

Before treating the case $q \notin \tau$, we make a preliminary remark. We have $\mathfrak{B} \subseteq \tilde{\mathfrak{P}} \cap \mathfrak{E}$ by **(5**, Lemma 5.5**)**, and hence $\mathfrak{H}_1 = \mathbf{N}_{\mathfrak{G}}(\tilde{\mathfrak{P}} \cap \mathfrak{E}) \subseteq \mathfrak{N}_1$ by **(5**, Lemma 5.4**)**. Since $\mathfrak{T} \subseteq \mathfrak{H}_1$, we conclude that $\mathfrak{T} \subseteq \mathfrak{N}_1$.

Now let $\mathfrak{Q}$ be a maximal element of $\mathsf{M}(\tilde{\mathfrak{P}}; q)$, $q \neq p$, and set $\mathfrak{N} = \mathbf{N}(\mathfrak{Q})$. Then $\mathfrak{N}_1 = \mathbf{O}_{p'}(\mathfrak{N}_1)(\mathfrak{N}_1 \cap \mathfrak{N})$ by **(5**, Lemma 4.2**)**. Let $\mathfrak{L} = \mathbf{N}_{\mathfrak{N}_1}(\mathfrak{P}^*)$. Then by Sylow's Theorem, we have

$$\mathbf{O}_{p'}(\mathfrak{N}_1)\mathfrak{L} = \mathbf{O}_{p'}(\mathfrak{N}_1)(\mathfrak{L} \cap \mathfrak{N}).$$

Since $\mathfrak{T} \subseteq \mathfrak{L}$, it follows that $\mathfrak{T}^* = \mathfrak{T}^Y \subseteq \mathfrak{L} \cap \mathfrak{N}$ for some Y in $\mathbf{O}_{p'}(\mathfrak{N}_1)$. Furthermore, $\mathfrak{P}^* = \gamma\mathfrak{T}\mathfrak{P}^*$ by **(4**, Lemma 8.11**)**, since $\mathfrak{P}^* = \gamma\mathfrak{T}\mathfrak{P}$ by definition. But then $\bar{\mathfrak{P}}^* = \gamma\bar{\mathfrak{T}}\bar{\mathfrak{P}}^*$, where $\bar{\mathfrak{T}}$, $\bar{\mathfrak{P}}^*$ denote the images of $\mathfrak{T}$ and $\mathfrak{P}^*$ in $\bar{\mathfrak{N}} = \mathfrak{N}_1/\mathbf{O}_{p'}(\mathfrak{N}_1)$. Since $\mathfrak{T}^*$ and $\mathfrak{T}$ have the same images in $\bar{\mathfrak{N}}_1$, we conclude at once that $\gamma\mathfrak{T}^*\mathfrak{P}^* = \mathfrak{P}^*$.

Now $\mathfrak{T}^* \subseteq \mathfrak{L} \cap \mathfrak{N}$ and hence $\mathfrak{T}^*$ normalizes $\mathfrak{Q}$. If $\mathfrak{T}^*$ centralizes $\mathfrak{Q}$, then $\gamma\mathfrak{T}^*\mathfrak{P}^* \subseteq \mathbf{C}(\mathfrak{Q})$. Since $\gamma\mathfrak{T}^*\mathfrak{P}^* = \mathfrak{P}^*$, $\mathfrak{P}^*$ centralizes $\mathfrak{Q}$ in this case. Suppose that $\mathfrak{T}^*$ does not centralize $\mathfrak{Q}$. Since $\mathfrak{T} \subseteq [\mathbf{N}_{\mathfrak{H}}(\mathfrak{T}), \mathbf{N}_{\mathfrak{H}}(\mathfrak{T})]$, $\mathfrak{T}$ is certainly non-cyclic, and hence also $\mathfrak{T}^*$ is non-cyclic. But then $\mathfrak{Q} = \langle \mathbf{C}_{\mathfrak{Q}}(T) | T \in \mathfrak{T}^{*\#} \rangle$. Since each $\mathbf{C}_{\mathfrak{Q}}(T)$ is invariant under $\mathfrak{T}^*$, there must therefore exist an element $T \neq 1$ in $\mathfrak{T}^*$ such that $\mathfrak{T}^*$ does not centralize $\mathfrak{Q}_0 = \mathbf{C}_{\mathfrak{Q}}(T)$. Set $\mathfrak{C} = \mathbf{C}(T)$. Then $\mathfrak{C}$ is solvable and hence $\mathfrak{Q}_1 = \gamma\mathfrak{Q}_0\mathfrak{T}^* \subseteq \mathbf{O}(\mathfrak{C})$ by Lemma 2.4(iii). Let $\mathfrak{Q}_2$ be a $\mathfrak{T}^*$-invariant S_q-subgroup of $\mathbf{O}(\mathfrak{C})$ containing $\mathfrak{Q}_1$, and let $\mathfrak{S}$ be an S_2-subgroup of $\mathfrak{C}$ containing $\mathfrak{T}^*$ and normalizing $\mathfrak{Q}_2$. Now $\mathfrak{T}^*$ does not centralize $\mathfrak{Q}_1$; otherwise $\mathfrak{T}^*$ would centralize $\mathfrak{Q}_0$ by **(4**, Lemma 8.11**)**. Since $\mathfrak{T}^* \subseteq \mathfrak{S}$ and $\mathfrak{Q}_1 \subseteq \mathfrak{Q}_2$, $\mathfrak{S}$ does not centralize $\mathfrak{Q}_2$. But $\mathfrak{S}$ is an S_2-subgroup of $\mathfrak{G}$, and $\mathfrak{Q}_2 \in \mathsf{M}(\mathfrak{S}; q)$.

We conclude therefore from the definition that $q \in \sigma$. But either $\sigma = \sigma^*$ or $\sigma = \sigma^* \cup \{3\}$. Since $\mathfrak{P}^*$ centralizes every element of $\mathsf{M}(\tilde{\mathfrak{P}}; q)$ for q in τ by the first part of the proof, the lemma follows.

Let p be defined as in Lemma 7.3. Then we have

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PROPOSITION 10. $\mathfrak{G}$ *satisfies the uniqueness condition for* p.

Proof. We preserve the notation of Lemma 7.3. We have

$$\mathfrak{T} \subseteq [\mathbf{N}_{\mathfrak{H}}(\mathfrak{T}), \mathbf{N}_{\mathfrak{H}}(\mathfrak{T})],$$

$\mathfrak{P}^* = \gamma\mathfrak{T}\mathfrak{P}$, and $\mathfrak{P} \not\subseteq \mathbf{C}(\mathfrak{T})$. Furthermore, since $\mathfrak{P} \lhd \mathfrak{H}$, $\mathbf{N}_{\mathfrak{H}}(\mathfrak{T})$ normalizes $\mathfrak{P}^*$. Suppose $\mathfrak{P}^*$ were cyclic. Then $[\mathbf{N}_{\mathfrak{H}}(\mathfrak{T}), \mathbf{N}_{\mathfrak{H}}(\mathfrak{T})]$ would centralize $\mathfrak{P}^*$, whence $\gamma\mathfrak{T}\mathfrak{P}^* = 1$. But then $\mathfrak{P} \subseteq \mathbf{C}(\mathfrak{T})$ by (**4**, Lemma 8.11), a contradiction. Thus $\mathfrak{P}^*$ is non-cyclic, and therefore the normal closure $\mathfrak{B}^*$ of $\mathfrak{P}^*$ in $\tilde{\mathfrak{P}}$ is non-cyclic. But then $\mathfrak{B}^*$ contains a subgroup $\mathfrak{B}$ of type (p, p) which is normal in $\tilde{\mathfrak{P}}$.

Now let $\mathfrak{Q}$ be an element of $\mathsf{M}(\tilde{\mathfrak{P}}; q)$. Then by the preceding lemma, either $\mathfrak{P}^*$ centralizes $\mathfrak{Q}$ or else $q = 3$. However, in the first case, $\mathfrak{P}^* \subseteq \mathbf{O}_p(\mathfrak{Q}\tilde{\mathfrak{P}})$, by (**9**, Lemma 1.2.3), whence $\mathfrak{B}^* = \mathfrak{B}^{\tilde{\mathfrak{P}}} \subseteq \mathbf{O}_p(\mathfrak{Q}\tilde{\mathfrak{P}})$. But this implies that $\mathfrak{B}^*$ centralizes $\mathfrak{Q}$. We conclude that $\mathfrak{B}$ centralizes every element of $\mathsf{M}(\tilde{\mathfrak{P}}; q)$ for all but at most the prime $q = 3$. But now the hypotheses of (**5**, Theorems C and E) are satisfied, and therefore $\mathfrak{G}$ satisfies the uniqueness condition for p.

This leads to the following result.

PROPOSITION 11. $\mathfrak{G}$ *does not contain an* A_1*-subgroup of characteristic* 3^t, $t > 1$. *In particular,* $\sigma = \sigma^*$.

Proof. By Proposition 6 and the definition of σ^*, either $\sigma = \sigma^*$ or $\sigma = \sigma^* \cup \{3\}$ and $\mathfrak{G}$ possesses an A_1-subgroup of characteristic 3^t, $t \geqslant 3$. Thus the second assertion of the proposition will follow at once from the first.

Suppose first that σ^* is non-empty. Then $\mathfrak{G}$ satisfies the uniqueness condition for some prime p in $\sigma^* \cap \pi_3$ by the preceding proposition. But in this case Proposition 5(ii) implies that every A_1-subgroup of $\mathfrak{G}$ has characteristic 2^n for some n. But then by Lemma 2.6(i) no A_1-subgroup of $\mathfrak{G}$ is of characteristic 3^t, $t > 1$.

Suppose next that σ^* is empty. Since σ is non-empty and either $\sigma = \sigma^*$ or $\sigma = \sigma^* \cup \{3\}$, this implies that $\sigma = \{3\}$. Assume by way of contradiction that $\mathfrak{G}$ possesses an A_1-subgroup $\mathfrak{H}$ of characteristic 3^t, $t > 1$. Let $\mathfrak{T}$ be an S_2-subgroup of $\mathfrak{H}$. Then by Lemma 2.6(vii) there exists a prime $q \neq 3$ such that if $\mathfrak{Q}$ is a maximal element of $\mathsf{M}_{\mathfrak{H}}(\mathfrak{T}; q)$, then $\mathfrak{T}$ does not centralize $\mathfrak{Q}$. Arguing now as at the end of the proof of Lemma 7.3, we see that there exists an element T in $\mathfrak{T}^\#$ such that $\mathfrak{T}$ normalizes, but does not centralize, $\mathfrak{Q}_0 = \mathbf{C}_{\mathfrak{Q}}(T)$. Setting $\mathfrak{C} = \mathbf{C}(T)$, we have $\mathfrak{Q}_1 = \gamma\mathfrak{Q}_0\mathfrak{T} \subseteq \mathbf{O}(\mathfrak{C})$ by Lemma 2.4(iii), and also $\mathfrak{Q}_1 \neq 1$. Then if $\mathfrak{Q}_2$ is a $\mathfrak{T}$-invariant S_q-subgroup of $\mathbf{O}(\mathfrak{C})$ containing $\mathfrak{Q}_1$ and if $\mathfrak{S}$ is an S_2-subgroup of $\mathfrak{C}$ containing $\mathfrak{T}$ and normalizing $\mathfrak{Q}_2$, we have $\mathfrak{Q}_2 \in \mathsf{M}(\mathfrak{S}; q)$ and $\mathfrak{Q}_2 \not\subseteq \mathbf{C}(\mathfrak{S})$. But then $q \in \sigma$, contrary to the fact that $\sigma = \{3\}$ and $q \neq 3$.

Combining our results, we prove finally

PROPOSITION 12. σ *is empty.*

Proof. First of all, $\sigma = \sigma^*$ by the preceding proposition; and secondly

$\sigma^* \cap \pi_4$ is empty by Proposition 9. Thus $\sigma \subseteq \pi_3$. Suppose by way of contradiction that σ is non-empty. Since $\sigma = \sigma^* \subseteq \pi_3$, we can choose a prime p in σ so that the conditions of Lemma 7.3 hold for $\mathfrak{P}^*$, where $\bar{\mathfrak{P}}$, $\mathfrak{H}$, $\mathfrak{T}$, $\mathfrak{P}$, and $\mathfrak{P}^*$ are defined as in that lemma. Then $\mathfrak{G}$ satisfies the uniqueness condition for p by Proposition 10. Let $\mathfrak{M}$ be the corresponding subgroup of $\mathfrak{G}$ which is unique subject to containing an element of $\mathscr{A}_4(\bar{\mathfrak{P}})$ and $p \in \pi_s(\mathfrak{M})$. Then $\mathfrak{H} \subseteq \mathfrak{M}$ since $\bar{\mathfrak{P}} \subseteq \mathfrak{H}$ and $\mathfrak{P} = \mathbf{O}_p(\mathfrak{H}) \neq 1$. Thus $\mathfrak{T} \subseteq [\mathbf{N}_{\mathfrak{M}}(\mathfrak{P}), \mathbf{N}_{\mathfrak{M}}(\mathfrak{P})]$. If $\mathfrak{M}$ is an A_1-group of odd characteristic, then $\mathfrak{T}$ is an S_2-subgroup of $\mathfrak{M}$, and it follows from Lemma 2.6(vi) that $\mathfrak{P} = (\mathfrak{P} \cap \mathbf{O}(\mathfrak{M}))\mathbf{C}_{\mathfrak{P}}(\mathfrak{T})$. Since $\mathfrak{P}^* = \gamma\mathfrak{P}\mathfrak{T}$, this implies that $\mathfrak{P}^* \subseteq \mathbf{O}(\mathfrak{M})$. On the other hand, if $\mathfrak{M}$ is an A_0-group or an A_1-group of characteristic 2^n, the same conclusion follows from Lemma 2.4(iii) and 2.5(iii). But $\bar{\mathfrak{P}}$ normalizes an S_q-subgroup $\mathfrak{Q}$ of $\mathbf{O}_{p'}(\mathbf{F}(\mathfrak{M}))$ for each $q \neq p$ in $\pi(\mathbf{F}(\mathfrak{M}))$. Since $\sigma = \sigma^*$, $\mathfrak{P}^*$ centralizes $\mathfrak{Q}$ by Lemma 7.3, and consequently $\mathfrak{P}^*$ centralizes $\mathbf{O}_{p'}(\mathbf{F}(\mathfrak{M}))$. This implies that $\mathfrak{T} \not\subseteq \mathfrak{C} = \mathbf{C}_{\mathfrak{M}}(\mathbf{O}_p(\mathfrak{M}))$; for otherwise, $\mathfrak{P}^* = \gamma\mathfrak{T}\mathfrak{P}^* \subseteq \mathfrak{C}$, whence $\mathfrak{P}^*$ centralizes $\mathbf{F}(\mathfrak{M})$. But then $\mathfrak{P}^* \subseteq \mathbf{O}_p(\mathfrak{M})$. Since $\mathfrak{T}$ does not centralize $\mathfrak{P}^*$, we conclude that $\mathfrak{T}$ does not centralize $\mathbf{O}_p(\mathfrak{M})$. But now if $\mathfrak{S}$ is an S_2-subgroup of $\mathfrak{M}$ containing $\mathfrak{T}$, $\mathfrak{S}$ does not centralize $\mathbf{O}_p(\mathfrak{M})$. However, $\mathfrak{S}$ centralizes $\mathbf{O}_p(\mathfrak{M})$ by Proposition 5. This contradiction completes the proof of the proposition.

Since σ is non-empty by Proposition 3, Proposition 12 shows that no simple group exists satisfying the assumptions of Theorem 3. Thus Theorem 3, and consequently also Theorem 1, is proved.

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