ON THE ÉTALE $K$-THEORY OF AN ELLIPTIC CURVE WITH COMPLEX MULTIPLICATION FOR REGULAR PRIMES

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ABSTRACT. Generalizing a result of Soulé we prove that for an elliptic curve $E$ defined over an imaginary quadratic field $K$ with complex multiplication having good ordinary reduction at the prime number $p > 3$ which is regular for $E$ and the extension $F$ of $K$ contained in $K(E_p)$ the dimensions of the étale $K$-groups are equal to the numbers predicted by Bloch and Beilinson, i.e.,

$$\dim K_i^{ét}(E \times_K F, \mathbb{Q}_p/\mathbb{Z}_p) = [F : \mathbb{Q}] \quad \text{for all} \quad i \geq 2.$$ 

Let $E$ be an elliptic curve defined over a number field $F$ with potential good reduction. Then the rank of the $K$-group $K_{2j-2}(E)$ for an integer $j \geq 2$ should conjecturally be equal to the degree $[F : \mathbb{Q}]$ (Bloch, Beilinson) which is conjecturally the order of vanishing of the $L$-function $L(E, s)$ at $s = 2 - j$ (Serre). In [9] Soulé proved that the $\mathbb{Z}_p$-corank of the étale $K$-group $K_2^{ét}(E, \mathbb{Q}_p/\mathbb{Z}_p)$, which is isomorphic to $K_2(E, \mathbb{Q}_p/\mathbb{Z}_p)$ by the theorem of Merkujew and Suslin, is exactly $[F : \mathbb{Q}]$ if $E$ has complex multiplication and $p$ is assumed to be regular for $E/F$ in the sense of Yager [11].

Using the Dwyer-Friedlander and the Hochschild-Serre spectral sequence it is easy to see that for $j \geq 2$ the equality

$$\dim K_{2j-2}^{ét}(E, \mathbb{Q}_p/\mathbb{Z}_p) = [F : \mathbb{Q}]$$

is equivalent to the vanishing of a certain Galois cohomology group:

$$H^2(\text{Gal}(F_S/F), H^1(\bar{E}, \mathbb{Q}_p/\mathbb{Z}_p(j))) = 0.$$ 

Here $S$ is a finite set of primes of $F$ containing $S_p = \{v \mid p\}$ and all primes where $E$ has bad reduction; $F_S$ denotes the maximal $S$-ramified extension and $\bar{E}$ is $E \times_F \bar{F}$.

The last assertion is a special case of a conjecture of Jannsen concerning arbitrary smooth projective varieties over number fields [2].

Our aim is to generalize Soulé’s result to all $j \geq 2$. Let $K$ be an imaginary quadratic field and let $E$ be an elliptic curve defined over $K$ with complex multiplication by an order of $K$. Let $p > 3$ be a prime number which splits in $K$, i.e., $p = \wp\bar{p}$, and where

Received by the editors June 15, 1988.
AMS 1980 Subject Classification: 11G05, 19E08, 19E20.
E has good (ordinary) reduction. Let $\mathcal{F} = K(E_p)$ and let $F$ be a finite extension of $K$ contained in $\mathcal{F}$. Let $\chi_1$ and $\chi_2$ be the canonical characters with values in $\mathbb{Z}_p^\times$ given by the action of $\text{Gal}(\mathcal{F}/K)$ on the $p$ and $\bar{p}$ division points of $E$ respectively.

If $\psi$ denotes the Hecke character of $E$, $\bar{\psi}$ its conjugate, and $L(\bar{\psi}^k, s)$ the primitive $L$-function attached to the powers of $\bar{\psi}$ ($k \in \mathbb{Z}$, $s \in \mathbb{C}$), then by Damerell's theorem the complex numbers

$$L_\infty(\bar{\psi}^{k+j}, k) = \left( \frac{2\pi}{\sqrt{d_k}} \right)^j \Omega_\infty^{-k+j} L(\bar{\psi}^{k+j}, k)$$

lie in $\bar{K}$ when $k \geq 1$ and $j \geq 0$ (here $\Omega_\infty$ denotes the complex period). If $0 \leq j \leq p - 1$ and $1 < k \leq p$ then the numbers are $p$-integral. By definition, $p$ is regular for $E$ and $F$ if $p$ does not divide the numbers $L_\infty(\bar{\psi}^{k+j}, k)$ for all integers $j, k$ with $1 \leq j < p - 1$ and $1 < k \leq p$ such that $\chi_1^j \chi_2^{-j}$ is a non-trivial character belonging to $F$, i.e., $\chi_1^k \chi_2^{-j}$ is trivial when restricted to $\text{Gal}(\mathcal{F}/F)$.

According to a theorem of Yager [11] we know:

- $p$ is regular for $E$ and $F$ $\iff$ $F_{S_p}(p)$ is a $\mathbb{Z}_p$-extension of $F$;
- here $F_{S_p}(p)$ denotes the maximal $p$-extension of $F$ unramified outside $S_p = \{ v \mid p \}$.

If $F_\nu$ denotes the completion of $F$ with respect to a prime $\nu$ then by the theorem of Grunwald-Hasse-Wang the maximal $p$-extension $F_\nu(p)$ of $F_\nu$ coincides with the completion of the maximal $p$-extension $F(p)$ of $F$ with respect to $\nu$:

$$F_\nu(p) = (F_\nu)(p),$$

(see the proof of Theorem 11.3 in [5]). Consider now the compositum of maps

$$\varphi_\nu : \text{Gal}(F_\nu(p)/F_\nu) \hookrightarrow \text{Gal}(F(p)/F) \twoheadrightarrow \text{Gal}(F_{S_p}(p)/F)$$

where the first map is the inclusion of a decomposition group with respect to an extension of $\nu$ to $F(p)$ in the global group and the second map is the canonical surjection on the Galois group of the maximal $p$-extension $F_{S_p}(p)$ of $F$ unramified outside $S_p$.

We say: The Galois group $\text{Gal}(F_{S_p}(p)/F)$ is purely local with respect to $\nu$ if $\varphi_\nu$ is an isomorphism:

$$\text{Gal}(F_\nu(p)/F_\nu) \xrightarrow{\varphi_\nu \sim} \text{Gal}(F_{S_p}(p)/F).$$

**Theorem.** The prime $p$ is regular for $E$ and $F$ if and only if $\text{Gal}(F_{S_p}(p)/F)$ is purely local with respect to $\bar{p}$.

**Corollary 1.** Let $p$ be regular for $E$ and $F$, let $S \supseteq S_p$ be a set of primes of $F$ and let $j \in \mathbb{Z}$. Furthermore let $M$ be a $p$-primary divisible $\text{Gal}(F_{S_p}(p)/F)$-module of cofinite type such that for all $\nu \in S \setminus S_p$ with $\mu_p \subset F_\nu$ the $\text{Gal}(F_\nu(p)/F_\nu)$-coinvariants of $M(j - 1)$ are zero:

$$M(j - 1)_{\text{Gal}(F_\nu(p)/F_\nu)} = 0.$$
Then

\[ H^2(\text{Gal}(F_S/F), M(j)) = 0. \]

**Corollary 2.** Let \( p \) be regular for \( E \) and \( \mathcal{F} \), i.e., \( p \) and \( \mathcal{F} \) are regular for \( E/\mathcal{F} \), let \( F \) be an extension of \( K \) inside \( \mathcal{F} \) and let \( S \) be a set of primes of \( F \) containing \( S_p \) and all primes where \( E \times_K F \) has bad reduction, then

\[ H^2(\text{Gal}(F_S/F), H^1(\mathcal{E}, \mathbb{Q}_p/\mathbb{Z}_p(j))) = 0 \]

for all \( j \in \mathbb{Z} \).

**Corollary 3.** Let \( p \) be regular for \( E \) and \( \mathcal{F} \). Then for an extension \( F \) of \( K \) contained in \( \mathcal{F} \)

\[ \dim K^\text{ét}_i(E \times_K F, \mathbb{Q}_p/\mathbb{Z}_p) = [F : \mathbb{Q}] \]

for all \( i \geq 2 \).

**Proof of the Theorem.** Consider the commutative and exact diagram

\[
\begin{align*}
1 & \longrightarrow \text{Gal}(F_S(p)/F_S(p)) \longrightarrow \text{Gal}(F_S(p)/F) \longrightarrow \text{Gal}(F_S(p)/F) \longrightarrow 1 \\
& \text{\quad \quad \quad} \uparrow \varphi_{\tilde{p}} \quad \quad \quad \uparrow \psi_{\tilde{p}} \\
1 & \longrightarrow I(F_{\tilde{p}}(p)/F_{\tilde{p}}) \longrightarrow \text{Gal}(F_{\tilde{p}}(p)/F_{\tilde{p}}) \longrightarrow \text{Gal}(F_{\tilde{p}}^{nr}(p)/F_{\tilde{p}}) \longrightarrow 1
\end{align*}
\]

where \( F_{\tilde{p}}^{nr}(p) \) is the maximal unramified \( p \)-extension of \( F_{\tilde{p}} \) and \( I(F_{\tilde{p}}(p)/F_{\tilde{p}}) \) denotes the inertia subgroup of \( \text{Gal}(F_{\tilde{p}}(p)/F_{\tilde{p}}) \). Now, if \( \varphi_{\tilde{p}} \) is an isomorphism then \( \psi_{\tilde{p}} \) is surjective, hence \( F_S(p)/F \) is a \( \mathbb{Z}_p \)-extension. By the result of Yager \( p \) is regular for \( E \) and \( F \).

Conversely, the induced map \( \psi_{\tilde{p}} \) is an isomorphism if \( p \) is regular. Therefore \( \varphi_{\tilde{p}} \) is surjective, since its restriction to the inertia subgroup is surjective; indeed the normal subgroup generated by the image of \( I(F_{\tilde{p}}(p)/F_{\tilde{p}}) \) is the whole group \( \text{Gal}(F_S(p)/F_S(p)) \), since there is only one prime of the \( \mathbb{Z}_p \)-extension \( F_S(p) \) above \( \tilde{p} \) and \( F_S(p) \) has no \( p \)-extension unramified outside \( S_p \). But \( p \)-groups are nilpotent, hence the assertion follows.

Now let \( R \) be the kernel of \( \varphi_{\tilde{p}} \). The Hochschild-Serre spectral sequence implies an exact sequence

\[
0 \longrightarrow H^1(\text{Gal}(F_S(p)/F), \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow H^1(\text{Gal}(F_{\tilde{p}}(p)/F_{\tilde{p}}), \mathbb{Q}_p/\mathbb{Z}_p) \]

\[
\longrightarrow H^1(R, \mathbb{Q}_p/\mathbb{Z}_p) \xrightarrow{\text{Gal}(F_{\tilde{p}}/F)} 0
\]

because \( H^2(\text{Gal}(F_S(p)/F), \mathbb{Q}_p/\mathbb{Z}_p) = 0 \), i.e., the Leopoldt conjecture is true for abelian extensions of \( K \). The (in)-equalities

\[
\text{corank}_{\mathbb{Z}_p} H^1(\text{Gal}(F_S(p)/F), \mathbb{Q}_p/\mathbb{Z}_p) = [F : K] + 1
\]

\[
= \text{corank}_{\mathbb{Z}_p} H^1(\text{Gal}(F_{\tilde{p}}(p)/F_{\tilde{p}}), \mathbb{Q}_p/\mathbb{Z}_p)
\]
and

\[ \dim_{\mathbb{F}_p} H^1(\text{Gal}(F_{S_p}(p)/F), \mathbb{Z}/p\mathbb{Z}) \geq [F : K] + 1 + \delta \]

\[ = \dim_{\mathbb{F}_p} H^1(\text{Gal}(F_{\overline{p}}(p)/F_{\overline{p}}), \mathbb{Z}/p\mathbb{Z}) \]

(\(\delta = 1\) if \(F_{\overline{p}}\) contains the group \(\mu_p\) of \(p\)-th roots of unity and \(\delta = 0\) otherwise). [3] Satz 11.8, show that

\[ H^1(R, \mathbb{Q}_p/\mathbb{Z}_p)^{\text{Gal}(F_{S_p}(p)/F)} = 0 \]

and therefore \(R = 0\). This finishes the proof of the theorem. \(\square\)

**Proof of Corollary 1.** According to [6] Theorem 1

\[ H^2(\text{Gal}(F_{S_p}(p)/F), M(j)) = H^2(\text{Gal}(F_{S_p}(p)/F), M(j)). \]

Furthermore \(\text{Gal}(F_{S_p}(p)/F_{S_p}(p))\) is the free pro-\(p\)-product of all inertia groups with respect to primes \(v\) of \(F_{S_p}(p)\) above \(S \setminus S_p\), in particular \(\text{Gal}(F_{S_p}(p)/F_{S_p}(p))\) is a free pro-\(p\)-group (see [10], Theorem 2.2, which goes back on a slightly weaker theorem of Neumann and also Neukirch in the case \(F = \mathbb{Q}\)). Therefore the Hochschild-Serre spectral sequence yields an exact sequence

\[ H^2(\text{Gal}(F_{S_p}(p)/F), M(j)) \rightarrow H^2(\text{Gal}(F_{S_p}(p)/F), M(j)) \]

\[ \rightarrow H^1(\text{Gal}(F_{S_p}(p)/F), H^1(\text{Gal}(F_{S_p}(p)/F_{S_p}(p)), M(j))). \]

Since \(M(j)\) is a trivial \(G(F_{S_p}(p)/F_{S_p}(p))\)-module the group on the right is equal to

\[ \bigoplus_{v \in S \setminus S_p} H^1(\text{Gal}(F_{S_p}(p)/F_v), H^1(I(F_v(p)/F_v), M(j))) \]

\[ = \bigoplus_{v \in S \setminus S_p} H^2(\text{Gal}(F_v(p)/F_v), M(j)) \]

by [4] Satz 4.1 and Shapiro’s lemma. If \(\mu_p\) is not contained in \(F_v\) then \(\text{Gal}(F_v(p)/F_v)\) is free; otherwise it is a Poincaré group of dimension two with dualizing module \(\mathbb{Q}_p/\mathbb{Z}_p(1)\), hence

\[ H^2(\text{Gal}(F_v(p)/F_v), M(j)) = \lim_{m} H^0(\text{Gal}(F_v(p)/F_v), \text{Hom}(\rho_{m}M(j), \mathbb{Q}_p/\mathbb{Z}_p(1))^*) \]

\[ = M(j - 1)_{\text{Gal}(F_v(p)/F_v)} = 0 \]

(\(\rho_{m}M := \{x \in M \mid p^m x = 0\}\)).

Therefore we have reduced the corollary to the case \(S = S_p\). But, since \(\text{Gal}(F_{S_p}(p)/F)\) is purely local with respect to \(\overline{p}\), we obtain

\[ H^2(\text{Gal}(F_{S_p}(p)/F), M(j)) = H^2(\text{Gal}(F_{\overline{p}}(p)/F_{\overline{p}}), M(j)) \]

https://doi.org/10.4153/CMB-1990-025-x Published online by Cambridge University Press
which is zero if $\mu_p \not\subseteq F_p$ and otherwise equal to $M(j-1)_{\text{Gal}(F_p/F_p)}$ which is zero by our assumption. This proves Corollary 1.

**Proof of Corollary 2.** Observing that

$$H^1(\bar{E}, \mathbb{Q}_p / \mathbb{Z}_p(1)) = E_{p, \infty} = E_{p, \infty} \oplus E_{p, \infty}$$

and that the order of $\text{Gal}(\mathcal{F}/F)$ is prime to $p$ it is enough to show that

$$H^2(\text{Gal}(\mathcal{F}/F), E_{p, \infty}(j)) = 0$$

for all $j \in \mathbb{Z}$. But $E \times_K \mathcal{F}$ has good reduction everywhere, hence $E_{p, \infty}$ is a $p$-primary divisible $\text{Gal}(\mathcal{F}_S(p)/\mathcal{F})$-module. Now Corollary 1 implies the result because the $\text{Gal}(\mathcal{F}_S(p)/\mathcal{F}_\nu)$-coinvariants of $E_{p, \infty}(j-1)$ are zero for all $j \in \mathbb{Z}$ and all $\nu \in S$. 

**Proof of Corollary 3.** From the Dwyer-Friedlander spectral sequence [1]

$$E_2^{i,j} = \left\{ \begin{array}{ll} H^i(E \times_K F, \mathbb{Q}_p / \mathbb{Z}_p(j)), & t = -2j \ 
0, & t \text{ odd} \end{array} \right\} \Rightarrow K^\text{et}_{-s-t}(E \times_K F, \mathbb{Q}_p / \mathbb{Z}_p)$$

and the Hochschild-Serre spectral sequence

$$E_2^{i,j} = H^i(F, H^j(\bar{E}, \mathbb{Q}_p / \mathbb{Z}_p(j))) \Rightarrow H^{i+j}(E \times_K F, \mathbb{Q}_p / \mathbb{Z}_p(j))$$

we obtain for $j \geq 2$

$$\dim K^\text{et}_{2j-2}(E \times_K F, \mathbb{Q}_p / \mathbb{Z}_p) = \dim H^2(E, \mathbb{Q}_p / \mathbb{Z}_p(j))$$

$$= \dim H^1(F, H^1(\bar{E}, \mathbb{Q}_p / \mathbb{Z}_p(j)))$$

and

$$\dim K^\text{et}_{2j-1}(E \times_K F, \mathbb{Q}_p / \mathbb{Z}_p) = \dim H^1(E, \mathbb{Q}_p / \mathbb{Z}_p(j)) + \dim H^3(E, \mathbb{Q}_p / \mathbb{Z}_p(j+1))$$

$$= 2 \dim H^1(F, \mathbb{Q}_p / \mathbb{Z}_p(j)) + \dim H^2(F, H^1(\bar{E}, \mathbb{Q}_p / \mathbb{Z}_p(j)))$$

(using $H^2(\bar{E}, \mathbb{Q}_p / \mathbb{Z}_p(1)) = \mathbb{Q}_p / \mathbb{Z}_p$ and $H^2(F, \mathbb{Q}_p / \mathbb{Z}_p(j)) = 0$ for $j \neq 1$, [7] Satz 4.1 (ii)). Since

$$\dim H^1(F, \mathbb{Q}_p / \mathbb{Z}_p(j)) = [F : K] + \dim H^2(\text{Gal}(F_{S_p}/F), \mathbb{Q}_p / \mathbb{Z}_p(j)),$$

[7] 4.5 (iii), Satz 4.6,

$$\dim H^1(F, H^1(\bar{E}, \mathbb{Q}_p / \mathbb{Z}_p(j))) = \dim H^1(\text{Gal}(F_{S_p}/F), H^1(\bar{E}, \mathbb{Q}_p / \mathbb{Z}_p(j))),$$

[2] Lemma 2.4 (or see the proof of Proposition 1 in [8]) and

$$\sum_{k=0}^{2} (-1)^k \dim H^k(\text{Gal}(F_{S_p}/F), H^1(\bar{E}, \mathbb{Q}_p / \mathbb{Z}_p(j))) = -[F : Q]$$
where $S$ is a finite set of primes of $F$ containing $S_p$ and all primes where $E \times_K F$ has bad reduction, [8] Proposition 2, we obtain ($j \geq 2$):

\[
\dim K_{2j-2}^p(E \times_K F, \mathbb{Q}_p/\mathbb{Z}_p) = [F : \mathbb{Q}] + \dim H^2(\text{Gal}(F_S/F), H^1(\bar{E}, \mathbb{Q}_p/\mathbb{Z}_p(j))),
\]

\[
\dim K_{2j-1}^p(E \times_K F, \mathbb{Q}_p/\mathbb{Z}_p) = [F : \mathbb{Q}] + 2 \dim H^2(\text{Gal}(F_{S_p}/F), \mathbb{Q}_p/\mathbb{Z}_p(j)) + \dim H^2(F, H^1(\bar{E}, \mathbb{Q}_p/\mathbb{Z}_p(j))).
\]

Now Corollary 2 completes the proof because as in the proof of Corollary 1 for $j \neq 1$

\[
\dim H^2(\text{Gal}(F_{S_p}/F), \mathbb{Q}_p/\mathbb{Z}_p(j)) = \dim H^2(\text{Gal}(F_S/F), \mathbb{Q}_p/\mathbb{Z}_p(j)) \]

\[
= \dim H^2(\text{Gal}(F_S/F), \mathbb{Q}_p/\mathbb{Z}_p(j))^{\text{Gal}(\bar{F}/F)}
\]

\[
= \dim H^2(\text{Gal}(F_{S_p}(p)/\bar{F}), \mathbb{Q}_p/\mathbb{Z}_p(j))^{\text{Gal}(\bar{F}/F)}
\]

\[
= \dim H^2(\text{Gal}(F_{S_p}(p)/\bar{F}, \mathbb{Q}_p/\mathbb{Z}_p(j))^{\text{Gal}(\bar{F}/F})
\]

\[
= 0.
\]

\textbf{REFERENCES}


