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ON THE ÉTALE K-THEORY OF AN ELLIPTIC CURVE WITH COMPLEX MULTIPLICATION FOR REGULAR PRIMES

BY

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ABSTRACT. Generalizing a result of Soulé we prove that for an elliptic curve E defined over an imaginary quadratic field K with complex multiplication having good ordinary reduction at the prime number $p > 3$ which is regular for E and the extension F of K contained in $K(E_p)$ the dimensions of the étale K-groups are equal to the numbers predicted by Bloch and Beilinson, i.e.,

$$\dim K_i^{ét}(E \times_K F, \mathbf{Q}_p/\mathbf{Z}_p) = [F : \mathbf{Q}] \quad \text{for all} \quad i \geqq 2.$$

Let E be an elliptic curve defined over a number field F with potential good reduction. Then the rank of the K-group $K_{2j-2}(E)$ for an integer $j \geqq 2$ should conjecturally be equal to the degree $[F : \mathbf{Q}]$ (Bloch, Beilinson) which is conjecturally the order of vanishing of the L-function $L(E, s)$ at $s = 2 - j$ (Serre). In [9] Soulé proved that the $\mathbf{Z}_p$-corank of the étale K-group $K_2^{ét}(E, \mathbf{Q}_p/\mathbf{Z}_p)$, which is isomorphic to $K_2(E, \mathbf{Q}_p/\mathbf{Z}_p)$ by the theorem of Merkujew and Suslin, is exactly $[F : \mathbf{Q}]$ if E has complex multiplication and p is assumed to be regular for E/F in the sense of Yager [11].

Using the Dwyer-Friedlander and the Hochschild-Serre spectral sequence it is easy to see that for $j \geqq 2$ the equality

$$\dim K_{2j-2}^{ét}(E, \mathbf{Q}_p/\mathbf{Z}_p) = [F : \mathbf{Q}]$$

is equivalent to the vanishing of a certain Galois cohomology group:

$$H^2(\mathrm{Gal}(F_S/F),\ H^1(\bar{E}, \mathbf{Q}_p/\mathbf{Z}_p(j))) = 0.$$

Here S is a finite set of primes of F containing $S_p = \{v \mid p\}$ and all primes where E has bad reduction; F_S denotes the maximal S-ramified extension and $\bar{E}$ is $E \times_F \bar{F}$.

The last assertion is a special case of a conjecture of Jannsen concerning arbitrary smooth projective varieties over number fields [2].

Our aim is to generalize Soulé's result to all $j \geqq 2$. Let K be an imaginary quadratic field and let E be an elliptic curve defined over K with complex multiplication by an order of K. Let $p > 3$ be a prime number which splits in K, i.e., $p = \mathfrak{p}\bar{\mathfrak{p}}$, and where

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E has good (ordinary) reduction. Let $\mathcal{F} = K(E_p)$ and let F be a finite extension of K contained in $\mathcal{F}$. Let χ_1 and χ_2 be the canonical characters with values in $\mathbf{Z}_p^\times$ given by the action of $\mathrm{Gal}(\mathcal{F}/K)$ on the $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ division points of E respectively.

If ψ denotes the Hecke character of E, $\bar{\psi}$ its conjugate and $L(\bar{\psi}^k, s)$ the primitive L-function attached to the powers of $\bar{\psi}$ ($k \in \mathbf{Z}$, $s \in \mathbf{C}$), then by Damerell's theorem the complex numbers

$$L_\infty(\bar{\psi}^{k+j}, k) = \left(\frac{2\pi}{\sqrt{d_k}}\right)^j \Omega_\infty^{-(k+j)} L(\bar{\psi}^{k+j}, k)$$

lie in $\bar{K}$ when $k \geqq 1$ and $j \geqq 0$ (here Ω_∞ denotes the complex period). If $0 \leqq j \leqq p-1$ and $1 < k \leqq p$ then the numbers are $\mathfrak{p}$-integral. By definition, $\mathfrak{p}$ is regular for E and F if $\mathfrak{p}$ does not divide the numbers $L_\infty(\bar{\psi}^{k+j}, k)$ for all integers j, k with $1 \leqq j < p-1$ and $1 < k \leqq p$ such that $\chi_1^k \chi_2^{-j}$ is a non-trivial character belonging to F, i.e., $\chi_1^k \chi_2^{-j}$ is trivial when restricted to $\mathrm{Gal}(\mathcal{F}/F)$.

According to a theorem of Yager [11] we know:

$\mathfrak{p}$ is regular for E and $F \Leftrightarrow F_{S_\mathfrak{p}}(p)$ is a $\mathbf{Z}_p$-extension of F;

here $F_{S_\mathfrak{p}}(p)$ denotes the maximal p-extension of F unramified outside $S_\mathfrak{p} = \{v \mid \mathfrak{p}\}$.

If F_v denotes the completion of F with respect to a prime v then by the theorem of Grunwald-Hasse-Wang the maximal p-extension $F_v(p)$ of F_v coincides with the completion of the maximal p-extension $F(p)$ of F with respect to v:

$$F_v(p) = (F_v)(p),$$

(see the proof of Theorem 11.3 in [5]). Consider now the compositum of maps

$$\varphi_v : \mathrm{Gal}(F_v(p)/F_v) \hookrightarrow \mathrm{Gal}(F(p)/F) \twoheadrightarrow \mathrm{Gal}(F_{S_p}(p)/F)$$

where the first map is the inclusion of a decomposition group with respect to an extension of v to $F(p)$ in the global group and the second map is the canonical surjection on the Galois group of the maximal p-extension $F_{S_p}(p)$ of F unramified outside S_p.

We say: The Galois group $\mathrm{Gal}(F_{S_p}(p)/F)$ is *purely local* with respect to v if φ_v is an isomorphism:

$$\mathrm{Gal}(F_v(p)/F_v) \xrightarrow[\sim]{\varphi_v} \mathrm{Gal}(F_{S_p}(p)/F).$$

THEOREM. *The prime $\mathfrak{p}$ is regular for E and F if and only if $\mathrm{Gal}(F_{S_p}(p)/F)$ is purely local with respect to $\bar{\mathfrak{p}}$.*

COROLLARY 1. *Let $\mathfrak{p}$ be regular for E and F, let $S \supseteq S_p$ be a set of primes of F and let $j \in \mathbf{Z}$. Furthermore let M be a p-primary divisible $\mathrm{Gal}(F_{S_p}(p)/F)$-module of cofinite type such that for all $v \in S \backslash S_\mathfrak{p}$ with $\mu_p \subset F_v$ the $\mathrm{Gal}(F_v(p)/F_v)$-coinvariants of $M(j-1)$ are zero:*

$$M(j-1)_{\mathrm{Gal}(F_v(p)/F_v)} = 0.$$

Then

$$H^2(\mathrm{Gal}(F_S/F), M(j)) = 0.$$

COROLLARY 2. *Let p be regular for E and $\mathcal{F}$, i.e., $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ are regular for $E/\mathcal{F}$, let F be an extension of K inside $\mathcal{F}$ and let S be a set of primes of F containing S_p and all primes where $E \times_K F$ has bad reduction, then*

$$H^2(\mathrm{Gal}(F_S/F),\ H^1(\bar{E}, \mathbf{Q}_p/\mathbf{Z}_p(j))) = 0$$

for all $j \in \mathbf{Z}$.

COROLLARY 3. *Let p be regular for E and $\mathcal{F}$. Then for an extension F of K contained in $\mathcal{F}$*

$$\dim K_i^{\acute{e}t}(E \times_K F, \mathbf{Q}_p/\mathbf{Z}_p) = [F : \mathbf{Q}]$$

for all $i \geqq 2$.

PROOF OF THE THEOREM. Consider the commutative and exact diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Gal}(F_{S_p}(p)/F_{S_{\mathfrak{p}}}(p)) & \longrightarrow & \mathrm{Gal}(F_{S_p}(p)/F) & \longrightarrow & \mathrm{Gal}(F_{S_{\mathfrak{p}}}(p)/F) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow{\scriptstyle \varphi_{\bar{\mathfrak{p}}}} & & \uparrow{\scriptstyle \psi_{\bar{\mathfrak{p}}}} & & \\
1 & \longrightarrow & I(F_{\bar{\mathfrak{p}}}(p)/F_{\bar{\mathfrak{p}}}) & \longrightarrow & \mathrm{Gal}(F_{\bar{\mathfrak{p}}}(p)/F_{\bar{\mathfrak{p}}}) & \longrightarrow & \mathrm{Gal}(F_{\bar{\mathfrak{p}}}^{nr}(p)/F_{\bar{\mathfrak{p}}}) & \longrightarrow & 1
\end{array}$$

where $F_{\bar{\mathfrak{p}}}^{nr}(p)$ is the maximal unramified p-extension of $F_{\bar{\mathfrak{p}}}$ and $I(F_{\bar{\mathfrak{p}}}(p)/F_{\bar{\mathfrak{p}}})$ denotes the inertia subgroup of $\mathrm{Gal}(F_{\bar{\mathfrak{p}}}(p)/F_{\bar{\mathfrak{p}}})$. Now, if $\varphi_{\bar{\mathfrak{p}}}$ is an isomorphism then $\psi_{\bar{\mathfrak{p}}}$ is surjective, hence $F_{S_{\mathfrak{p}}}(p)/F$ is a $\mathbf{Z}_p$-extension. By the result of Yager $\mathfrak{p}$ is regular for E and F.

Conversely, the induced map $\psi_{\bar{\mathfrak{p}}}$ is an isomorphism if $\mathfrak{p}$ is regular. Therefore $\varphi_{\bar{\mathfrak{p}}}$ is surjective, since its restriction to the inertia subgroup is surjective; indeed the normal subgroup generated by the image of $I(F_{\bar{\mathfrak{p}}}(p)/F_{\bar{\mathfrak{p}}})$ is the whole group Gal $(F_{S_p}(p)/F_{S_{\mathfrak{p}}}(p))$, since there is only one prime of the $\mathbf{Z}_p$-extension $F_{S_{\mathfrak{p}}}(p)$ above $\bar{\mathfrak{p}}$ and $F_{S_{\mathfrak{p}}}(p)$ has no p-extension unramified outside $S_{\mathfrak{p}}$. But p-groups are nilpotent, hence the assertion follows.

Now let R be the kernel of $\varphi_{\bar{\mathfrak{p}}}$. The Hochschild-Serre spectral sequence implies an exact sequence

$$0 \longrightarrow H^1(\mathrm{Gal}(F_{S_p}(p)/F), \mathbf{Q}_p/\mathbf{Z}_p) \longrightarrow H^1(\mathrm{Gal}(F_{\bar{\mathfrak{p}}}(p)/F_{\bar{\mathfrak{p}}}), \mathbf{Q}_p/\mathbf{Z}_p)$$
$$\longrightarrow H^1(R, \mathbf{Q}_p/\mathbf{Z}_p)^{\mathrm{Gal}(F_{S_p}/F)} \longrightarrow 0$$

because $H^2(\mathrm{Gal}(F_{S_p}(p)/F), \mathbf{Q}_p/\mathbf{Z}_p) = 0$, i.e., the Leopoldt conjecture is true for abelian extensions of K. The (in)-equalities

$$\begin{aligned}\mathrm{corank}_{\mathbf{Z}_p} H^1(\mathrm{Gal}(F_{S_p}(p)/F), \mathbf{Q}_p/\mathbf{Z}_p) &= [F : K] + 1 \\ &= \mathrm{corank}_{\mathbf{Z}_p} H^1(\mathrm{Gal}(F_{\bar{\mathfrak{p}}}(p)/F_{\bar{\mathfrak{p}}}), \mathbf{Q}_p/\mathbf{Z}_p)\end{aligned}$$

and

$$\dim_{\mathbf{F}_p} H^1(\mathrm{Gal}(F_{S_p}(p)/F), \mathbf{Z}/p\mathbf{Z}) \geqq [F:K]+1+\delta$$
$$= \dim_{\mathbf{F}_p} H^1(\mathrm{Gal}(F_{\bar{\mathfrak{p}}}(p)/F_{\bar{\mathfrak{p}}}), \mathbf{Z}/p\mathbf{Z})$$

($\delta = 1$ if $F_{\bar{\mathfrak{p}}}$ contains the group μ_p of p-th roots of unity and $\delta = 0$ otherwise), [3] Satz 11.8, show that

$$H^1(R, \mathbf{Q}_p/\mathbf{Z}_p)^{\mathrm{Gal}(F_{S_p}(p)/F)} = 0$$

and therefore $R = 0$. This finishes the proof of the theorem. □

PROOF OF COROLLARY 1. According to [6] Theorem 1

$$H^2(\mathrm{Gal}(F_S/F), M(j)) = H^2(\mathrm{Gal}(F_S(p)/F), M(j)).$$

Furthermore $\mathrm{Gal}(F_S(p)/F_{S_p}(p))$ is the free pro-p-product of all inertia groups with respect to primes v of $F_{S_p}(p)$ above $S\backslash S_p$, in particular $\mathrm{Gal}(F_S(p)/F_{S_p}(p))$ is a free pro-p-group (see [10], Theorem 2.2, which goes back on a slightly weaker theorem of Neumann and also Neukirch in the case $F = \mathbf{Q}$). Therefore the Hochschild-Serre spectral sequence yields an exact sequence

$$H^2(\mathrm{Gal}(F_{S_p}(p)/F), M(j)) \longrightarrow H^2(\mathrm{Gal}(F_S(p)/F), M(j))$$
$$\longrightarrow H^1(\mathrm{Gal}(F_{S_p}(p)/F), H^1(\mathrm{Gal}(F_S(p)/F_{S_p}(p)), M(j))).$$

Since $M(j)$ is a trivial $G(F_S(p)/F_{S_p}(p))$-module the group on the right is equal to

$$\bigoplus_{v \in S\backslash S_p} H^1(\mathrm{Gal}(F_v^{nr}(p)/F_v), H^1(I(F_v(p)/F_v), M(j)))$$
$$= \bigoplus_{v \in S\backslash S_p} H^2(\mathrm{Gal}(F_v(p)/F_v), M(j))$$

by [4] Satz 4.1 and Shapiro's lemma. If μ_p is not contained in F_v then $\mathrm{Gal}(F_v(p)/F_v)$ is free; otherwise it is a Poincaré group of dimension two with dualizing module $\mathbf{Q}_p/\mathbf{Z}_p(1)$, hence

$$H^2(\mathrm{Gal}(F_v(p)/F_v), M(j)) = \varinjlim_{m} H^0(\mathrm{Gal}(F_v(p)/F_v), \mathrm{Hom}({}_{p^m}M(j), \mathbf{Q}_p/\mathbf{Z}_p(1))^*$$
$$= M(j-1)_{\mathrm{Gal}(F_v(p)/F_v)} = 0$$

(${}_{p^m}M := \{x \in M \mid p^m x = 0\}$).

Therefore we have reduced the corollary to the case $S = S_p$. But, since $\mathrm{Gal}(F_{S_p}(p)/F)$ is purely local with respect to $\bar{\mathfrak{p}}$, we obtain

$$H^2(\mathrm{Gal}(F_{S_p}(p)/F), M(j)) = H^2(\mathrm{Gal}(F_{\bar{\mathfrak{p}}}(p)/F_{\bar{\mathfrak{p}}}),\ M(j))$$

which is zero if $\mu_p \not\subset F_{\bar{\mathfrak{p}}}$ and otherwise equal to $M(j-1)_{\mathrm{Gal}(F_{\bar{\mathfrak{p}}}(p)/F_{\bar{\mathfrak{p}}})}$ which is zero by our assumption. This proves Corollary 1. □

PROOF OF COROLLARY 2. Observing that

$$H^1(\bar{E}, \mathbf{Q}_p/\mathbf{Z}_p(1)) = E_{p^\infty} = E_{\mathfrak{p}^\infty} \oplus E_{\bar{\mathfrak{p}}^\infty}$$

and that the order of $\mathrm{Gal}(\mathcal{F}/F)$ is prime to p it is enough to show that

$$H^2(\mathrm{Gal}(\mathcal{F}_S/\mathcal{F}),\ E_{\mathfrak{p}^\infty}(j)) = 0$$

for all $j \in \mathbf{Z}$. But $E \times_K \mathcal{F}$ has good reduction everywhere, hence $E_{\mathfrak{p}^\infty}$ is a p-primary divisible $\mathrm{Gal}(\mathcal{F}_{S_p}(p)/\mathcal{F})$-module. Now Corollary 1 implies the result because the $\mathrm{Gal}(\mathcal{F}_v(p)/\mathcal{F}_v)$-coinvariants of $E_{\mathfrak{p}^\infty}(j-1)$ are zero for all $j \in \mathbf{Z}$ and all $v \in S$. □

PROOF OF COROLLARY 3. From the Dwyer-Friedlander spectral sequence [1]

$$E_2^{s,t} = \left\{ \begin{matrix} H^s(E \times_K F, \mathbf{Q}_p/\mathbf{Z}_p(j)), & t = -2j \\ 0 \qquad , & t \text{ odd} \end{matrix} \right\} \Rightarrow K_{-s-t}^{\acute{e}t}(E \times_K F, \mathbf{Q}_p/\mathbf{Z}_p)$$

and the Hochschild-Serre spectral sequence

$$E_2^{s,t} = H^s(F, H^t(\bar{E}, \mathbf{Q}_p/\mathbf{Z}_p(j))) \Rightarrow H^{s+t}(E \times_K F, \mathbf{Q}_p/\mathbf{Z}_p(j))$$

we obtain for $j \geqq 2$

$$\begin{aligned} \dim K_{2j-2}^{\acute{e}t}(E \times_K F, \mathbf{Q}_p/\mathbf{Z}_p) &= \dim H^2(E, \mathbf{Q}_p/\mathbf{Z}_p(j))) \\ &= \dim H^1(F, H^1(\bar{E}, \mathbf{Q}_p/\mathbf{Z}_p(j)) \end{aligned}$$

and

$$\begin{aligned} \dim K_{2j-1}^{\acute{e}t}(E \times_K F, \mathbf{Q}_p/\mathbf{Z}_p) &= \dim H^1(E, \mathbf{Q}_p/\mathbf{Z}_p(j)) + \dim H^3(E, \mathbf{Q}_p/\mathbf{Z}_p(j+1)) \\ &= 2 \dim H^1(F, \mathbf{Q}_p/\mathbf{Z}_p(j)) + \dim H^2(F, H^1(\bar{E}, \mathbf{Q}_p/\mathbf{Z}_p(j))) \end{aligned}$$

(using $H^2(\bar{E}, \mathbf{Q}_p/\mathbf{Z}_p(1)) = \mathbf{Q}_p/\mathbf{Z}_p$ and $H^2(F, \mathbf{Q}_p/\mathbf{Z}_p(j)) = 0$ for $j \neq 1$, [7] Satz 4.1 (ii)). Since

$$\dim H^1(F, \mathbf{Q}_p/\mathbf{Z}_p(j)) = [F : K] + \dim H^2(\mathrm{Gal}(F_{S_p}/F), \mathbf{Q}_p/\mathbf{Z}_p(j)),$$

[7] 4.5 (iii), Satz 4.6,

$$\dim H^1(F, H^1(\bar{E}, \mathbf{Q}_p/\mathbf{Z}_p(j))) = \dim H^1(\mathrm{Gal}(F_S/F),\ H^1(\bar{E}, \mathbf{Q}_p/\mathbf{Z}_p(j))),$$

[2] Lemma 2.4 (or see the proof of Proposition 1 in [8]) and

$$\sum_{k=0}^{2} (-1)^k \dim H^k(\mathrm{Gal}(F_S/F),\ H^1(\bar{E}, \mathbf{Q}_p/\mathbf{Z}_p(j))) = -[F : \mathbf{Q}]$$

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where S is a finite set of primes of F containing S_p and all primes where $E \times_K F$ has bad reduction, [8] Proposition 2, we obtain ($j \geqq 2$):

$$\begin{aligned}
\dim K^{\acute{e}t}_{2j-2}(E \times_K F, \mathbf{Q}_p/\mathbf{Z}_p) &= [F:\mathbf{Q}] + \dim H^2(\mathrm{Gal}(F_S/F), H^1(\bar{E}, \mathbf{Q}_p/\mathbf{Z}_p(j))),\\
\dim K^{\acute{e}t}_{2j-1}(E \times_K F, \mathbf{Q}_p/\mathbf{Z}_p) &= [F:\mathbf{Q}] + 2\dim H^2(\mathrm{Gal}(F_{S_p}/F), \mathbf{Q}_p/\mathbf{Z}_p(j))\\
&\quad + \dim H^2(F, H^1(\bar{E}, \mathbf{Q}_p/\mathbf{Z}_p(j))).
\end{aligned}$$

Now Corollary 2 completes the proof because as in the proof of Corollary 1 for $j \neq 1$

$$\begin{aligned}
\dim H^2(\mathrm{Gal}(F_{S_p}/F), \mathbf{Q}_p/\mathbf{Z}_p(j)) &= \dim H^2(\mathrm{Gal}(F_S/F), \mathbf{Q}_p/\mathbf{Z}_p(j))\\
&= \dim H^2(\mathrm{Gal}(\mathcal{F}_S/\mathcal{F}), \mathbf{Q}_p/\mathbf{Z}_p(j))^{\mathrm{Gal}(\mathcal{F}/F)}\\
&= \dim H^2(\mathrm{Gal}(\mathcal{F}_{S_p}(p)/\mathcal{F}), \mathbf{Q}_p/\mathbf{Z}_p(j))^{\mathrm{Gal}(\mathcal{F}/F)}\\
&= \dim H^2(\mathrm{Gal}(\mathcal{F}_{\bar{\mathfrak{p}}}(p)/\mathcal{F}_{\bar{\mathfrak{p}}}), \mathbf{Q}_p/\mathbf{Z}_p(j))^{\mathrm{Gal}(\mathcal{F}_{\bar{\mathfrak{p}}}/F_{\bar{\mathfrak{p}}})}\\
&= 0. \qquad \square
\end{aligned}$$

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