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GENERATORS OF CHEVALLEY GROUPS OVER Z

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1. Introduction. Let $\mathcal{L}(K)$ be the universal Chevalley group ([**1**], p. 197) of type $\mathcal{L}$ over a field K and $\mathcal{L}_z = \langle x_r(1) \mid r \in \Phi \rangle \subseteq (\mathcal{L}(K))$ where Φ is the set of roots of $\mathcal{L}$. Let $\Pi = \{\alpha_1, \alpha_2, \dots, a_n\}$ be a fundamental system of roots of $\mathcal{L}$ and put

$$W = \phi_{\alpha_1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\phi_{\alpha_2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dots \phi_{\alpha_n}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$M = \phi_{\alpha_1}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}\phi_{\alpha_2}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \dots \phi_{\alpha_n}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then we know from [**2**] (p. 950) that

$$W^h = M^{h+1} = \pm 1$$

where h is the Coxeter number of $\mathcal{L}$. We call an element of $\mathcal{L}(K)$ conjugate to W a Coxeter element and an element conjugate to M a Kac element. The purpose of this note is to prove:

THEOREM. *The group $\mathcal{L}_z$ is generated by a Coxeter element and a Kac element if $\mathcal{L} \neq B_2, C_2$.*

This theorem may be regarded as a generalization of the well-known fact that the second order unimodular group is generated by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

The proof will be carried out by case-by-case computation. The classical types and type G_2 will be settled in Section 2 using matrix representations ([**1**], pp. 183-188) of $\mathcal{L}(K)$. For types E_n and F_4, our arguments will depend solely on the commutator formulas and the effect of the conjugation by W on $x_r(\pm 1)$ and these will be done in Section 3.

In this note, the conjugation by Y means $X \to YXY^{-1}$, and as in [**1**],

$$[X, Y] = XYX^{-1}Y^{-1}.$$

Denote by w the Coxeter element in the Weyl group defined by

$$Wx_r(t)W^{-1} = x_{w(r)}(\pm t).$$

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By the *w-closure* of a subset S of Φ, we mean the minimal subset T of Φ such that

(i) $S \subseteq T$,

(ii) $r, s \in T, r + s \in \Phi \Rightarrow r + s \in T$ and

(iii) $r \in T \Rightarrow w(r) \in T$.

Note that, if the Dynkin diagram of $\mathscr{L}$ is simply laced, so that the commutator formula is

$$[x_r(t), x_s(u)] = x_{r+s}(\pm tu) \text{ or } 1,$$

then the subgroup generated by $\{x_r(1) | r \in S\}$ and W contains

$$\langle x_r(1) | r \in w\text{-closure of } S \rangle.$$

2. The classical types.

2.1. $\mathscr{L} = A_n$. We choose the fundamental system

$$\Pi = \{\alpha_1, \dots, \alpha_n\}$$

in the usual way, i.e., $\alpha_i = e_{i-1} - e_n$, $i = 1, \dots, n$ where $\{e_0, \dots, e_n\}$ is an orthonormal basis of R^{n+1} ([**1**], p. 46), and use the identification $A_n(K) = SL_{n+1}(K)$ ([**1**], p. 184). Then

$$W = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & -1 & 0 \end{pmatrix},$$

$$M = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & -1 & 0 \end{pmatrix}.$$

Unfortunately M and W do not generate $\mathscr{L}_z$ and we need a slight modification.

(2.1) If $\mathscr{L} = A_n$, $n \geqq 2$ then $\mathscr{L}_z$ is generated by W and VMV^{-1}, where $V = x_{\alpha_1+\alpha_2}(1)$.

Proof. Let $H = \langle W, VMV^{-1} \rangle$. The action of w is given by

$$w: e_0 \rightarrow e_1 \rightarrow \dots \rightarrow e_n \rightarrow e_0,$$

and we can see easily that the w-closure of any α in Π is the whole set Φ. Therefore, we need only to show that $x_\alpha(1) \in H$ (or $x_{-\alpha}(1) \in H$) for some $\alpha \in \Pi$.

Let $M' = VMV^{-1}$, $X_1 = M'W^{-1}$, $X_2 = WX_1W^{-1}$, $X_3 = W^2X_1W^{-2}$ and $Y = X_1X_2$. Then matrix computations show that

(2.1.a) $[Y, X_1][Y^2, X_1^{-1}] = x_{-\alpha_1}(1)$, if $n = 2$

(2.1.b) $[X_2, [X_2, [X_1, X_3]]] = x_{\alpha_1}(-1)$, if $n = 3$

(2.1.c) $YW^{-1}ZWY^{-1}Z = x_{\alpha_2}(-1)$, if $n \geqq 4$

where $Z = [Y, X_1][Y^2, X_1^{-1}]$.

(Note that $V = I + e_{02}$ in the notation of [**1**]. One can obtain

$$Z = x_{e_1-e_3}(-1)x_{e_1-e_4}(1)$$

and (2.1.c) for $n = 4$ by direct computation. Then, for $n \geqq 5$, these expressions may be verified inductively by conjugating X_1 and Y with $x_{\alpha_n}(1)$ which will bring them "into $A_{n-1}(K)$".)

2.2. $\mathcal{L} = B_n$, C_n, $n \geqq 3$. Here again, Π is chosen as in [**1**], p. 47 and use the matrix representations of $B_n(K)$ and $C_n(K)$ given in [**1**], pp. 185-187. Thus

$$x_{\alpha_1}(t) = I + t(e_{12} - e_{-2,-1})$$
$$\dots$$
$$x_{\alpha_n}(t) = I + t(2e_{n0} - e_{0,-n}) - t^2e_{n,-n}$$

in $B_n(K)$ and

$$x_{\alpha_1}(t) = I + t(e_{12} - e_{-2,-1})$$
$$\dots$$
$$x_{\alpha_n}(t) = I + te_{n,-n}$$

in $C_n(K)$. The action of w is given by

$$w:e_1 \to e_2 \to \dots \to e_n \to -e_1$$

in both B_n and C_n. We show that

(2.2) *If* $\mathcal{L} = B_n$ *or* C_n *and* $n \geqq 3$ *then* $\mathcal{L}_z$ *is generated by* W *and* M.

Proof. Let $X_1 = MW^{-1}$, $X_2 = WX^{-1}$, $X_3 = W^{-2}MW^{-2}$ and $H = \langle W, M\rangle$. In $B_n(K)$, we have

(2.2.a) $X_2^{-1}X_1X_2[[X_1, X_2], [X_2, X_3]] = x_{\alpha_1}(-1)$.

This can be verified by direct computation when $n = 3$ and $n = 4$. For $n \geqq 5$, we may see it inductively as follows. We have, from the matrix representation,

$$X_1 = x_{e_1-e_2} \dots x_{e_1-e_{n-2}}x_{e_1-e_{n-1}}x_{e_1-e_n}x_{e_1},$$
$$X_2 = x_{e_2-e_3} \dots x_{e_2-e_{n-1}}x_{e_2-e_n}x_{e_2+e_1}x_{e_2},$$
$$X_3 = x_{e_3-e_4} \dots x_{e_3-e_n}x_{e_2+e_1}x_{e_3+e_2}x_{e_3}$$

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where $x_r(-1)$ is denoted by x_r for brevity. Then the conjugation by $V = x_{e_{n-1}-e_n}(1)$ eliminates $x_{e_1-e_n}$, $x_{e_2-e_n}$ and $x_{e_3-e_n}$ in X_1, X_2 and X_3, respectively (and thus X_1, X_2 and X_3 are brought "into $B_{n-1}(K)$"). Then by the induction hypothesis,

$$VYV^{-1} = x_{\alpha_1}(-1)$$

where Y is the left hand side of (2.2.a). Hence

$$Y = x_{\alpha_1}(-1).$$

Once we have $x_{\alpha_1}(1) \in H$, we can prove (2.2) again inductively as follows. First, we can show easily that

$$x_{\alpha_1}(1) \in H \Rightarrow x_r(1) \in H$$

for all $r \in \Phi$, in $B_3(K)$. Then for $n \geqq 4$,

$$x_{\alpha_1}(1) \in H \Rightarrow W^{h/2}x_{\alpha_1}(1)W^{-h/2} = x_{-\alpha_1}(\pm 1) \in H.$$

Hence

$$\phi_{\alpha_1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \phi_{\alpha_1}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \in H, \quad \text{and}$$

$$\phi_{\alpha_2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ldots \phi_{\alpha_n}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\phi_{\alpha_2}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \ldots \phi_{\alpha_n}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \in H.$$

Then by the induction hypothesis $x_r(1) \in H$ for all r in the root system of B_{n-1} spanned by $\{\alpha_2, \ldots, \alpha_n\}$, and hence $x_r(1) \in H$ for all $r \in \Phi$.

An argument almost identical to the above will settle the case when $\mathscr{L} = C_n$. The key identity in this case is

$$X_2^{-1}X_1^{-1}X_2[[X_1, X_2], [X_2, X_3]] = x_{\alpha_1}(1).$$

2.3. $\mathscr{L} = D_n$. Once again Π is as in [**1**], p. 47 and use the matrix representation in [**1**], p. 185. Thus

$$x_{\alpha_1}(t) = x_{e_1-e_2}(t) = I + t(e_{1,2} - e_{-2,-1})$$

$$\ldots$$

$$x_{\alpha_n}(t) = x_{e_{n-1}+e_n}(t) = I + t(e_{n-1,-n} - e_{n,-n+1})$$

and the action of w is given by

$$w{:}e_1 \to e_2 \to \ldots \to e_{n-1} \to -e_1, e_n \to -e_n.$$

(2.3) If $\mathscr{L} = D_n$, $n \geqq 4$, then $\mathscr{L}_z$ is generated by W and VMV^{-1} where

$$V = x_{e_1+e_n}(1).$$

Proof. Let

$$X_1 = VMV^{-1}W^{-1},\ X_2 = WX_1W^{-1},$$
$$Y_1 = [X_1, X_2],\ Y_2 = [X_1, X_2^{-1}],\ Y_3 = [X_1^{-1}, X_2]$$
$$Y_4 = [X_1^{-1}, X_2^{-1}] \quad \text{and} \quad H = \langle W, VMV^{-1}\rangle.$$

From the matrix representation, we have

$$X_1 = x_{e_1-e_2} \dots x_{e_1-e_{n-2}} x_{e_1-e_{n-1}} x_{e_1-e_n} x_{e_2-e_n}$$
$$X_2 = x_{e_2-e_3} \dots x_{e_2-e_{n-1}} x_{e_2+e_1}^{-1} x_{e_2+e_n} x_{e_3+e_n}$$

where x_r denotes $x_r(-1)$. Here the key identities are:

$$Y_1Y_2 = x_{e_1+e_n}(1),$$
$$Y_1Y_3 = x_{e_1+e_2}(-3)x_{e_1+e_3}(1),$$
$$Y_2Y_4 = x_{e_1+e_2}(2)x_{e_1+e_3}(-1).$$

Again, these identities can be verified by direct computation for $n = 4$, then use induction for $n \geqq 5$. So, when $n \geqq 5$, by conjugating X_1 and X_2 with $x_{e_{n-2}-e_{n-1}}(1)$ we can eliminate $x_{e_1-e_{n-1}}$ and $x_{e_2-e_{n-1}}$, then conjugating them again with

$$\phi_{e_{n-1}-e_n}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

the e_n's are brought to e_{n-1}. We can then use the induction hypothesis to obtain the desired result.

The above identities show that $x_{e_1+e_2}(1)$, $x_{e_1+e_n}(1)$ and $x_{e_1+e_n}(1) \in H$. Then it is easy to check that the w-closure of $\{e_1 + e_2, e_1 + e_3, e_1 + e_n\}$ is the whole set Φ and this proves (2.3).

2.4. $\mathscr{L} = G_2$. In this case we have the following matrix representation:

$$\phi_{\alpha_1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b & & & & & \\ c & d & & & & & \\ & & a^2 & 2ab & -b^2 & & \\ & & ac & ad+bc & -bd & & \\ & & -c^2 & -2cd & d^2 & & \\ & & & & & a & -b \\ & & & & & -c & d \end{pmatrix}$$

$$\phi_{\alpha_2}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & & & & & & \\ & a & b & & & & \\ & c & d & & & & \\ & & & 1 & & & \\ & & & & a & -b & \\ & & & & -c & d & \\ & & & & & & 1 \end{pmatrix}$$

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Let

$$X_1 = MW^{-1},\ X_2 = WX_1W^{-1},\ X_3 = W^2X_1W^{-2}$$
$$\text{and } H = \langle W, M\rangle.$$

Then using the above representation, we can show that

$$[X_3,\ X_1X_2X_3X_1^{-1}] = x_{3\alpha_1+2\alpha_2}(-1) = W^{-1}x_{\alpha_2}(-1)W.$$

Since $w^3(r) = -r$,

$$x_{\alpha_2}(1) \in H \Rightarrow x_{-\alpha_2}(1) \in H.$$

Hence

$$\phi_{\alpha_2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \phi_{\alpha_2}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \in H,$$

which implies also that

$$\phi_{\alpha_1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \phi_{\alpha_1}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \in H.$$

Hence $x_{\alpha_1}(1) \in H$. Now w acts transitively on the short (and the long) roots. Therefore $x_r(1) \in H$ for all $r \in \Phi$, i.e., $\mathscr{L}_z = \langle W, M\rangle$.

3. Exceptional types.

3.1. $\mathscr{L} = F_4$. Choose the fundamental system Π as in [**1**], p. 47. Thus

$$\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3,\ \alpha_3 = e_3,$$
$$\alpha_4 = (-e_1 - e_2 - e_3 + e_4)/2.$$

Let Φ_l and Φ_s be the sets of long roots and short roots, respectively, and recall the commutator formulas:

(3.1.a)
$$\begin{aligned}[x_r(t), x_s(u)] &= x_{r+s}(\pm tu) \quad \text{if } r, s, r+s \in \Phi_l \\ &= x_{r+s}(\pm tu)x_{r+2s}(\pm tu^2) \\ &\qquad\qquad \text{if } r \in \Phi_l,\ s, r+s \in \Phi_s \\ &= x_{r+s}(\pm 2tu) \quad \text{if } r, s \in \Phi_s,\ r+s \in \Phi_l \\ &= x_{r+s}(\pm tu) \quad \text{if } r, s, r+s \in \Phi_s \\ &= 1 \quad \text{if } r+s \notin \Phi. \end{aligned}$$

The action of w is given by

$$w{:}e_1 \to (e_1 + e_2 - e_3 + e_4)/2,$$
$$e_2 \to (e_1 - e_2 + e_3 + e_4)/2,$$
$$e_3 \to (-e_1 - e_2 - e_3 + e_4)/2,$$
$$e_4 \to (-e_1 + e_2 + e_3 + e_4)/2.$$

Let $X_1 = MW^{-1}$, $X_2 = WX_1W^{-1}$, $X_3 = W^2X_1W^{-2}$ and $H = \langle W, M\rangle$. Then

$$X_1 = x_{e_1-e_2}(\pm 1)x_{e_1-e_3}(\pm 1)x_{e_1}(\pm 1)x_{(e_1-e_2-e_3+e_4)/2}(\pm 1).$$

We note also that

$$(3.1.b)\quad [ab, c] = [a, [b, c]][b, c][a, c],$$
$$[a, cd] = [a, c][c, [a, d]][a, d].$$

In particular,

$$(3.1.c)\quad [ab, c] = [a, c] \quad \text{if } [b, c] = 1,$$
$$= [b, c] \quad \text{if } [a, b] = [a, c] = 1.$$
$$[a, cd] = [a, c] \quad \text{if } [a, d] = 1$$
$$= [a, d] \quad \text{if } [c, d] = [a, c] = 1.$$

Now using (3.1.a), (3.1.b) and (3.1.c) we compute:

$$\begin{aligned}[X_1, X_2] = {} & [x_{(e_1-e_2-e_3+e_4)/2}(\pm 1), x_{e_2}(\pm 1)] \\ & [x_{e_1}(\pm 1), x_{e_2}(\pm 1)] \\ & [x_{e_1-e_2}(\pm 1), x_{e_2-e_3}(\pm 1)] \\ & [x_{e_1-e_2}(\pm 1), x_{e_2}(\pm 1)] \\ = {} & x_{(e_1+e_2-e_3+e_4)/2}(\pm 1)x_{e_1+e_2}(\pm 2) \\ & x_{e_1-e_3}(\pm 1)x_{e_1}(\pm 1)x_{e_1+e_2}(\pm 1).\end{aligned}$$

Similarly, we get

$$[X_1, X_3] = x_{e_1+e_4}(\pm 2).$$
$$[X_1, W^2[X_1, X_2]W^{-2}] = x_{e_1+e_4}(\pm 1 \pm 2 \pm 2 \pm 2).$$

These two identities imply that $x_{e_1+e_4}(1) \in H$. Then it is easy to check that the w-closure of $e_1 + e_4$ is Φ_l which implies, together with the first formula of (3.1.a), that

$$x_r(1) \in H \quad \text{for all } r \in \Phi_l.$$

As for the short roots, we know that

$$Y = x_{e_1}(\pm 1)x_{(e_1-e_2-e_3+e_4)/2}(\pm 1)$$
$$(= X_1 \text{ less } x_{e_1-e_2}(\pm 1)x_{e_1-e_3}(\pm 1))$$

is in H. Hence

$$[Y, WYW^{-1}] = x_{(e_1+e_2-e_3+e_4)/2}(\pm 1)x_{e_1+e_2}(\pm 1) \in H.$$

Hence

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$$x_{(e_1+e_2-e_3+e_4)/2}(1) \in H.$$

The w-orbit of $r = (e_1 + e_2 - e_3 + e_4)/2$ contains one-half of the short roots and the other half is the w-orbit of

$$r - w(r) = r + w^7(r).$$

Hence $x_r(1) \in H$ for all $r \in \Phi_s$, too. Hence $H = \mathscr{L}_z$.

3.2. $\mathscr{L} = E_n$, $n = 6, 7, 8$. For E types, we find it most convenient to use the description of the roots of E_8 given in [**3**], p. 19-09. So, let $\omega_1, \ldots, \omega_9$ be vectors in R^8 such that $\omega_1 + \ldots + \omega_9 = 0$, $(\omega_i, \omega_i) = 8/9$, $1 \leqq i \leqq 9$ and $(\omega_i, \omega_j) = -1/9$, $i \neq j$, $1 \leqq i, j \leqq 9$, where (,) is the inner product in R^8. Then the fundamental system of E_8 is given by

$$\alpha_i = \omega_i - \omega_{i+1}, \quad 1 \leqq i \leqq 7 \quad \text{and}$$

$$\alpha_8 = \omega_6 + \omega_7 + \omega_8.$$

We choose $\{\alpha_3, \ldots, \alpha_8\}$ and $\{\alpha_2, \alpha_3, \ldots, \alpha_8\}$ for the fundamental systems of E_6 and E_7, respectively.

The set of roots of E_8 is given by

$$\Phi = \{\omega_i - \omega_j, \pm(\omega_i + \omega_j + \omega_k) | 1 \leqq i, j, k \leqq 9, \quad i \neq j \neq k, i \neq k\}.$$

Let w_r be the reflection defined by $r \in \Phi$. Then $w_{\omega_i - \omega_j}$ interchanges ω_i and ω_j and leaves ω_k, $k \neq i, j$ unchanged, and $w_{\omega_i+\omega_j+\omega_k}$ sends ω_m to

$$\omega_m + 1/3(\omega_i + \omega_j + \omega_k) \quad \text{if } m \neq i, j, k,$$

and to

$$\omega_m - 2/3(\omega_i + \omega_j + \omega_k) \quad \text{if } m = i, j \text{ or } k.$$

Now we show that

(3.2). If $\mathscr{L} = E_6$, E_7 or E_8, then L_z is generated by W and M.

Proof. Assume that $\mathscr{L} = E_6$. Let $X = MW^{-1}$. Then

$$X = x_{\omega_3-\omega_4}x_{\omega_3-\omega_5}x_{\omega_3-\omega_6}x_{\omega_3-\omega_7}x_{\omega_3-\omega_8}x_{\omega_3+\omega_7+\omega_8}.$$

Here, and in what follows, $x_r, y_r, z_r, \ldots$ denote $x_r(\pm 1)$. Now, we look for a conjugate W^iXW^{-i} of X such that, in computing $[X, W^iXW^{-i}]$, we can take advantage of (3.1.c) as much as possible. We find that

$$W^3XW^{-3} = x_{\omega_3+\omega_6+\omega_8}x_{\omega_3+\omega_4+\omega_6}x_\sigma x_{\omega_3+\omega_5+\omega_6}x_{\omega_3+\omega_6+\omega_7}x_{\omega_4+\omega_6+\omega_8}$$

where $\sigma = -(\omega_1 + \omega_2 + \omega_9) =$ the maximal root of E_6, is a reasonably good one. So

$$Y = [X, W^3XW^{-3}] = [x_{\omega_3-\omega_4}x_{\omega_3-\omega_6}x_{\omega_3-\omega_8}, x_{\omega_4+\omega_6+\omega_8}]$$
$$= x_{\omega_3+\omega_6+\omega_8}x_{\omega_3+\omega_4+\omega_8}x_{\omega_3+\omega_4+\omega_6}.$$

Then

$$[Y, WYW^{-1}] = y_{\omega_3+\omega_4+\omega_6}y_\sigma$$
$$[Y, W^2YW^{-2}] = z_\sigma.$$

Hence $x_{\omega_3+\omega_4+\omega_6}(1), x_\sigma(1) \in H$. Now

$$(\omega_3 + \omega_4 + \omega_6) + w^5(\omega_3 + \omega_4 + \omega_6) = \omega_6 - \omega_7,$$
$$\sigma + w^5\sigma = \omega_6 + \omega_7 + \omega_8 = \alpha_8.$$

The w-orbit of $\omega_6 - \omega_7 = \alpha_6$ contains all the α_i, $3 \leqq i \leqq 7$. Hence the w-closure of $\{\omega_3 + \omega_4 + \omega_6, \sigma\}$ is the whole set Φ. Hence $H = \mathcal{L}_z$.

Next, let $\mathcal{L} = E_7$. We have

$$X = MW^{-1} = x_{\omega_2-\omega_3}x_{\omega_2-\omega_5}x_{\omega_2-\omega_6}x_{\omega_2-\omega_7}x_{\omega_2-\omega_8}x_{\omega_2+\omega_7+\omega_8}.$$

Here, again we compute:

$$Y = [X, W^3XW^{-3}]$$
$$= [x_{\omega_2-\omega_3}x_{\omega_2-\omega_5}x_{\omega_2-\omega_8}, x_{\omega_5-\omega_6}x_{\omega_3+\omega_5+\omega_8}]$$
$$= y_{\omega_2-\omega_6}y_{\omega_2+\omega_5+\omega_8}y_{\omega_2+\omega_3+\omega_8}y_{\omega_2+\omega_3+\omega_5}.$$

Then we get

$$[Y, W^4YW^{-4}] = z_{\omega_2+\omega_3+\omega_5}.$$

Put $r = \omega_2 + \omega_3 + \omega_5$. Then

$$r + w^7(r) = \omega_2 + \omega_5 + \omega_6 = s, \quad w^{-5}(r) + s = \omega_2 - \omega_3.$$

In other words, we have

$$x_{\alpha_2}(1) \in \langle W, M\rangle = H.$$

Since $w^{h/2} = -1$ in E_7, we also have $x_{-\alpha_2}(1) \in H$ and, as in the proof of type B_n, the problem for E_7 is reduced to that of E_6. This proves $\mathcal{L}_z = H$ in E_7.

Finally, let $\mathcal{L} = E_8$. Here again, $w^{h/2} = -1$. Hence we only have to show that

$$x_{\alpha_1}(1) \in \langle W, M\rangle = H.$$

Let $X = MW^{-1}$, $Y = [X, W^3XW^{-3}]$, then

$$[Y, W^6YW^{-6}] = x_{\omega_1-\omega_9}.$$

Since

$$Wx_{\omega_1-\omega_9}W^{-1} = x_{\omega_2-\omega_9}$$

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(and $w^{h/2} = -1$) we have

$$x_{\omega_1 - \omega_2}(1) = x_{\alpha_1}(1) \in H.$$

This completes the proof of the theorem.

4. Remark. The condition that $\mathscr{L} \neq B_2, C_2$ in the theorem cannot be removed. This can be seen as follows. Let $G = B_2(2)$. Then $(G:G') = 2$ ([**1**], p. 176). Clearly, M (hence its conjugates) is in G' and

$$\begin{aligned} W &\equiv x_a(1)x_b(1) \equiv x_{a+b}(1)x_{2a+b}(1) \equiv [x_a(1), x_b(1)] \\ &\equiv 1 (\mathrm{mod}\ G') \end{aligned}$$

where a is a short root and b is a long root. Hence $\langle W, M \rangle = G'$. However, if char $K > 2$, then we still have $\mathscr{L}_z = \langle W, M \rangle$.

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