MATHEMATICAL NOTES.


In a recent article in Nature (Vol. 157, p. 571, May 4, 1946) entitled "Mathematics and the Civil Service", John Todd and D. H. Sadler expressed the view that training in numerical methods should be given to mathematical students at the post-graduate level. The chief reason given was that their experience had shown that a knowledge of the more advanced parts of mathematics often becomes a powerful tool in the hands of the computer. The object of this note is to draw attention to a further example which arose recently in connection with my research work, providing additional evidence for their views.

In the course of a certain investigation, it was necessary to tabulate the function given by the doubly infinite series

\[
\sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{\sin (2r-1) \pi \alpha/a \cdot \sin (2s-1) \pi \beta/b}{(2r-1)(2s-1) [(2r-1)^2 b^2 + (2s-1)^2 a^2]} \quad \cdots \cdots \cdots \cdots (1)
\]

where \(a = 160, b = 60\), for a range of values of \(\alpha\) and \(\beta\) given by

\[
\begin{align*}
\alpha &= 5.0, \ 6.0, \ 7.0, \ 8.0, \ 9.0, \ 10.0, \\
\beta &= 0.6, \ 0.7, \ 0.8, \ 0.9, \ 1.0, \ 1.1.
\end{align*}
\]

Consider the direct computation from (1) for the values \(\alpha = 5.0, \beta = 0.6\). After considerable numerical computation it was shown that the sum of the first hundred terms of the doubly infinite series (corresponding to \(1 \leq r \leq 10, 1 \leq s \leq 10\)) evaluated to \(3.528 \times 10^{-7}\). However, the series converged so slowly that even with this amount of laborious calculation little accuracy could be expected.

It may be shown, however, by a method the details of which I hope to publish elsewhere, that one of the double series may be summed, and the expression reduces to

\[
\frac{\pi}{4a^2} \sum_{s=1}^{\infty} \frac{\sin (2s-1) \pi \beta/b}{(2s-1)^3} \\
\times \{1 - \cosh (2s-1) \pi \alpha/b + \coth (2s-1) \pi \alpha/b \cdot \sinh (2s-1) \pi \alpha/b - \sinh (2s-1) \pi \alpha/b \cdot \cosech (2s-1) \pi \alpha/b\} \quad \cdots (2)
\]

Moreover, for the given values of \(a\) and \(b\), correct to at least five places of decimals we have \(\coth (2s-1) \pi \alpha/b\) equal to unity. With this simplification, the expression may be shown to reduce to

\[
\frac{\pi^2 \beta}{32a^2} \left(1 - \beta/b\right) - \frac{\pi}{4a^2} \sum_{s=1}^{\infty} \frac{\sin (2s-1) \pi \beta/b \cdot \exp (1-2s) \pi \alpha/b}{(2s-1)^3} \quad \cdots (3)
\]

I have been unable to complete the summation contained in (3), but the series converges sufficiently rapidly for a good approximation to be obtained from consideration of the first ten terms. For the particular case \(\alpha = 5.0, \beta = 0.6\), expression (3) gives a value \(3.712 \times 10^{-7}\) which differs appreciably from the value \(3.528 \times 10^{-7}\) found by the much more laborious method of direct computation. As it was required to evaluate the expression for the range of values of \(\alpha\) and \(\beta\), the total time saved was considerable.

This example shows that a knowledge of the more advanced parts of mathematics may be extremely useful in problems of numerical computation, and confirms the views previously expressed that training in numerical methods suitable for candidates proposing to make a career in the Scientific Civil Service should be given at the post-graduate level. T. J. WILLMORE.


Note 1825 suggests the theorem:
If $R(x)$ is rational and has no poles in the finite interval $a \leq x \leq b$, then

$$\int_a^b R(x) \, dx = \Sigma \text{res.} \left\{ R(z) \log \frac{z-b}{z-a} \right\}$$

where the right-hand side denotes the sum of all the residues of the function $R(z) \log \{(z-b)/(z-a)\}$.

**Proof.** Put $L(z) = \log \{(z-b)/(z-a)\}$, and let $\delta$ denote the "dumbbell" contour with end-points at $z = a$ and $z = b$. Then $L(z)$ is analytic, and $R(z)L(z)$ is analytic except for poles, at all points in the $z$-plane external to $\delta$, including the point at infinity. Consequently, if $C$ is a contour that surrounds $\delta$ and all the finite poles of $R(z)$, we have by Cauchy's theorem

$$\int_C = \int_\delta + 2\pi i \{\text{sum of finite residues of } R(z)L(z)\},$$

the integrand in both integrals being $R(z)L(z)$; and hence

$$\int_\delta = -2\pi i \{\text{sum of all residues, including that at infinity}\}.$$

The theorem follows, since, as may be easily shown,

$$\int_\delta = \int_\delta R(z) \log \frac{z-b}{z-a} \, dz = -2\pi i \int_a^b R(x) \, dx.$$

**Example.** Put $R(z) = z^n$, where $n$ is an integer, positive, negative or zero. Then, near $z = 0$,

$$z^n \log \{(z-b)/(z-a)\} = z^n \{\log (b/a) - (b-a)z - \frac{1}{2}(b^2-a^2)z^2 - \ldots \};$$

and near $z = \infty$,

$$z^n \log \{(z-b)/(z-a)\} = z^n \{- (b-a)/z - (b^2-a^2)/2z^2 - \ldots \}.$$

If $n \geq 0$, the only pole of $z^nL(z)$ is $z = \infty$, where the residue is $(b^{n+1} - a^{n+1})/(n+1)$. If $n \leq -1$, the only pole is $z = 0$, where the residue is $\log (b/a)$ if $n = -1$, or $-(b^{m+1} - a^{m+1})/(m-1)$ if $n = -m < -1$. The theorem now gives the well-known results.

**Note 1.** If we put $\zeta = (z-a)/(b-z)$, the theorem reduces to one given in Watson’s Cambridge Tract, *Complex Integration* (§ 29), but the form given here seems simpler.

**Note 2.** The theorem is given by Harkness and Morley, *Theory of Functions*, p. 197, under the severe restrictions that the poles of $R(z)$ should be simple and that $R(z) = O(1/z^2)$ at $z = \infty$, thus excluding, for instance, the case of $R(z) = z^n$ for every value of $n$.

1979. An area and a volume.

1. The area of a triangle.

With rectangular axes in the plane of the triangle, the equations to the sides can be written

$$a_r x + b_r y + c_r = 0, \quad r = 1, 2, 3.$$

If

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad D = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

where $A_1$ is the co-factor of $a$, in $\Delta$, and so on, then it is well known and easy to prove by determinant multiplication that $D = \Delta^3$; the coordinates of the vertices of the triangle can be found in terms of the co-factors and hence we have the known result for the area

$$S = \Delta^2 / 2C_1 C_2 C_3.$$