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RIGHT SEMIDEFINITE EIGENVALUE PROBLEMS

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Abstract The relationships between various notions of completeness of eigenvectors and root vectors of the eigenvalue problem $Af = \lambda Bf$ are investigated. Here A and B are self-adjoint operators in Hilbert space with B bounded and positive semidefinite, and with A having compact resolvent.

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1. Introduction

The purpose of this note is to compare various conditions related to eigenvector completeness for the right semidefinite generalized eigenvalue problem

$$Af = \lambda Bf, \quad 0 \neq f \in D(A), \tag{1.1}$$

where A and B are self-adjoint operators in a (finite- or infinite-dimensional) Hilbert space H. A complex number λ is called an eigenvalue if there exists a corresponding eigenvector f satisfying (1.1). The problem is right semidefinite in the sense that B is non-negative definite. We also assume that A has compact resolvent and that B is bounded. For non-triviality we assume

$$\mathbb{N}(A) \cap \mathbb{N}(B) = \{0\}, \tag{1.2}$$

$\mathbb{N}$ denoting nullspace, since otherwise every complex number is an eigenvalue. In our applications to differential equations, this assumption holds automatically unless $B = 0$.

We shall see in § 2 that the eigenvalues of (1.1) are all real, and there is at least one real number which is not an eigenvalue. This allows us to translate the λ origin and to produce a new equation where the corresponding A is invertible. Moreover, the set of eigenvalues of (1.1) has no finite point of accumulation and the multiplicity of each eigenvalue is finite. The linear space of all eigenvectors belonging to a given eigenvalue λ forms the corresponding (geometric) eigenspace. Its dimension is the multiplicity of λ. We shall say that the eigenvectors are complete if the eigenspaces have dense linear span, which in a Hilbert space setting means that there is an orthonormal basis of eigenvectors.

In the case in which B is positive definite, it is natural to discuss completeness in the Hilbert space H_b defined as the completion of H under the inner product $b[f,g] := (Bf,g)$. In the semidefinite case, b is no longer an inner product and in § 2 we shall define an appropriate analogue of H_b, which turns out to be related to (the closure of) $R(B)$. In § 3 we establish some necessary and sufficient conditions for completeness in H_b, and we also give some conditions which are sufficient but not necessary. The following simple example shows that the eigenvectors of (1.1) are not always complete in H_b.

Example 1.1. Let $H = \mathbb{C}^2$ and

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\dim H_b = \dim R(B) = 1$ but (1.1) has no eigenvalues.

From § 4 on, we assume that A is bounded below. This allows us to use variational methods to give a constructive proof of eigenvalue existence, and an interpretation of their indexing, via two parameter eigencurves. A new sufficient condition for eigenvector completeness is established in terms of eigencurve asymptotes. It is also a necessary condition if and only if $\dim H_b$ is finite. In § 5 we discuss the role of the self-adjoint operator $Q = A^{-1}B$ in the Pontryagin space H_a, which is defined via a form $a[f,g]$ which extends (Af,g). A standard notion of completeness in H_a concerns the root vectors of Q, but it turns out that this neither implies nor is implied by completeness of the eigenvectors in H_b. A related notion is completeness of the eigenvectors of Q in H_a, and this turns out to be equivalent to the eigencurve asymptote condition of § 5, and is thus strictly stronger than completeness in H_b.

We conclude with differential equations of the form (1.1), where B is a multiplication operator. In the case of Sturm–Liouville equations with L^1 coefficients, we show in § 6 that completeness in H_b (which we identify explicitly) is automatic. This is in sharp contrast to the case of indefinite B, where some condition (cf. [**4**]) on the weight function is necessary (cf. [**15**]). We also show that completeness may fail for such problems in H_a. For further discussion of semidefinite weight Sturm–Liouville problems we refer to [**10**]. In § 7 we consider certain elliptic partial differential equations with L^∞ coefficients, and again completeness in H_b is automatic. This improves on a result of Allegretto [**1**] where the weight function was assumed to vanish on a smooth domain.

2. Preliminaries

We start with the following elementary observation.

Lemma 2.1. *Equation (1.2) forces every eigenvector f of (1.1) to satisfy $Bf \neq 0$.*

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It is clear that if (1.2) fails, then every complex number is an eigenvalue. The converse is not difficult to establish, and in fact more is true.

Theorem 2.2. *Equation (1.2) is equivalent to the condition that no real interval of positive length consists of eigenvalues of (1.1).*

Proof. Suppose such a real interval I consists of eigenvalues of (1.1). Then by the Kato–Rellich Theorem [**11**, Theorem VII.3.9], there is an eigenvector $u(\lambda)$, of unit norm and analytic in λ, such that (1.1) holds with $f = u(\lambda)$, for all $\lambda \in I$. Differentiating we obtain $Au'(\lambda) = Bu(\lambda) + \lambda Bu'(\lambda)$. Taking the inner product with $u(\lambda)$ and recalling that A and B are self-adjoint, we have $(Bu(\lambda), u(\lambda)) = 0$ whence $u(\lambda) \in \mathbb{N}(B)$. This contradicts Lemma 2.1. The converse is immediate. $\square$

Suppose, then, that $\lambda_0 \in \mathbb{R}$ is not an eigenvalue. Then (1.1) can be rewritten

$$(A - \lambda_0 B)f = (\lambda - \lambda_0)Bf, \quad 0 \neq f \in D(A).$$

Since $A - \lambda_0 B$ has all the properties assumed of A, we may translate the λ origin to give the following corollary.

Corollary 2.3. *There is a change of eigenvalue parameter in (1.1) after which A becomes invertible.*

Below we shall assume this to have been carried out, unless otherwise stated.

We now consider the location of the eigenvalues.

Theorem 2.4. *The eigenvalues of (1.1) are real, non-zero, of finite multiplicity, and without finite accumulation.*

Proof. If f is an eigenvector, then $Bf \neq 0$ by Lemma 2.1, so $(Bf, f) > 0$. If the corresponding eigenvalue is non-real, however, a standard calculation gives $(Bf, f) = 0$, and we have a contradiction. Thus all eigenvalues are real. Zero cannot be an eigenvalue since A is invertible.

Now rewrite (1.1) in the form $A^{-1}Bf = \lambda^{-1}f$. Then the remaining contentions follow from compactness of $A^{-1}B$. $\square$

The next step is to construct the space H_b. The form $b[f, g] := (Bf, g)$ is positive semidefinite on H. We call two vectors $f, g \in H$ equivalent if $Bf = Bg$, so the set of equivalence classes is the quotient space $H/\mathbb{N}(B)$. We define H_b as the Hilbert space completion of $H/\mathbb{N}(B)$ under the form b.

As stated in § 1, H_b is related to the range $R(B)$ of B. Indeed, we have the following lemma.

Lemma 2.5. *The operator $B^{1/2}$ induces (in a natural way) an isometric isomorphism between H_b and the closure R of $R(B)$.*

Proof. Since $\mathbb{N}(B) = \mathbb{N}(B^{1/2})$, $B^{1/2}$ maps equivalent vectors to the same vector. Let $[f] \in H/\mathbb{N}(B)$ be the equivalence class of $f \in H$ and define $J[f] = B^{1/2}f$. If g is in the range $R(J)$, then $g = B^{1/2}f$ for some $f \in H$, and, for all $h \in \mathbb{N}(B)$, $(g, h) = (f, B^{1/2}h) = 0$. This shows that $R(J) \subset \mathbb{N}(B)^{\perp} = R$ and therefore $J : H/\mathbb{N}(B) \to R$.

Since $b[f, g] := (B^{1/2}f, B^{1/2}g)$, J is an isometry and thus has a continuous isometric extension $\bar{J} : H_b \to R$. It remains to prove that $R(\bar{J}) = R$, and this follows from $R(B) \subset R(B^{1/2}) \subset R(\bar{J}) \subset R$. □

Let p be the supremum of all integers $l \geqslant 0$ such that there exists a linear subspace L of H with $\dim L = l$ on which b is positive definite, that is, $b[f] := b[f, f] > 0$ for all non-zero $f \in L$. This number p (which can be finite or infinite) agrees with the dimension of H_b.

Let L be the linear span of all eigenvectors of (1.1). We have a natural map

$$E : H \to H_b \tag{2.1}$$

that assigns to every $f \in H$ its corresponding equivalence class. We say that the eigenvectors of (1.1) are complete in H_b if $E(L)$ is dense in H_b. In this case there exists an orthonormal basis for H_b consisting of (equivalence classes of) eigenvectors.

3. Completeness of eigenvectors

Our first result on completeness relates (1.1) to the symmetric operator

$$S := B^{1/2}A^{-1}B^{1/2} : H \to H.$$

Theorem 3.1. *The eigenvectors of (1.1) are complete in H_b if and only if $\mathbb{N}(S) = \mathbb{N}(B)$.*

Proof. Let $Af = \lambda Bf$, $0 \neq f \in D(A)$. Then $g := B^{1/2}f \neq 0$ satisfies $Sg = \lambda^{-1}g$. Conversely, let $Sg = \sigma g$, $0 \neq g \in H$ and $\sigma \neq 0$. Then $g = B^{1/2}f$ with $f := \sigma^{-1}A^{-1}B^{1/2}g \in D(A)$ and f satisfies $Af = \sigma^{-1}Bf$. Therefore, λ is an eigenvalue of (1.1) if and only if $\sigma = \lambda^{-1}$ is an eigenvalue of S, and $B^{1/2}$ maps the eigenspace of (1.1) belonging to the eigenvalue λ one-to-one onto the eigenspace of S belonging to the eigenvalue σ. Of course, S may have the eigenvalue 0, which does not correspond to an eigenvalue of (1.1).

Recall that R is the closure of $R(B)$, and that H is the orthogonal direct sum of its subspaces $\mathbb{N}(B)$ and R. Both subspaces $\mathbb{N}(B)$ and R are invariant under S, and $S|_{\mathbb{N}(B)}$ is the null operator. Hence the operator $S|_R : R \to R$ has the same non-zero eigenvalues as S with the same eigenspaces. Since $S|_R$ is compact and self-adjoint in R, its eigenvectors (including those belonging to the eigenvalue 0) are complete in R. Thus, using the connection between the eigenvectors of (1.1) and $S|_R$, the eigenvectors of (1.1) are complete in H_b if and only if $S|_R$ is one-to-one, that is, if and only if $\mathbb{N}(S) = \mathbb{N}(B)$. □

We turn now to another equivalence which will be used frequently in what follows. We write

$$Z_A := \mathbb{N}(B) \cap D(A).$$

Theorem 3.2. *The condition* $\mathbb{N}(S) = \mathbb{N}(B)$ *is equivalent to*

$$f \in Z_A,\ Af \in R(B^{1/2}) \text{ implies } f = 0. \tag{3.1}$$

Proof. Assume (3.1), and let $Sg = 0$. Define $f := A^{-1}B^{1/2}g$. Then $f \in Z_A$ and $Af = B^{1/2}g \in R(B^{1/2})$. Hence, by (3.1), $f = 0$ and so $g \in \mathbb{N}(B)$. This proves $\mathbb{N}(S) = \mathbb{N}(B)$. Now assume that $\mathbb{N}(S) = \mathbb{N}(B)$. Let $f \in Z_A$ and $Af \in R(B^{1/2})$. Choose $g \in H$ such that $Af = B^{1/2}g$. Then $0 = B^{1/2}f = Sg$. Hence $g \in \mathbb{N}(S) = \mathbb{N}(B)$ and so $Af = 0$. This implies $f = 0$. □

We now consider two further conditions, which both turn out to be sufficient for completeness:

$$f \in Z_A \text{ implies } Af \in \mathbb{N}(B) \tag{3.2}$$

and

$$f \in Z_A,\ (Af, g) = 0 \text{ for all } g \in \mathbb{N}(B) \text{ implies } f = 0. \tag{3.3}$$

In fact we have the following result.

Theorem 3.3. *Equation (3.2) implies (3.3), which in turn implies that the eigenvectors of (1.1) are complete in* H_b.

Proof. Suppose that (3.2) is satisfied. Let $f \in Z_A$ and $(Af, g) = 0$ for all $g \in \mathbb{N}(B)$. Thus $Af \in \mathbb{N}(B) \cap \mathbb{N}(B)^{\perp} = \{0\}$. Since A is invertible, $f = 0$ and so (3.3) also holds.

Now suppose that (3.3) is satisfied. Let $f \in Z_A$ and $Af = B^{1/2}h$. For all $g \in \mathbb{N}(B)$, $(Af, g) = (B^{1/2}h, g) = (h, B^{1/2}g) = 0$. Thus by (3.3), $f = 0$, and so (3.1) must hold. The result now follows from Theorems 3.1 and 3.2. □

Neither of the above implications is reversible. Indeed the first fails even in finite dimensions as follows.

Example 3.4. Let $H = \mathbb{C}^2$ and

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Writing e_1 and e_2 for the standard basis in $\mathbb{C}^2$, we see that $Be_1 = 0 \neq BAe_1$, so (3.2) fails. On the other hand, $\mathbb{N}(B)$ is spanned by e_1 but $(Ae_1, e_1) = 1$ so (3.3) holds trivially.

In finite dimensions $R(B^{1/2}) = R$ so (3.1) and (3.3) are the same, but the following shows that this is no longer true if $\dim H$ is infinite.

Example 3.5. Let $H = l^2$ with orthonormal basis e_n and

$$\begin{aligned} Ae_1 &= \sum_{n=2}^{\infty} n^{-1}e_n, \quad & Ae_n &= n^{-1}e_1 + ne_n, \quad & n > 1, \\ Be_1 &= 0, & Be_n &= n^{-1}e_n, & n > 1. \end{aligned}$$

Then $0 \neq e_1 \in Z_A$ and Ae_1 is orthogonal to $\mathbb{N}(B)$, so (3.3) fails. On the other hand, Ae_1 is not in $R(B^{1/2})$, so (3.1) holds trivially again.

4. Variational eigencurves

From now on we assume that A is bounded below. We summarize in Theorem 4.1 below some properties of such operators and their connections with sesquilinear forms. We recall that a form t is said to be closed if $u_n \in D(t)$, $u \in H$, $u_n \to u$ in H and $t[u_n - u_m, u_n - u_m] \to 0$ as $n, m \to \infty$ implies that $u \in D(t)$ and $t[u_n - u, u_n - u]$ as $n \to \infty$.

Theorem 4.1. *There is a one-to-one correspondence between the set of all self-adjoint operators T in H which are bounded below and the set of all densely defined, closed symmetric forms t in H which are bounded below. For a given operator T, the corresponding form t is the closure of the form (Tf, g), $f, g \in D(T)$. For a given form t, the corresponding operator T is uniquely determined by $D(T) \subset D(t)$ and $(Tf, g) = t[f, g]$ for every $f \in D(T)$ and $g \in D(t)$. Moreover, T has compact resolvent if and only if the corresponding form t has the following compactness property* (C).

> *Every bounded sequence $u_n \in D(t)$ for which $t[u_n, u_n]$ is bounded admits a subsequence that converges to a vector in $D(t)$.*

For the proof, we refer to Theorems 2.1, 2.7 and Corollary 2.2 in Chapter VI of [**11**] and Theorem XIII.64 of [**13**].

Let $a : D(a) \times D(a) \to \mathbb{C}$ be the form corresponding to A according to Theorem 4.1.

We denote the eigenvalues of $A - \lambda B$ in increasing order and counted according to multiplicity by

$$\mu_1(\lambda) \leqslant \mu_2(\lambda) \leqslant \mu_3(\lambda) \leqslant \cdots .$$

The 'eigencurves', which are the graphs of the $\mu_j : \mathbb{R} \to \mathbb{R}$, are continuous and non-increasing. A real number λ is an eigenvalue of (1.1) if and only if there exists j such that $\mu_j(\lambda) = 0$. We call j an index of λ. This index might not be unique but there is at most one eigenvalue for any given index because Theorem 2.2 implies that each μ_j can have at most one zero. The multiplicity of an eigenvalue λ equals the number of indices associated with λ.

The minimum–maximum principle states that

$$\mu_j(\lambda) = \min\{\max\{a[u] - \lambda b[u] : u \in F \cap U\} : F \subset D(a),\ \dim F = j\}, \tag{4.1}$$

where U denotes the unit sphere of H. If the eigenvalue λ_j of index j exists, we obtain

$$\lambda_j = \min\left\{\max\left\{\frac{a[u]}{b[u]} : u \in F \cap U,\ b[u] > 0\right\} : F \subset D(a),\ \dim F = j\right\}.$$

In order to decide on whether a given eigencurve μ_j has a zero we need the limits $\mu_j(\lambda)$ as $\lambda \to \pm\infty$. They can be determined as follows.

Let

$$Z_a := \mathbb{N}(B) \cap D(a), \quad z := \dim Z_a.$$

Then it follows that the form a restricted to Z_a is closed, symmetric, bounded below and has property (C). Let $\bar{Z}_a$ be the closure of Z_a in H. We now apply Theorem 4.1 to the

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form a restricted to Z_a considered as a form in the Hilbert space $\bar{Z}_a$. We obtain that there is a unique self-adjoint operator T, bounded from below, in $\bar{Z}_a$ with $D(T) \subset Z_a$ such that

$$(Tf, g) = a[f, g] \quad \text{for all } f \in D(T),\ g \in Z_a.$$

This operator T has compact resolvent. We denote its eigenvalues in increasing order counted according to multiplicity by

$$\tau_1 \leqslant \tau_2 \leqslant \tau_3 \leqslant \cdots .$$

By the minimum–maximum principle,

$$\tau_j = \min\{\max\{a[u] : u \in F \cap U\} : F \subset Z_a,\ \dim F = j\}.$$

Here j ranges over all positive integers if $z = \infty$ and from 1 to z if z is finite. To simplify notation, we define $\tau_j = -\infty$ for $j \leqslant 0$ and $\tau_j = \infty$ for $j > z$.

From [**5**, **8**] we know that

$$\lim_{\lambda \to -\infty} \mu_j(\lambda) = \tau_j, \qquad \lim_{\lambda \to \infty} \mu_j(\lambda) = \tau_{j-p}, \tag{4.2}$$

where we recall that $p = \dim H_b$. Using the intermediate-value theorem we conclude the following result.

Theorem 4.2.

(a) *Let $p = \infty$. Then (1.1) has an infinite number of eigenvalues. The eigenvalue with index j exists if and only if $\tau_j > 0$.*

(b) *Let $p < \infty$. Then (1.1) has at most p eigenvalues counted according to multiplicity. The eigenvalue with index j exists if and only if $\tau_j > 0$ and $\tau_{j-p} < 0$.*

By (4.2), the first z eigencurves have horizontal asymptotes as $\lambda \to -\infty$. The condition that no such asymptote is the λ-axis turns out to be a central one.

Theorem 4.3. *The following statements are equivalent.*

(i) *None of the eigencurves tends to $\mu = 0$ as $\lambda \to -\infty$.*

(ii) *Each $\tau_j \neq 0$.*

(iii) *The form a is non-degenerate on Z_a, i.e.*

$$f \in Z_a,\ a[f, g] = 0 \text{ for all } g \in Z_a \text{ implies that } f = 0. \tag{4.3}$$

Proof. All three statements are equivalent to injectivity of T. □

We now connect these ideas with completeness of eigenvectors.

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Theorem 4.4.

(a) *The conditions of Theorem 4.3 suffice for the eigenvectors of (1.1) to be complete in* H_b.

(b) *If* $p < \infty$, *then these conditions are also necessary.*

Proof.

(a) It is easy to see that (4.3) implies (3.3), so the result follows from Theorem 3.3.

(b) Since $\dim H_b = p$, eigenvector completeness is equivalent to the statement that exactly p eigencurves have zeros. By Theorem 4.2 (b), the latter statement is equivalent to condition (ii) of Theorem 4.3.

□

We note that (4.3) is sufficient for completeness by Theorem 4.4 but not necessary by Example 3.5. In fact, (4.3) is strictly stronger than (3.3) as the following example shows.

Example 4.5. Let $H = L^2[0,1]$, $D(A) = H^2[0,1] \cap H_0^1[0,1]$, $Af = -f'' - 12f$, $D(a) = H_0^1[0,1]$. Let $h(x) = x$ for $0 \leqslant x \leqslant \frac{1}{2}$ and $h(x) = 1 - x$ for $\frac{1}{2} < x \leqslant 1$. Let B be the orthogonal projection with $\mathbb{N}(B) = \operatorname{span} h$.

Then $Z_A = \{0\}$, so (3.3) holds. On the other hand, $Z_a = \mathbb{N}(B) = \operatorname{span} h$ and

$$a[h,h] = \int_0^1 h'(x)^2 \,\mathrm{d}x - 12 \int_0^1 h(x)^2 \,\mathrm{d}x = 1 - 1 = 0,$$

so (4.3) fails. (See [**11**, Chapter VI, Examples 2.16, 2.17] for a general discussion. Since $h \notin D(A)$, we use the above formula to calculate $a[h,h]$.)

5. Operators in Pontryagin spaces

In this section we discuss the relationship between (1.1) and self-adjoint operators on Pontryagin spaces.

The linear space $D(a)$ endowed with the (indefinite) inner product $a[f,g]$ becomes a Pontryagin space, which we denote by H_a (cf. [**2**, **9**]). Let $Q : D(a) \to D(a)$ be defined by $Qf = A^{-1}Bf$ so that $a[Qf,g] = b[f,g]$ for $f, g \in D(a)$. Note that $\mathbb{N}(Q) = Z_a$ of § 4.

Lemma 5.1. *The operator* Q *is self-adjoint, positive semidefinite and compact on* H_a.

Proof. Only the compactness of Q requires a proof. Choose γ so large that $A + \gamma I$ is (uniformly) positive definite. Then the topology of the Pontryagin space H_a is generated by the inner product $a[f,g] + \gamma(f,g)$. Since $(A+\gamma I)^{1/2}$ is a homeomorphism from H_a onto H, compactness of Q on H_a is equivalent to compactness of

$$(A+\gamma I)^{1/2} A^{-1} B (A+\gamma I)^{-1/2} = (A+\gamma I)^{-1/2}(I + \gamma A^{-1}) B (A+\gamma I)^{-1/2}$$

on H, and this follows because $(A+\gamma I)^{-1/2}$ is compact. □

Note that $Af = \lambda Bf$ is equivalent to $Qf = \lambda^{-1} f$ if $\lambda \neq 0$. Thus the non-zero eigenvalues of Q are all real by Theorem 2.4. Moreover, Q can also have eigenvalue 0, which does not correspond to an eigenvalue of (1.1). We shall need the following elementary result on the root space L_σ of Q belonging to an eigenvalue σ.

Lemma 5.2. *If $0 \neq \sigma \in \mathbb{R}$, then $L_\sigma = \mathbb{N}(Q - \sigma I)$.*

Proof. Suppose that a Jordan chain of length at least two exists for Q at σ, where $0 \neq \sigma \in \mathbb{R}$, so $Qf = \sigma f$, $(Q - \sigma)g = f$, say, with $f \neq 0$. Then

$$(f, Af) = a[f, f] = a[f, (Q - \sigma)g] = a[(Q - \sigma)f, g] = 0.$$

Since $\sigma Af = Bf$, we obtain $(f, Bf) = 0$. It then follows that $Bf = 0$, whence $f = \sigma^{-1} A^{-1} Bf = 0$, a contradiction. □

A similar argument shows that $L_0 = \mathbb{N}(Q^2)$. We shall also need a basic completeness result of Azizov and Iohvidov (see [**2**, Lemma 2.14, p. 230], also [**7**]).

Theorem 5.3. *The root vectors of Q are complete in H_a if and only if a is non-degenerate on L_0, i.e.*

$$f \in L_0,\ a[f, g] = 0 \text{ for all } g \in L_0 \text{ implies that } f = 0. \tag{5.1}$$

According to Lemma 5.1 and [**2**, Section 4.1], [**9**, Section VIII.6], Q obeys the Krein–Langer spectral theorem with a critical point at 0. The non-degeneracy condition of Theorem 5.3 can then be expressed as regularity of this critical point. For more on such connections we cite [**6**, **12**].

Unfortunately, the conditions of Theorem 5.3 neither imply nor are implied by completeness in H_b. This can be seen from our previous examples. Indeed, Example 1.1 satisfies (5.1) since $L_0 = \mathbb{C}^2$ and A is invertible, but as we have already remarked, completeness fails in H_b.

Also Example 4.5 satisfies (3.2), and hence completeness in H_b by Theorem 3.3. On the other hand, $Qg = h$ cannot be satisfied, so there are no Jordan chains of length greater than one for Q at 0, i.e.

$$L_0 = \mathbb{N}(Q) = Z_a. \tag{5.2}$$

Since (4.3) fails, so does (5.1), and with it root vector completeness in H_a by Theorem 5.3.

If (4.3) holds, however, then we can relate the spectral theory of Q to that of the previous sections.

Theorem 5.4. *The eigenvectors of (1.1) are complete in H_a if and only if (4.3) holds.*

Proof. If the eigenvectors are complete in H_a, then (5.2) holds since each root vector in L_0 must in fact be an eigenvector. Thus (4.3) follows from Theorem 5.3.

Conversely, suppose that (4.3) holds, and that a Jordan chain of length at least two exists for Q at 0, so $Qf = 0$, $Qg = f$, say, for $f \neq 0$. Then, for all $h \in \mathbb{N}(Q)$,

$$a[f, h] = a[Qg, h] = a[g, Qh] = 0,$$

so $f = 0$ by (4.3), a contradiction. Thus (5.2) also holds and we can apply Theorem 5.3 to complete the proof. □

We remark that Example 4.5 shows that (5.2) may not be substituted for (4.3) in Theorem 5.4. This result also allows us to give a second proof of Theorem 4.4 (a). Consider the natural map (2.1) as a mapping from the Pontryagin space H_a into the Hilbert space H_b. This map is continuous and its range is dense. The linear span of all eigenvectors of Q belonging to eigenvalues different from 0 agrees with the linear span L of all eigenvectors of (1.1). Assuming (4.3), we conclude that the linear span of all eigenvectors of Q is dense in H_a, and hence that $E(L)$ is dense in H_b. This completes the proof.

6. Application to Sturm–Liouville eigenvalue problems

We consider the Sturm–Liouville eigenvalue problem

$$-(py')' + qy = \lambda ry, \quad -\infty < a \leqslant x \leqslant b < \infty, \tag{6.1}$$

subject to the boundary conditions

$$\alpha_0 y(a) + \alpha_1 (py')(a) = 0, \qquad \beta_0 y(b) + \beta_1 (py')(b) = 0. \tag{6.2}$$

We assume that $1/p, q, r \in L^1[a,b]$ are real valued with $p > 0$, $r \geqslant 0$ and

$$(\alpha_0, \alpha_1), (\beta_0, \beta_1) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

By means of a standard change of measure [**15**] we may assume that $p = 1$ and then (6.1) can be written

$$(r+1)^{-1}(-y'' + qy) = \lambda (r+1)^{-1} ry,$$

which can be put in the form (1.1). The underlying Hilbert space is the weighted space $H = L^2_{r+1}[a,b]$ with norm given by

$$\|f\|^2 = \int_a^b (r(x)+1)|f(x)|^2 \, \mathrm{d}x.$$

The domain $D(A)$ consists of all functions f with $f, f' \in AC[a,b]$ that satisfy $(r+1)^{-1}(-f'' + qf) \in H$ and the boundary conditions (6.2). The operators A and B are defined by

$$Af = (r+1)^{-1}(-f'' + qf), \qquad Bf = (r+1)^{-1} rf,$$

and it is easily seen that they satisfy the conditions imposed in § 1, except for (1.2).

If $r = 0$ a.e., then λ does not even enter (1.1), so assume that $r > 0$ on a set S of positive measure. If $Bf = 0$, then $f = 0$ on S so $f(x) = f'(x) = 0$ for any accumulation point $x \in S$. Thus if in addition $Af = 0$, then $f = 0$ since A is a second-order operator, and so (1.2) holds automatically.

Lemma 6.1. *Let $f \in AC[a,b]$, and let M be a measurable subset of $[a,b]$. Assume that $f(x) = 0$ for $x \in M$ a.e. Then $f'(x)$ exists and $f'(x) = 0$ for $x \in M$ a.e.*

Proof. Write $M = A \cup N$, where A is a null set and $f(x) = 0$ for all $x \in N$. Let B be a null set such that $f'(x)$ exists for $x \in [a, b] \setminus B$. Write the closed set $\bar{N}$ as $\bar{N} = C \cup P$, where C is finite or countable and P is perfect. Since f is continuous, $f(x) = 0$ for all $x \in P$. Hence $f'(x) = 0$ for all $x \in P \setminus B$. Since $M \setminus (P \setminus B) \subset A \cup B \cup C$, we obtain that $f'(x) = 0$ for $x \in M$ a.e. □

We are now ready for the completeness result in H_b.

Theorem 6.2. *The eigenvectors of (6.1), (6.2) are complete in* H_b.

Proof. We show that condition (3.2) is satisfied. Let $f \in \mathbb{N}(B) \cap D(A)$. Then $f(x) = 0$ for $x \in M$ a.e., where $M := \{x \in [a, b] : r(x) > 0\}$, and $f, f' \in AC[a, b]$. Applying Lemma 6.1 twice, we obtain that $f''(x) = 0$ for $x \in M$ a.e. This implies that $(Af)(x) = 0$ for $x \in M$ a.e. and so $Af \in \mathbb{N}(B)$. □

Completeness in H_a may fail as follows.

Example 6.3. Let H and $D(A)$ be as in Example 4.5, with $Af = -f'' - 4\pi^2 f$ and $Bf(x) = 0$ for $x \in [0, \frac{1}{2}]$, $Bf(x) = r(x)f(x)$ otherwise, where $0 \neq r \in L^\infty$.

Then T of § 4 is the Dirichlet operator corresponding to A on $[0, \frac{1}{2}]$, so $\tau_1 = 0$ and by Theorem 4.3, (4.3) fails. We note that an eigenvalue shift as in Corollary 2.3 can be used to ensure that A is invertible.

7. Application to eigenvalue problems for elliptic partial differential operators

Let Ω be an open, connected and bounded subset of $\mathbb{R}^k$. On Ω we consider the eigenvalue problem

$$-\Delta f + q(x)f = \lambda r(x)f$$

subject to Dirichlet boundary conditions. We assume that $q, r \in L^\infty(\Omega)$ are real valued with $r(x) \geqslant 0$ for all $x \in \Omega$. The underlying Hilbert space is $H = L^2(\Omega)$ and the operator B is given by $(Bf)(x) = r(x)f(x)$. The self-adjoint operator $Af = -\Delta f + qf$ subject to Dirichlet boundary conditions is defined as follows (see [**3**]). Let a be the Dirichlet form defined on $D(a) = H_0^1(\Omega)$:

$$a[f, g] := \sum_{j=1}^{k} (\partial_j f, \partial_j g) + (qf, g),$$

where $(\cdot, \cdot)$ is the inner product in H. This form is densely defined, closed, symmetric and bounded from below in H. The operator A is the self-adjoint operator corresponding to this form via Theorem 4.1. By definition, a function $f \in D(a)$ belongs to $D(A)$ if there is $h \in H$ such that

$$a[f, g] = (h, g) \quad \text{for all } g \in D(a).$$

Then $Af := h$. The self-adjoint operator A is bounded from below with compact resolvent.

Note that the equation $Af = g$ is equivalent to the statement that f is a weak solution of $-\Delta f + qf = g$. The interior regularity result on such weak solutions [**3**, p. 141] leads to the following lemma.

Lemma 7.1. *Let $f \in D(A)$, and let Ω_0 be an open set with $\bar{\Omega}_0 \subset \Omega$. Then $f|_{\Omega_0}$ lies in $H^2(\Omega_0)$ and $Af = -\Delta f + qf$ on Ω_0.*

We now argue as in § 6 and assume for non-triviality that $r > 0$ on a set of positive measure. By Lemma 7.1 and a strong unique continuation property for elliptic differential equations [**14**, Theorem 1.2 and references] we again see that condition (1.2) holds automatically.

In order to verify condition (3.2), we will need the following result [**3**, Lemma 3.7.2].

Lemma 7.2. *Let $f \in H^1(\Omega)$, and let M be a measurable subset of Ω such that $f(z) = 0$ for almost all $z \in M$. Then $\partial_1 f(z) = \cdots = \partial_k f(z) = 0$ for almost all $z \in M$.*

We can now prove the main result of this section.

Theorem 7.3. *Let $Af = -\Delta f + qf$, $Bf = rf$ be the self-adjoint operators as defined at the beginning of this section. Then the eigenvectors of $Af = \lambda Bf$ are complete in H_b.*

Proof. We show that condition (3.2) is satisfied. Let $f \in D(A) \cap \mathbb{N}(B)$. Then $f(x) = 0$ for $x \in M$ a.e., where $M := \{x \in \Omega : r(x) > 0\}$. Let Ω_0 be an open set whose closure is contained in Ω. By Lemma 6.1, $f|_{\Omega_0} \in H^2(\Omega_0)$. Let $M_0 := \Omega_0 \cap M$ so $f(x) = 0$ for $x \in M_0$ a.e. Applying Lemma 7.2 several times, we obtain $\Delta f(x) = 0$ for $x \in M_0$ a.e. This implies that $Af(x) = 0$ for $x \in M_0$ a.e. Exhausting Ω by a sequence of open subsets whose closure is contained in Ω, we obtain that $Af(x) = 0$ for $x \in M$ a.e. This shows that $Af \in \mathbb{N}(B)$, which completes the proof. □

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