ON VARIOUS TYPES OF BARRELLEDNESS AND THE HEREDITARY PROPERTY OF (DF)-SPACES

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(Received 1 April, 1975; revised 20 October, 1975)

1. Introduction. Recently, Levin and Saxon [5], De Wilde and Houet [2] defined the \( \sigma \)-barrelledness while Husain [3] defined the countable barrelledness and countable quasi-barrelledness. It is well-known that barrelled spaces are countably barrelled, and countably barrelled spaces are \( \sigma \)-barrelled. It is natural to ask whether there is some condition for \( \sigma \)-barrelled (resp. countably barrelled) spaces to be countably barrelled (resp. barrelled). Using the concept of \( S \)-absorbent sequences of sets, we are able to give such conditions in Theorem 2.5 and Corollaries 2.6 and 2.7.

Valdivia [9], Saxon and Levin [8] have shown that every vector subspace with countable codimension of a barrelled space is barrelled. Also Levin and Saxon showed in [5] that this hereditary property is true for \( \sigma \)-barrelled spaces. In §3, we show that this hereditary property is also true for countably barrelled spaces as well as for \( \sigma \)-barrelled \((DF)\)-spaces, which is a generalization of Valdivia [10, Theorem 3].

The final section is devoted to some properties of \( S \)-absorbent sequences of sets which extend some results of Valdivia [9], De Wilde and Houet [2].

2. The relationship between various types of barrelledness. Let \((E, T)\) be a Hausdorff locally convex space whose topological dual is denoted by \( E' \). If \( B \) is a subset of \( E \) (resp. \( E' \)), then the polar of \( B \), taken in \( E' \) (resp. \( E \)), is denoted by \( B^0 \). By a topologizing family (t. family, for short) for \( E' \) (resp. \( E \)) we mean a family \( S \) consisting of (convex circled) \( a(E, E') \)-bounded subsets of \( E \) (resp. \( a(E', E) \)-bounded subsets of \( E' \)) such that \( \cup \{ B: B \in S \} = E \) (resp. \( E' \)). For a t. family \( S \) for \( E' \) (resp. \( E \)), the topology on \( E' \) (resp. \( E \)) of uniform convergence on \( S \) is denoted by \( T_S \).

Let \( S \) be a t. family for \( E' \). We denote by \( S^b \) the family of all \( T_S \)-bounded subsets of \( E' \). Clearly \( S^b \) is again a t. family for \( E \). The topology on \( E \) of uniform convergence on \( S^b \) is denoted by \( T_S^b \), therefore we have \( T_S^{b^b} = T_S^b \). Similarly we can define \( S^{b^b} \) and \( T_S^{b^b} \), where \( S^{b^b} = S^{b_1 \ldots b} \) and \( T_S^{b^b} = T_S^{b_1 \ldots b} \), the superscript \( b \) being repeated \( n \) times in each case; consequently we have \( T_S^{b^b} = T_S^{b_n} \) for all \( n \geq 1 \).

If \( S \) is a t. family for \( E' \), let us say temporarily that \( S^b \) (resp. \( S^{b^b} \)) is the bounded-polar (resp. bounded-bipolar) family of \( S \), and that \( T_S^b \) (resp. \( T_S^{b^b} \)) is the bounded-polar (resp. bounded-bipolar) topology of \( T_S \). It is clear that \( \{ S^0 : S \in S^b \} \) forms a neighbourhood base at 0 for the bounded-polar topology \( T_S^b \), and that \( \{ B^0 : B \in S^{b^b} \} \) forms a neighbourhood base at 0 for the bounded-bipolar topology \( T_S^{b^b} \). If \( S_1 \) and \( S_2 \) are two t. families for \( E' \) with \( S_1 \subseteq S_2 \), then \( S_2^b \subseteq S_1^b \).

**Lemma 2.1.** For a t. family \( S \) for \( E' \), we have:

(a) \( S \subseteq S^{b^b} \),

† This research was supported by an N.R.C. grant.
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(b) $T_s \leq T^{bb}_s$; 
(c) $S^b = S^{3b}$ and $T^b_s = T^{3b}_s$.

The proof is straightforward and will be omitted. We shall see that the inclusion in (a) may be strict.

In the sequel we denote by $\beta(E, E')$ the strong topology on $E$, i.e., the topology of uniform convergence on all $\sigma(E', E)$-bounded subsets of $E'$, and by $\beta^*(E, E')$ the topology of uniform convergence on all $\beta(E', E)$-bounded subsets of $E'$. As usual, $\tau(E, E')$ denotes the Mackey topology on $E$. Clearly,

$$\tau(E, E') \leq \beta^*(E, E') \leq \beta(E, E')$$

and

$$\tau(E', E) \leq \beta^*(E', E) \leq \beta(E', E).$$

It is not hard to see that each $\sigma(E, E')$-bounded subset of $E$ is $\beta^*(E, E')$-bounded, and that dually each $\sigma(E', E)$-bounded subset of $E'$ is $\beta^*(E', E)$-bounded.

EXAMPLES. (1) If $S_f$ is the family of all finite subsets of $E$, then we have that $T_{S_f} = \sigma(E', E)$; $T^{bb}_{S_f} = \beta(E, E')$; $T^b_{S_f} = \beta^*(E', E)$. Therefore we conclude that $S_f \neq S^{bb}_f$ and $T_{S_f} \neq T^{bb}_{S_f}$, in general.

(2) If $S_\beta$ is the family of all $\beta(E, E')$-bounded subsets of $E$, then we have $T_{S_\beta} = \beta^*(E', E)$, $S^{bb}_\beta$ is the family of all $\sigma(E', E)$-bounded subsets of $E'$ and $T^b_{S_\beta} = \beta(E, E')$. Therefore we conclude that $S_\beta = S^{bb}_\beta$ and $T_{S_\beta} = T^{bb}_{S_\beta}$.

(3) If $S_\sigma$ is the family of all $\sigma(E, E')$-bounded subsets of $E$, then we have $T_{S_\sigma} = \beta(E', E)$, $T_{S_\sigma}^b = \beta^*(E', E')$ and $S^{bb}_\sigma$ is the family of all $\sigma(E', E')$-bounded subsets of $E$.

(4) Let $S_c$ be the family of all $T$-compact convex circled subsets of $E$ and let $c(E', E)$ be the topology on $E'$ of uniform convergence on $S_c$. Then $\sigma(E', E) \leq c(E', E) \leq \tau(E', E)$; furthermore we have $T_{S_c} = c(E', E)$, $T^{bb}_{S_c} = \beta(E, E')$ and $T^b_{S_c} = \beta^*(E', E)$.

DEFINITION 2.2. Let $(E, T)$ be a locally convex space and $S$ a t. family for $E'$. Then $E$ is said to be

1) S-barreled if each member in $S^b$ is $T$-equicontinuous;
2) countably S-barreled if each member of $S^b$ which is the countable union of $T$-equicontinuous subsets of $E$ is $T$-equicontinuous;
3) $\sigma$-S-barreled if each member in $S^b$ which is a countable set is $T$-equicontinuous.

If $S$ is the family of all finite subsets of $E$, then $E$ is S-barreled (resp. countably S-barreled, $\sigma$-S-barreled) if and only if it is barreled (resp. countably barreled, $\sigma$-barreled) under the usual terminology of [4] and [6] (resp. [3], [2]). $\sigma$-barreled spaces are also called $\omega$-barreled by Levin and Saxon [5]. Clearly each $\beta(E', E)$-bounded set is in $S^b$ for any t. family for $E'$. Hence $E$ is quasi-barreled (or countably quasi-barreled or $\sigma$-evaluable) if $E$ is S-barreled (or countably S-barreled or $\sigma$-S-barreled).

If $S$ is the family of all $\sigma(E, E')$-bounded subsets of $E$, then $E$ is S-barreled (resp. countably S-barreled, $\sigma$-S-barreled) if and only if $E$ is quasi-barreled (resp. countably
quasibarrelled, σ-evaluable) under the usual terminology of [4] [6] (resp. [3], [2]). Here we call σ-evaluable spaces σ-infrabarrelled.

If \((E, T)\) is a locally convex Riesz space and if \(S\) is the family of all order-bounded subsets of \(E\), then \(E\) is \(S\)-barrelled if and only if it is order-infrabarrelled under the usual definition of [11].

As a consequence of Lemma 2.1, we have the following result.

**Lemma 2.3.** Let \(S\) be a t. family for \(E'\). \(E\) is \(S\)-barrelled (resp. countably \(S\)-barrelled, σ-\(S\)-barrelled) if and only if \(E\) is \(S^{\text{bb}}\)-barrelled (resp. countably \(S^{\text{bb}}\)-barrelled, σ-\(S^{\text{bb}}\)-barrelled).

In particular, \((E, T)\) is barrelled if and only if each \(σ(E, E')\)-closed convex circled subset of \(E\) which absorbs all \(β(E, E')\)-bounded subsets of \(E\) is a \(T\)-neighbourhood of 0.

Using a standard argument, for instance, see Schaefer [6] and Köthe [4, p. 396], it is easily seen that \(E\) is \(S\)-barrelled if and only if each closed convex circled subset of \(E\) which absorbs all members of \(S\) is a \(T\)-neighbourhood of 0, and that \(E\) is countably \(S\)-barrelled if and only if for any sequence \((V_n)\) of closed convex circled \(T\)-neighbourhoods of 0, if \(V = \bigcap_{n=1}^\infty V_n\) absorbs all members in \(S\) then \(V\) is a \(T\)-neighbourhood of 0.

In order to give a dual characterization of the σ-\(S\)-barrelledness, we require the following terminology. Let \(S\) be a t. family for \(E'\). By an \(S\)-absorbent sequence (of closed sets) in \(E\) we mean a sequence \(\{V_n: n \geq 1\}\) of (closed) convex circled sets in \(E\) for which the following two conditions are satisfied:

1. \(V_n \subseteq V_{n+1}\) for all \(n \geq 1\);
2. each member in \(S\) is absorbed by some \(V_n\).

If \(S\) is the family of all finite subsets of \(E\), then \(\{V_n: n \geq 1\}\) is an \(S\)-absorbent sequence if and only if it is an absorbent sequence in \(E\) in the sense of [2]; and if \(S\) is the family of all \(σ(E, E')\)-bounded subsets of \(E\), then \(\{V_n: n \geq 1\}\) is a σ-absorbent sequence if and only if it is a bounded-absorbent sequence in the sense of [2].

**Proposition 2.4.** Let \(S\) be a t. family for \(E'\). Then \(E\) is σ-\(S\)-barrelled if and only if for any \(S\)-absorbent sequence \(\{V_n: n \geq 1\}\) in \(E\), the sequence \(\{f_n: n \geq 1\}\) is equicontinuous, where \(f_n \in V_n^0\) for all \(n \geq 1\).

**Proof.** Necessity. For any \(S \in S\) there exists \(λ > 0\) and \(n_0 > 1\) such that \(S \subseteq λV_n\) for all \(n \geq n_0\). For each \(n \geq 1\), let \(f_n \in V_n^0\). Then \(|f_n(x)| \leq λ\) for all \(x \in S\) and \(n \geq n_0\). Since \(S\) is \(σ(E, E')\)-bounded, there exists \(μ > 0\) with \(|f_n(x)| \leq μ\) for all \(x \in S\) and \(n = 1, \ldots, n_0 - 1\). Thus \(\sup \{|f_n(x)|: x \in S, n \geq 1\} \leq \max (λ, μ) < \infty\) and so \(\{f_n: n \geq 1\}\) is equi-continuous.

Sufficiency. Let \(\{h_n: n \geq 1\}\) be a \(T_S\)-bounded sequence in \(E'\). For each \(k \geq 1\), we define \(V_k = \{x \in E: |h_n(x)| \leq 1\text{ for all }n \geq k\}\).

Then \(\{V_k: n \geq 1\}\) is an \(S\)-absorbent sequence in \(E\). As \(h_k \in V_k^0\) for all \(k \geq 1\), we conclude from the hypothesis that \(\{h_k: k \geq 1\}\) is equi-continuous. This shows that \(E\) is σ-\(S\)-barrelled.
If $S_1$ and $S_2$ are two t. families for $E'$ such that $S_1 \subseteq S_2$, then the following implications hold:

$S_1$-barrelledness $\Rightarrow$ countably $S_1$-barrelledness $\Rightarrow$ $\sigma$-$S_1$-barrelledness

$\sigma$-$S_2$-barrelledness $\Rightarrow$ countably $S_2$-barrelledness $\Rightarrow$ $\sigma$-$S_2$-barrelledness.

Therefore it is natural to ask under what conditions on $E$ (or $E'$) the corresponding converse implications hold. We have the following result.

**Theorem 2.5.** Let $S_1$ and $S_2$ be two t. families for $E'$ such that $S_1 \subseteq S_2$. Then $(E, T)$ is $\sigma$-$S_1$-barrelled (resp. countably $S_1$-barrelled, $S_1$-barrelled) if and only if the following two conditions hold:

(i) $E$ is $\sigma$-$S_2$-barrelled (resp. countably $S_2$-barrelled, $S_2$-barrelled);

(ii) each $S_1$-absorbent sequence of closed sets in $E$ is $S_2$-absorbent.

**Proof.** Suppose that $E$ is $\sigma$-$S_1$-barrelled and that $\{V_n : n \geq 1\}$ is an $S_1$-absorbent sequence of closed sets in $E$ which is not $S_2$-absorbent. Then there exists $B \in S_2$ such that $B \not\subseteq nV_n$ is false for all natural numbers $n \geq 1$. For each $n \geq 1$, let $x_n$ in $B$, be such that $x_n \not\in nV_n$. As $V_n$ is closed convex and circled, the bipolar theorem ensures that there exists $f_n \in V_n^0$ such that 

$$|f_n(x_n)| > n. \quad (1)$$

As $E$ is $\sigma$-$S_1$-barrelled and $\{V_n : n \geq 1\}$ is an $S_1$-absorbent sequence of closed sets in $E$, it follows from Proposition 2.4 that $\{f_n : n \geq 1\}$ is $T$-equicontinuous sequence, and hence that $\{f_n : n \geq 1\}$ is $T_{S_2}$-bounded; consequently $\{f_n : n \geq 1\}$ must be absorbed by $B^0$, contrary to the inequality (1). Therefore the conditions are necessary. We show that the conditions are also sufficient.

Let $\{f_n : n \geq 1\}$ be a $T_{S_1}$-bounded sequence in $E'$. For each $k \geq 1$, let

$$V_k = \{x \in E : |f_n(x)| \leq 1 \text{ for all } n \geq k\}.$$ 

The $T_{S_1}$-boundedness of $\{f_n : n \geq 1\}$ ensures that $\{V_n : n \geq 1\}$ is an $S_1$-absorbent sequence of closed sets in $E$, and hence $\{V_n : n \geq 1\}$ is $S_2$-absorbent by the hypotheses. On the other hand, since $E$ is assumed to be $\sigma$-$S_2$-barrelled and since $f_n \in V_n^0$ for all $n \geq 1$, it follows from Proposition 2.4 that $\{f_n : n \geq 1\}$ is $T$-equicontinuous, and hence that $E$ is $\sigma$-$S_1$-barrelled.

The necessity part of the proof for countably $S_1$-barrelled and $S_1$-barrelled spaces is similar and so is omitted. The sufficiency part for all cases can be handled as follows. Observe that $S_1^b \supset S_2^b$. To show that (ii) implies $S_1^b = S_2^b$, let $A \in S_1^b$ and $A \not\subseteq S_2^b$. Then there is $B \in S_2$, a sequence $\{x_n\} \subseteq B$ and a sequence $\{f_n\} \subseteq A$ such that $|f_n(x_n)| > n$ for all $n \geq 1$. Since $V_n = \{x \in E : |f_n(x)| \leq 1 \text{ for } m \geq n\}$ is an $S_1$-absorbent sequence of closed sets in $E$, it follows by (ii) that it is also $S_2$-absorbent. Hence there exist $n$ and $\lambda$ such that $B \subseteq \lambda V_n$, a contradiction.

**Remark.** $E$ is $\sigma$-barrelled (resp. countably barrelled, barrelled) if and only if it is $\sigma$-infrabarrelled (resp. countably quasibarrelled, infrabarrelled) and each absorbent sequence of closed sets in $E$ is bounded-absorbent.
Corollary 2.6. Let $S_1$ and $S_2$ be two $t$. families for $E'$ such that $S_1 \subset S_2$. Then:

(a) $E$ is countably $S_1$-barrelled if and only if it is countably $S_2$-barrelled as well as $\sigma$-$S_1$-barrelled;

(b) $E$ is $S_1$-barrelled if and only if it is $\sigma$-$S_1$-barrelled as well as $S_2$-barrelled.

Proof. If $E$ is countably $S_1$-barrelled, then it is obvious that $E$ is countably $S_2$-barrelled as well as $\sigma$-$S_1$-barrelled. Conversely, if $E$ is countably $S_2$-barrelled and if $E$ is $\sigma$-$S_1$-barrelled, then by Theorem 2.5, each $S_1$-absorbent sequence of closed sets in $E$ is $S_2$-absorbent. We conclude from Theorem 2.5 again that $E$ is countably $S_1$-barrelled. This proves the assertion (a). The proof of (b) is similar.

Remark. $E$ is countably barrelled if and only if it is $\sigma$-barrelled and countably quasi-barrelled. $E$ is barrelled if and only if it is countably barrelled and quasibarrelled.

Corollary 2.7. Let $S_1$ and $S_2$ be two $t$. families for $E'$ such that $S_1 \subset S_2$. Then the following assertions hold:

(a) Let $E$ be countably $S_2$-barrelled. Then $E$ is countably $S_1$-barrelled if and only if each $S_1$-absorbent sequence of closed sets in $E$ is $S_2$-absorbent.

(b) Let $E$ be $S_2$-barrelled. Then $E$ is $S_1$-barrelled if and only if $E$ is $\sigma$-$S_1$-barrelled, and this is the case if and only if each $S_1$-absorbent sequence of closed sets in $E$ is $S_2$-absorbent.

Proof. (a) follows from Theorem 2.5 and Corollary 2.6 (a), while (b) follows from Corollary 2.6 (b) and the assertion (a) of this corollary.

Let $E$ be a locally convex space. A convex circled $\sigma(E', E)$-bounded subset $B$ of $E$ is said to be infracomplete if the normed space $E(B) = \bigcup_n nB$ equipped with the norm $\| \cdot \|_B$ defined by

$$\| x \|_B = \inf \{ \lambda \geq 0 : x \in \lambda B \} \ (x \in E(B))$$

is complete. It is clear that every convex circled $\sigma(E, E')$-bounded and $\tau(E, E')$-sequentially complete subset of $E$ is infracomplete. By the Banach–Mackey theorem, we see that every infracomplete subset $B$ of $E$ is $\beta(E, E')$-bounded (see [4, §20, 11(3)]).

Levin and Saxon [5] say that a locally convex space $E$ has the property (C) (resp. the property (S)) if every $\sigma(E', E)$-bounded subset of $E'$ is $\sigma(E', E)$-relatively countably compact (resp. $E'$ is $\sigma(E', E)$-sequentially complete). As a consequence of the result mentioned above ([4, §20, 11(3)]), we obtain the following result which gives a connection between $\sigma$-barrelledness and the property (S).

Proposition 2.8. For a $\sigma$-infrabarrelled locally convex space $E$, the following statements are equivalent:

(a) $E$ is $\sigma$-barrelled;

(b) $E$ has the property (C);

(c) $E$ has the property (S);

(d) each $\sigma(E', E)$-bounded, $\sigma(E', E)$-closed subset of $E'$ is $\sigma(E', E)$-sequentially complete.

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Proof. The implications (a) ⇒ (b) ⇒ (c) ⇒ (d) are obvious. Finally, if the statement (d) holds, then by the Banach–Mackey theorem each \( \sigma(E', E) \)-bounded subset of \( E' \) is \( \beta(E', E) \)-bounded, and thus the implication (d) ⇒ (a) follows.

Consider the vector space \( m \) of all bounded sequences with the Mackey topology \( \tau(m, l_1) \). Levin and Saxon have shown in [5, Proposition 6] that \( (m, \tau(m, l_1)) \) is a Mackey space with the property (S) but not property (C). According to this result and Proposition 2.8, we conclude that Mackey spaces are, in general, not \( \sigma \)-infrabarrelled spaces.

As another consequence of the Banach–Mackey theorem, we have the following result.

**Proposition 2.9.** Let \( E \) be a locally convex space for which every \( \sigma(E', E) \)-bounded closed set is \( \sigma(E', E) \)-sequentially complete (equivalently, \( E \) has the property (S)). Then the following assertions hold.

1. If \( E \) is infrabarrelled (in particular, bornological) then it is barrelled.
2. If \( E \) is countably infrabarrelled then it is countably barrelled.

Proof. According to the Banach–Mackey theorem each \( \sigma(E', E) \)-bounded subset of \( E' \) is \( \beta(E', E) \)-bounded, and the result follows.

Since metrizable locally convex spaces are infrabarrelled, part (1) of the preceding result is a generalization of Saxon [7, Theorem 2.7]. The following corollary is now immediate.

**Corollary 2.10.** Let \( E \) be a locally convex space in which every \( \sigma(E, E') \)-bounded closed set is \( \tau(E, E') \)-sequentially complete (in particular, \( E \) is either \( \tau(E, E') \)-sequentially complete or quasi-complete). Then the following assertions hold.

1. If \( E \) is \( \sigma \)-infrabarrelled then \( E \) is \( \sigma \)-barrelled and a fortiori has the property (S).
2. If \( E \) is countably infrabarrelled (resp. barrelled) then it is countably barrelled (resp. barrelled).

3. The hereditary property. Saxon, Levin [8] and Valdivia [9] have shown independently that a vector subspace with countable codimension of a barrelled space is barrelled. Also Saxon and Levin [5] have shown that a vector subspace with countable codimension of a \( \sigma \)-barrelled space is \( \sigma \)-barrelled. The same is true for countably barrelled spaces as shown by Webb [12]. We give a different and direct proof of this fact.

**Theorem 3.1.** Let \( M \) be a countable codimensional vector subspace of a countably barrelled space \( E \). Then \( M \) is countably barrelled when furnished with the relative topology.

Proof. In our proof we consider three cases.

(a) \( M \) is dense in \( E \). In this case, the topological dual \( M' \) of \( M \) can be canonically identified with \( E' \). Let \( S \) be a \( \sigma(M', M) \)-bounded subset of \( M' \) and let \( \{ S_n : n \geq 1 \} \) be a sequence of equicontinuous subsets of \( M' \) for which \( S = \bigcup_{n=1}^{\infty} S_n \). Since \( M \) is dense in \( E \), it follows from [5, Lemma 2] that \( S \) is \( \sigma(E', E) \)-bounded. Further we show that each \( S_n \) is an equicontinuous subset of \( E' \).
In fact, let $S_0$ denote the polar of $S$ taken in $E$. Since $S$ is an equicontinuous subset of $M'$, $S_0 \cap M$ is a 0-neighbourhood in $M$; then there exists an open 0-neighbourhood $U_n$ in $E$ such that $U_n \cap M \subset S_0 \cap M \subset S_0$. The density of $M$ ensures that $U_n \subset \overline{U_n \cap M} \subset S_0$, and hence $S_0$ is an equicontinuous subset of $E$.

Now the countable barrelledness of $E$ implies that $S$ is an equicontinuous subset of $E'$ and surely an equicontinuous subset of $M'$. This shows that $M$ is countably barrelled.

(b) $M$ is closed in $E$. Let $N$ be any algebraic complement to $M$ in $E$. Since countably barrelled spaces are $\sigma$-barrelled, it follows from [7, Theorem 1.1] that $N$ is a topological complement and has the strongest locally convex topology. Hence $N$ is closed in $E$, $M$ and $E/N$ are topologically isomorphic. Since $E$ is countably barrelled, by [3, Corollary 14], $E/N$ is countably barrelled and therefore $M$ must be countably barrelled.

(c) General case. Since $\overline{M}$ is a closed vector subspace of $E$ with countable codimension, it follows from (b) that $\overline{M}$ is countably barrelled. As $M$ is dense in $\overline{M}$, we conclude from (a) that $M$ is countably barrelled. This completes the proof of the theorem.

Corollary 3.2. Let $E$ be a $\sigma$-barrelled (DF)-space. Then any vector subspace $M$ of $E$ with countable codimension is a countably barrelled (DF)-space.

Proof. By Corollary 2.6, $E$ is a countably barrelled (DF)-space, and hence $M$ is a countably barrelled space by the preceding theorem. Since $E$ has a countable fundamental system of bounded sets, and since $M$ is a subspace, it follows that $M$ contains a countable fundamental system of bounded subsets of $\overline{M}$. Therefore $M$ is a countably barrelled (DF)-space.

The preceding result was proved by Valdivia [10, Theorem 3] in the special case when $E$ is barrelled.

4. Various types of absorbent sequences. Let $E$ be a vector space. By an increasing sequence of sets in $E$ we mean a sequence $\{V_n: n \geq 1\}$ of convex circled subsets of $E$ such that $V_n \subset V_{n+1}$ for all $n \geq 1$. Let $\{V_n: n \geq 1\}$ be an increasing sequence of sets in $E$. It is clear that $\{nV_n: n \geq 1\}$ is an increasing sequence of sets in $E$, and that if $E$ is a locally convex space then $\overline{V_n}$ is the closure of $\overline{V_n}$. An increasing sequence $\{V_n: n \geq 1\}$ of sets in $E$ is called an increasing sequence of $(P)$ sets in $E$ if each $V_n$ has the property $(P)$; for instance, $\{V_n: n \geq 1\}$ is an increasing sequence of closed (resp. complete, compact, metrizable etc.) sets in $E$ if each $V_n$ is closed (resp. complete, compact, metrizable etc.).

It is known from §2 that the concept of $S$-absorbent sequences is useful for studying the relationship between various types of barrelledness. It is not hard to give an example of an increasing sequence of sets in $E$ which is not $S$-absorbent. Therefore it is interesting to find some sufficient and necessary condition to ensure that increasing sequences are $S$-absorbent.

Proposition 4.1. Let $S$ be a $t$.family for $E'$ and suppose that $\{V_n: n \geq 1\}$ is an increasing sequence of closed sets in $E$. Then it is an $S$-absorbent sequence if and only if for any $f_n \in V'_n (n \geq 1)$, the sequence $\{f_n: n \geq 1\}$ is $T_S$-bounded.
Proof. Suppose that $\{V_n : n \geq 1\}$ is $S$-absorbent and that $\{f_n : n \geq 1\}$ is not $T^s_{\beta}$-bounded for some $f_n \in V^0_n \ (n \geq 1)$. Then there exists $B \in S$ such that $\{f_n : n \geq 1\} \nsubseteq k^2B^0$ is false for all natural numbers $k \geq 1$. For each $k \geq 1$, there exists $n_k$ such that $f_{n_k} \notin k^2B^0$. On the other hand, since $\{V_n : n \geq 1\}$ is $S$-absorbent, there exists $\lambda > 0$ and $n_0 \geq 1$ such that

$$V^0_{n_0} \subseteq V^0_{n_0} \subseteq \lambda B^0 \quad \text{for all} \quad n \geq n_0,$$

it then follows that $f_n \in \lambda B^0$ for all $n \geq n_0$, which contradicts the fact that $f_{n_k} \notin k^2B^0$. Therefore the condition is necessary.

Conversely, if $\{V_n : n \geq 1\}$ is not an $S$-absorbent sequence, then there exists $B \in S$ such that $B \nsubseteq nV^0_n$ is false for all natural numbers $n \geq 1$. For each $n$, let $x_n \in B \setminus (nV^0_n)$ and let $f_n$ in $V^0_n$, be such that $|f_n(x_n)| > n$. Then the sequence $\{f_n : n \geq 1\}$ is not $T^s_{\beta}$-bounded. This completes the proof.

In the sequel we always assume that $E$ is a locally convex space and that $S$ is a topologizing family for $E'$. If $S_1$ is another topologizing family for $E'$ such that $S \subseteq S_1$, then each $S_1$-absorbent sequence in $E$ must be $S$-absorbent. The converse is true for $S_1 = S^{bb}$ as the following result shows.

**Corollary 4.2.** $\{V_n : n \geq 1\}$ is an $S$-absorbent sequence of closed sets in $E$ if and only if it is an $S^{bb}$-absorbent sequence.

**Proof.** This follows from Proposition 4.1 and Lemma 2.1.

The preceding result was proved by De Wilde and Houet [2, Theorem 1] in the case when $S$ is the family of all finite subsets of $E$.

**Corollary 4.3.** Let $S_1$ and $S_2$ be two t. families for $E'$ such that $S_1 \subseteq S_2$. Then the following statements are equivalent:

(i) each $T^s_{S_1}$-bounded subset of $E'$ is $T^s_{S_2}$-bounded;

(ii) each $S_1$-absorbent sequence of closed sets in $E$ is $S_2$-absorbent.

**Proof.** The implication (i) $\Rightarrow$ (ii) follows from Proposition 4.1, while the implication (ii) $\Rightarrow$ (i) has been observed in Theorem 2.5.

When $S_1$ is the family of all finite subsets of $E$ and $S_2$ is the family of all $\sigma(E, E')$-bounded subsets of $E$, then the implication (i) $\Rightarrow$ (ii) in the preceding result was proved by Valdivia [9, Theorem 6] in the case when $E$ is barrelled, and was proved by De Wilde and Houet [2, Corollary 1] in the case when $E$ is $\sigma$-barrelled.

By making use of Theorem 2.5, for a $\sigma$-barrelled space $E$, each $\sigma(E', E)$-bounded subset of $E'$ is $\beta(E', E)$-bounded.

**Corollary 4.4.** Let $S_1$ and $S_2$ be two t. families for $E'$ such that $S_1 \subseteq S_2$, and let $E$ satisfy one of the equivalent conditions (i) and (ii) of Corollary 4.3. If $S_2$ has a sequence $\{B_n : n \geq 1\}$ such that each member of $S_1$ is absorbed by some $B_n$, then the saturated hull ([6], p. 81) of $S_2$ contains a countable fundamental subfamily.

**Proof.** For each $n$, let $V_n$ be the closed convex circled hull of $\bigcup_{j=1}^{\kappa} B_j$. Then $V_n$ is in the saturated hull of $S_2$, and $\{V_n : n \geq 1\}$ is an $S_1$-absorbent sequence of closed sets in $E$, so by
the hypothesis, \( \{V_n : n \geq 1\} \) is \( S_2 \)-absorbent. Consequently \( \{nV_n : n \geq 1\} \) is a countable fundamental subfamily of the saturated hull of \( S_2 \) because a member of the saturated hull of \( S_2 \) is either a subset, scalar multiple or an absolute convex hull of a finite number of elements of \( S_2 \).

**Remark.** If \( E \) is a countably barrelled space with a sequence \( \{B_n : n \geq 1\} \) of bounded sets such that \( \bigcup_{n=1}^{\infty} B_n \) is absorbing, then \( E \) is a \( (DF) \)-space.

Corollary 4.4 was proved by Valdivia [9, Corollary 2.6] in the case when \( E \) is barrelled. A trivial modification of De Wilde and Houet’s argument in [2] yields the following more general result, but for completeness we shall give the entire proof.

**Theorem 4.5.** Let \( E \) be a \( \sigma \)-\( S \)-barrelled space and \( \{V_n : n \geq 1\} \) an \( S \)-absorbent sequence in \( E \). Then

\[
\bigcup_{m=1}^{\infty} V_m \subset (1+\varepsilon) \bigcup_{m=1}^{\infty} V_m^0 \quad \text{for all} \quad \varepsilon > 0.
\]

**Proof.** If \( x \notin (1+\varepsilon) \bigcup_{m=1}^{\infty} V_m \) for some \( \varepsilon > 0 \), then \( x \notin (1+\varepsilon) \bigcup_{m=1}^{\infty} V_m \) for all \( m \geq 1 \), and thus, for any \( m \geq 1 \), there exists \( f_m \in V_m^0 \) such that \( f_m(x) > 1+\varepsilon \). Since \( E \) is \( \sigma \)-\( S \)-barrelled, by Proposition 2.4, \( \{f_m : m \geq 1\} \) has a \( (E', E) \)-cluster point \( f \), say, in \( E' \); hence \( f(x) \geq 1+\varepsilon \). On the other hand, since \( V_n \) is increasing and \( f_n \in V_n^0 \), it follows that \( f \in V_n^0 \) for all \( n \geq 1 \) or, equivalently \( f \in \bigcap_{n=1}^{\infty} V_n^0 = \left( \bigcup_{n=1}^{\infty} V_n \right)^0 \). However the inequality \( f(x) \geq 1+\varepsilon \) shows that \( x \notin \bigcup_{m=1}^{\infty} V_m \). This completes the proof.

**Remarks.** (1) As De Wilde in [1, p. 212] pointed out, the condition in Theorem 4.5 that \( E \) be \( \sigma \)-\( S \)-barrelled can be replaced by the following condition: \( \{V_n : n \geq 1\} \) is an \( S \)-absorbent sequence in \( E \) such that for each \( f_n \in V_n^0 \) \( (n \geq 1) \), the sequence \( \{f_n : n \geq 1\} \) is equicontinuous.

(2) According to the preceding theorem, Corollaries 2.a–2.d in [2] hold for a \( \sigma \)-\( S \)-barrelled space.

**Acknowledgement.** The authors are grateful to the referee for many helpful comments.

**References**


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