Canad. Math. Bull. Vol. **44** (1), 2001 pp. 12–18

A Technique of Studying Sums of Central Cantor Sets

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Abstract. This paper is concerned with the structure of the arithmetic sum of a finite number of central Cantor sets. The technique used to study this consists of a duality between central Cantor sets and sets of subsums of certain infinite series. One consequence is that the sum of a finite number of central Cantor sets is one of the following: a finite union of closed intervals, homeomorphic to the Cantor ternary set or an M-Cantorval.

1 Introduction

Related with some problems in dynamical systems, J. Palis [8] asked if the arithmetic difference (or sum) of two Cantor sets, both with zero Lebesgue measure, is either of Lebesgue measure zero or it contains an interval. For regular Cantor sets negative answers were given in [1] and [9]. This problem initiated the investigation of such arithmetic sums.

In this paper we present a technique of dealing with the arithmetic sum of a finite number of central Cantor sets, which will enable us to study the topological structure and to describe the sum in some cases. This technique is based on the duality between central Cantor sets and sets of subsums of certain infinite series. In a very particular case, this type of argument was used in [5] (see Theorem D).

In the main result of this note, Theorem 2, we show that the sum of a finite number of central Cantor sets is a finite union of closed intervals or homeomorphic to the Cantor ternary set or an M-Cantorval. For sums of two symmetric homogeneous Cantor sets similar results were obtained in [5]; although these two classes of Cantor sets intersect, neither of them is contained in the other.

Another consequence of our technique is Theorem 3, which gives necessary and sufficient conditions for $\underbrace{\mathbf{C}+\cdots+\mathbf{C}}_{n\text{-times}}$ to be an interval or a finite union of closed intervals. This theorem improves upon [2].

2 Preliminaries

A *central Cantor* set in $\mathbf{R}$ is constructed in the following way: choose an arbitrary closed interval K_0 and delete a middle open interval leaving two intervals. Let K_1 be the union of the remaining two intervals. Repeat the process for each of the two intervals of K_1 (we request now that both open middle intervals which are removed be of the same length) and we obtain a compact set K_2 which is the union of 2^2

Received by the editors February 3, 1999.
AMS subject classification: 11B05.

intervals of the same length. Proceeding inductively we construct for each $n \in \mathbf{N}$ a set K_n that is the union of 2^n intervals of the same length. The central Cantor set is given by the intersection of these sets. That is $\mathbf{C} = \bigcap_{i\geq 0} K_i$.

Note that in the process of constructing K_{n+1} from K_n it is sufficient to know the middle interval which is removed from the first of the 2^n intervals of K_n, the others being deleted by symmetry.

To simplify the proofs, we will consider only central Cantor sets with the first interval K_0 starting from 0. It is clear that this assumption does not present any loss of generality.

The main tools used to study sums of central Cantor sets will be some properties of the set of subsums of an infinite series.

Let $\sum_{n\geq 0} a_n$ be a convergent series with $0 < a_{n+1} \leq a_n$ for all n and let

$$\mathcal{M}\Bigl(\sum_{n\geq 0} a_n\Bigr) = \Bigl\{\sum_{n\in F} a_n : F \subseteq \mathbf{N}\Bigr\}$$

denote its set of subsums. Also, let $r_n = \sum_{s\geq n+1} a_s$ denote the n-th tail of the series.

The following results are known regarding the set $\mathcal{M}(\sum_{n\geq 0} a_n)$:

Theorem 1 **([3], [4], [6])** *Let $\sum_{n\geq 0} a_n$ be a convergent series with $0 < a_{n+1} \leq a_n$ and let*

$$A_1 = \Bigl\{n \in \mathbf{N} \Bigm| a_n > \sum_{s>n} a_s\Bigr\}$$

$$A_2 = \Bigl\{n \in \mathbf{N} \Bigm| a_n \leq \sum_{s>n} a_s\Bigr\}.$$

Then

(i) *If $A_2 = \mathbf{N}$ (equivalent with $A_1 = \varnothing$) we have*

$$\mathcal{M}\Bigl(\sum_{n\geq 0} a_n\Bigr) = \Bigl[0, \sum_{n\geq 0} a_n\Bigr]$$

Also, $\mathcal{M}(\sum_{n\geq 0} a_n)$ is an interval if and only if $A_2 = \mathbf{N}$.

(ii) *If $A_1 = \mathbf{N}$ (equivalent with $A_2 = \varnothing$) then $\mathcal{M}(\sum_{n\geq 0} a_n)$ is a central Cantor set, which can be explicitly described.*

(iii) *The set $\mathcal{M}(\sum_{n\geq 0} a_n)$ is one of the following: a finite union of closed intervals, homeomorphic to the Cantor ternary set or homeomorphic to the set E of subsums of $\sum_{n\geq 1} b_n$ where $b_{2n-1} = \frac{3}{4^n}$ and $b_{2n} = \frac{2}{4^n}$, $n \geq 1$.*

We can also describe $\mathcal{M}(\sum_{n\geq 0} a_n)$ in the case when A_1 or A_2 are finite, as seen from the following remark: if $\mathcal{M}_1$ is the set of subsums of some tail of $\sum_{n\geq 0} a_n$, then $\mathcal{M}(\sum_{n\geq 0} a_n)$ is a finite union of translates of $\mathcal{M}_1$.

The relation between central Cantor sets and sets of subsums of infinite series, that is essential in the sequel, will follow naturally from the description of the set which appears in Theorem 1(ii). Under the assumptions of this case the set $\mathcal{M}(\sum_{n\geq 0} a_n)$

https://doi.org/10.4153/CMB-2001-002-8 Published online by Cambridge University Press

is the central Cantor set which is obtained in the following way: the first interval K_0 will be $[0, \sum_{n\geq 0} a_n]$, the set K_1 will be obtained from K_0 by removing the middle interval $(\sum_{s>0} a_s, a_0)$; the set K_2 will be obtained from K_1 by removing the interval $(\sum_{s>1} a_s, a_1)$ from the first of the 2^1 intervals of K_1 and then, by symmetry, the corresponding intervals from the remaining $2^1 - 1$ intervals of K_1; inductively, the set K_{n+1} will be obtained from K_n by removing the interval $(\sum_{s>n} a_s, a_n)$ from the first of the 2^n intervals of K_n and then, by symmetry, the corresponding intervals from the remaining $2^n - 1$ intervals of K_n.

3 Sums of Central Cantor Sets

We present now a type of duality between central Cantor sets and some infinite series. That is, for each central Cantor set we correspond a unique infinite series with positive terms with the property that its set of subsums is our initial Cantor set.

Let **C** be a central Cantor set, with the notations from the beginning of Section 2. Let us denote by a the endpoint of K_0 and by r_0, respectively a_0, the left endpoint, respectively the right endpoint of the middle interval deleted from K_0.

Inductively, denote by r_n and a_n the endpoints of the middle interval deleted from the first of the 2^n intervals of K_n ($r_n < a_n$).

We have that $\sum_{n\geq 0} a_n = a$ and $r_n = \sum_{s>n} a_s$. Moreover the series $\sum_{n\geq 0} a_n$ satisfies (ii) from Theorem 1 ($a_n > r_n$ and $0 < a_{n+1} < a_n$, $n \in \mathbf{N}$, by construction). It follows from the end of Section 2 that $\mathcal{M}(\sum_{n\geq 0} a_n)$ is a central Cantor set, in fact exactly **C**.

Note that the Lebesgue measure of the set **C** can be calculated in a simple way. Indeed,

$$\lambda(\mathbf{C}) = a - \sum_{n\geq 0} 2^n \Big(a_n - \sum_{s>n} a_s \Big) = \lim_{n\to\infty} 2^{n+1} r_n.$$

The proof of the following result is straightforward.

Proposition 1 *Let $\mathbf{C_1}, \mathbf{C_2}, \dots, \mathbf{C_k}$ be central Cantor sets. Then $\mathbf{C_1} + \mathbf{C_2} + \cdots + \mathbf{C_k}$ can be written as the set of subsums of a series with positive terms $\sum_{n\geq 0} a_n$, which satisfies $a_n \geq a_{n+1}$ ($n \geq 0$). Therefore all the results from Theorem 1 can be applied for $\mathbf{C_1} + \mathbf{C_2} + \cdots + \mathbf{C_k}$.*

As a corollary it follows that the sum of a finite number of central Cantor sets is of one of the three types described in Theorem 1(iii). In fact, we can give a stronger description, as it will follow from next Theorem. First, recall that an M-Cantorval is defined as a perfect subset of **R**, such that any gap is accumulated on both sides by infinitely many intervals and gaps.

Theorem 2 *If $\mathbf{C_1}, \mathbf{C_2}, \dots, \mathbf{C_k}$ are central Cantor sets, then $\mathbf{C_1} + \mathbf{C_2} + \cdots + \mathbf{C_k}$ is one of the following:*

(i) a finite union of closed intervals,
(ii) homeomorphic to the Cantor ternary set,
(iii) an M-Cantorval.

Proof From Proposition 1, $\mathbf{C} := \mathbf{C_1} + \mathbf{C_2} + \cdots + \mathbf{C_k}$ is the set of subsums of a series with positive terms $\sum_{n\geq o} a_n$, which satisfies $a_n \geq a_{n+1}$ $(n \geq 0)$.

Suppose that $\mathbf{C}$ is neither a finite union of closed intervals nor homeomorphic to the Cantor ternary set. This case, as was proved in [3], will result in $\mathbf{C}$ having infinitely many intervals and infinitely many gaps. Moreover, for each point $x \in \mathbf{C}$ there are intervals in $\mathbf{C}$ arbitrarily close to x. In order to prove that $\mathbf{C}$ is an M-Cantorval it remains to show that we cannot have in $\mathbf{C}$ a gap followed by a non-trivial interval of $\mathbf{C}$ or a non-trivial interval of $\mathbf{C}$ followed by a gap.

Suppose that the former is true. That is, there exists in $\mathbf{C}$ (which is contained in the interval $[0, \sum_{n\geq o} a_n]$) a gap denoted by (u, v), followed by a non-trivial interval of $\mathbf{C}$ denoted by $[v, w]$ $(u, v, w \in \mathbf{R})$.

Let $N \in \mathbf{N}$ be such that $a_{N+1} < v - u$.

Denote with z the minimum of the set

$$\Big\{\sum_{n\in F} a_n : F \subset \{0, \dots, N\} \text{ and } \sum_{n\in F} a_n > v\Big\}$$

if this set is non void, otherwise put $z = w$. In both cases we have $z > v$. Suppose that there exists $l \in \mathbf{N}$ such that $a_l < \min\{z - v, w - v, a_{N+1}\}$ and

$$a_l > \sum_{n>l} a_n.$$

Then

$$\Big(v + \sum_{n>l} a_n, v + a_l\Big) \subset (v, w) \subset \mathbf{C}$$

For an arbitrary $x \in (v + \sum_{n>l} a_n, v + a_l)$ we cannot have in the decomposition of x (as an element of the set of subsums of $\sum_{n\geq 0} a_n$) terms a_n with $N + 1 \leq n \leq l$, otherwise

$$x - a_n < v + a_l - a_n \leq v$$

and

$$x - a_n > v - a_n > u$$

which contradicts $x - a_n \in \mathbf{C}$. So that there exist $F_1 \subset \{l + 1, l + 2, \dots\}$ and $F_2 \subset \{0, \dots, N\}$ such that

$$x = \sum_{n\in F_1} a_n + \sum_{n\in F_2} a_n$$

But now

$$\sum_{n\in F_2} a_n = x - \sum_{n\in F_1} a_n > v + \sum_{n>l} a_n - \sum_{n\in F_1} a_n \geq v$$

and

$$\sum_{n\in F_2} a_n = x - \sum_{n\in F_1} a_n < x < v + a_l < z$$

contradiction with the definition of z.

In conclusion there exists a tail of the series $\sum_{n\geq o} a_n$ such that each term of the tail satisfies $a_k \leq \sum_{n>k} a_n$, which implies (see Theorem 1) that $\mathbf{C}$ is a finite union of closed intervals, contradiction.

The case when a non-trivial interval of $\mathbf{C}$ is followed by a gap results from the previous one using the remark that the set $\mathbf{C}$ is symmetric with respect to the middle point of $[0, \sum_{n\geq 0} a_n]$. ∎

Remark Note that we have obtained above that if $\mathbf{C} := \mathbf{C_1} + \mathbf{C_2} + \cdots + \mathbf{C_k}$ does not satisfy (i) and (ii), then $\mathbf{C}$ is an M-Cantorval in which, as it was mentioned in [3], the endpoints are trivial intervals in $\mathbf{C}$. In particular, all the sets of this type are homeomorphic (see Appendix in [5]).

Theorem 3 *Let $\mathbf{C}$ be a central Cantor set and let $\sum_{n\geq 0} a_n$ be the series corresponded to $\mathbf{C}$. The positive integer m has the property that $\underbrace{\mathbf{C} + \cdots + \mathbf{C}}_{m\text{-times}}$ is an interval (respectively a finite union of closed intervals) if and only if $a_n/r_n \leq m$ for all n (respectively $a_n/r_n \leq m$ for all but a finite number of n).*

Proof The statement follows from Theorem 1 (similarly as in [7]).

Remark Note that Theorem 3 gives a stronger result than Theorem 2.5 presented in [2], showing that the condition of the statement from [2] is not only sufficient but also necessary (here we use the fact that the ratio of dissection of step k—see [2] for definition—is exactly $\xi_k = \frac{r_k}{a_k + r_k}$).

Regarding the cases when the problem of Palis has a negative answer, we present a natural example, constructed using the same technique as above. This example is close in spirit to the earlier example from [9]; we can now obtain with a quite short proof the arithmetic sum to be not only a Cantor set but also a central Cantor set.

Proposition 2 *For any $\epsilon > 0$ there exist two central Cantor sets corresponding to some intervals $[0, a]$ and $[0, b]$ respectively ($a, b \geq \frac{1}{5}$) of zero Lebesgue measure whose arithmetic sum is a central Cantor set of Lebesgue measure larger than $a + b - \epsilon$.*

Proof We will go backwards. First we will construct a central Cantor set which we want to be the result of the arithmetic sum and then we will show how we can write it as a sum of two central Cantor sets of Lebesgue measure zero.

Let $\frac{1}{5} > \epsilon > 0$. There exists $k \geq 0$ such that $\epsilon > \frac{1}{2^{k+1}}$. Consider the sequence $(\alpha_n)_{n\geq 0}$ defined by $\alpha_n = \frac{1}{2^{n+k+2}}$, $n \geq 0$. Clearly $\sum_{n\geq 0} \alpha_n = \frac{1}{2^{k+1}} < \epsilon$.

Consider the following central Cantor set obtained from the interval $[0, 1]$. Delete from $K_0 = [0, 1]$ the middle open interval of length α_0. From each of the intervals of K_1 delete the corresponding middle open interval of length $\frac{\alpha_1}{2}$. Inductively, from each of the 2^n intervals of K_n delete the corresponding middle open interval of length $\frac{\alpha_n}{2^n}$. We will obtain a central Cantor set $\mathbf{C}$ with the Lebesgue measure

$$\lambda(\mathbf{C}) = 1 - \sum_{n\geq 0} 2^n \frac{\alpha_n}{2^n} = 1 - \sum_{n\geq 0} \alpha_n > 1 - \epsilon.$$

Inductively, is not hard to see that the terms of the series corresponded (in the sense of the duality presented above) to **C** are

$$a_0 = \frac{1}{2} + \frac{\alpha_0}{2},$$
$$a_n = \frac{1}{2^{n+1}} - \frac{\alpha_0}{2^{n+1}} - \frac{\alpha_1}{2^{n+1}} - \cdots - \frac{\alpha_{n-1}}{2^{n+1}} + \frac{\alpha_n}{2^{n+1}}, \quad n \geq 1.$$

which will give

(1) $$a_n = \frac{1}{2^{n+1}} A + 3\frac{1}{4^{n+1}} B \quad (n \geq 0),$$

with $A := 1 - \frac{1}{2^{k+1}}$ and $B := \frac{1}{2^{k+1}}$.

We have

$$\mathbf{C} = \mathcal{M}\Big(\sum_{n\geq 0} a_n\Big) = \mathcal{M}\Big(\sum_{n\geq 0} a_{2n}\Big) + \mathcal{M}\Big(\sum_{n\geq 0} a_{2n+1}\Big).$$

Since the series $\sum_{n\geq 0} a_n$ satisfies (see the beginning of the section)

$$a_n > r_n = \sum_{s>n} a_s \quad (n \geq 0),$$

we obtain that

$$\mathbf{C_1} := \mathcal{M}\Big(\sum_{n\geq 0} a_{2n}\Big), \quad \mathbf{C_2} := \mathcal{M}\Big(\sum_{n\geq 0} a_{2n+1}\Big)$$

are two central Cantor sets (Theorem 1(ii)). Note that $\mathbf{C_1}$ and $\mathbf{C_2}$ correspond to the intervals $[0, a]$ and $[0, b]$ respectively, where $a := \sum_{n\geq 0} a_{2n} \geq a_0 > \frac{1}{2}$ and $b := \sum_{n\geq 0} a_{2n+1} \geq a_1 > \frac{1}{5}$ and $a + b = \sum_{n\geq 0} a_n = 1$.

By an earlier remark, the Lebesgue measures of $\mathbf{C_1}$ and $\mathbf{C_2}$ are

(2) $$\lambda(\mathbf{C_1}) = \lim_{n\to\infty} 2^{n+1} r_n^{(1)}$$

(3) $$\lambda(\mathbf{C_2}) = \lim_{n\to\infty} 2^{n+1} r_n^{(2)}$$

where we denoted $r_n^{(1)} := \sum_{s>n} a_{2s}$ and $r_n^{(2)} := \sum_{s>n} a_{2s+1}$.

For $n \geq 0$, we have

$$r_n^{(1)} = \sum_{s>n} a_{2s} = \sum_{s>n} \frac{1}{2^{2s+1}} A + \sum_{s>n} \frac{3}{4^{2s+1}} B = \frac{A}{3 \cdot 2^{2n+1}} + \frac{B}{5 \cdot 4^{2n+1}}$$

and

$$r_n^{(2)} = \sum_{s>n} a_{2s+1} = \sum_{s>n} \frac{1}{2^{2s+2}} A + \sum_{s>n} \frac{3}{4^{2s+2}} B = \frac{A}{3 \cdot 2^{2n+2}} + \frac{B}{5 \cdot 4^{2n+2}}.$$

From (2) and (3) we obtain $\lambda(\mathbf{C_1}) = \lambda(\mathbf{C_2}) = 0$. ∎

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https://doi.org/10.4153/CMB-2001-002-8 Published online by Cambridge University Press