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EXPONENTIAL POLYNOMIAL APPROXIMATION OF WEIGHTED BANACH SPACE ON $\mathbb{R}^n$

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Abstract. Necessary and sufficient conditions for the incompleteness of exponential system in C_α are characterised, where C_α is the weighted Banach space of complex continuous functions f defined on $\mathbb{R}^n$ with $f(t)\exp(-\alpha(t))$ vanishing at infinity in the uniform norm.

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1. Introduction and notations. In this paper points of $\mathbb{C}^n$ will be denoted by $z = (z_1, \dots, z_n)$, where $z_k \in \mathbb{C}$. If $z_k = x_k + iy_k$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, then we write $z = x + iy$. The vectors $x = \Re z$ and $y = \Im z$ are the real and imaginary parts of z, respectively, $\mathbb{R}^n$ will be thought of as the set of all $z \in \mathbb{C}^n$ with $\Im z = 0$. The set of non-negative integers will be denoted by $\mathbb{Z}_+$. The notations

$$|z| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2},$$
$$|\Re z| = (|x_1|^2 + \cdots + |x_n|^2)^{1/2},$$
$$|\Im z| = (|y_1|^2 + \cdots + |y_n|^2)^{1/2},$$
$$z^\beta = z_1^{\beta_1} \cdots z_n^{\beta_n},$$
$$\langle z, t\rangle = z_1 t_1 + \cdots + z_n t_n,$$
$$e_z(t) = \exp(\langle z, t\rangle)$$

will be used for any multi-index β, any $t \in \mathbb{R}^n$ and $z \in \mathbb{C}^n$. Let A denote positive constants, it may be different at each occurrence.

Let $\alpha(t)$ be a non-negative continuous function defined on $\mathbb{R}^n$, henceforth called a *weight*, satisfying

$$\lim_{|t|\to\infty} |t|^{-1}\alpha(t) = \infty. \tag{1}$$

Given a weight $\alpha(t)$, the weighted Banach space C_α consists of complex continuous functions f defined on $\mathbb{R}^n$ with $f(t)\exp(-\alpha(t))$ vanishing at infinity, normed by

$$\|f\|_\alpha = \sup\{|f(t)\exp(-\alpha(t))| : t \in \mathbb{R}^n\}.$$

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Our space C_α is rooted from [**1–4**, **11**, **12**], in which the exponential polynomial approximation problem is investigated.

By a *complete system* of elements $\{e_k\}$ of a Banach space B, we mean $\overline{Span}\{e_k\} = B$, i.e. the completeness is equivalent to the possibility of an arbitrary good approximation of any element of the space by linear combination of elements of this system.

Denote by $\overline{Span}\{e^{\lambda t}\}$ the closure of exponential polynomials that are finite linear combinations of exponential system $\{e^{\lambda t} : \lambda \in \Lambda\}$, where $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ is a sequence of complex numbers that have no limit points in the complex plane $\mathbb{C}$.

It is well known that various aspects of analysis and applied mathematics often lead to know approximate properties of exponential systems $\{e^{\lambda t}\}$ on subsets of real line. The problem of incompleteness of $\overline{Span}\{e^{\lambda t}\}$ in C_α in the norm $\|.\|_\alpha$ is the so-called exponential polynomial approximation, which is similar to the classical Bernstein problem on polynomial approximation in [**5**].

The approach in the references just mentioned can be interpreted as an application of uniqueness of analytical functions. In the celebrating work [**1**], Malliavin's uniqueness theorem in [**8**] is employed to investigate incompleteness of $\overline{Span}\{e^{\lambda t}\}$ in C_α where $\alpha(t)$ is a non-negative convex function defined on the real axis $\mathbb{R}$, satisfying (1) for $t \in \mathbb{R}$, the closure of $\overline{Span}\{e^{\lambda t}\}$ is also characterised by the Dirichlet series. The incomplete theorems in [**4**] and [**11**] generalise the work in [**1**] to the case that the sequence $\{\lambda_k\}_{k=1}^{\infty}$ has an infinite upper density. It is not hard to understand that some classical results of the approximation theory appear to be very difficult and interesting (see [**6**], for example). This fact can be explained by the known connection between approximation problems and zero distribution of certain entire function. The distribution of zeros of entire functions is of course much more complex for $n > 1$. In the present paper we will continue the investigation of the multivariate exponential polynomial approximation problem.

Motivated by [**1–4**, **11**, **12**], in this paper we will investigate the incompleteness of $\overline{Span}\{e^{\lambda t}\}$ in C_α where $\alpha(t)$ is a non-negative continuous function defined in $\mathbb{R}^n$ for $t \in \mathbb{R}^n$, satisfying (1) and $\lambda \in \Lambda$, where

$$\Lambda = \bigcup_{k=1}^{\infty} \Lambda_k \tag{2}$$

and $\Lambda_k = \{\lambda : \langle \lambda, a^k \rangle = 0\}$ is a hyperplane. Our result can be thought of as a generalisation of results in [**1–3**, **11**] to multivariable case. Our main result depends upon the theory of entire functions with 'planar' zeros. As is well known that the zeros of entire functions in $\mathbb{C}^n (n \geq 2)$ are never discrete. The multivariable case may be different from a single variable case. That is why it needs to be treated separately.

To present our main result, we need some background material. For a sequence of points $\{a^k = (a_1^k, a_2^k, \dots, a_n^k)\}_{k=1}^{\infty}$ in $\mathbb{C}^n$, we denote the number of points a^k in the ball of radius of t by $n(t)$, which is called *the counting function*. The order of the counting function is defined by

$$\rho = \limsup_{r \to +\infty} \frac{\ln n(r)}{\ln r}.$$

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The integer number p is defined by the condition

$$\sum_{n=1}^{\infty} |a^k|^{-p} = \infty, \sum_{n=1}^{\infty} |a^k|^{-p-1} < \infty. \tag{3}$$

We say that for a sequence $\{a^k\}$ with convergence exponent $\rho = p$ *condition I* is satisfied if

$$\forall (s_1, \dots, s_n) \in \mathbb{Z}_+^n : s_1 + \dots + s_n = p$$

the quantity

$$\limsup_{r\to\infty} \left| \sum_{|a^k|<r} \frac{(a_1^k)^{s_1} \dots (a_n^k)^{s_n}}{|a^k|^{2p}} \right| < \infty.$$

Then the main result of this paper is as follows.

THEOREM 1.1. *Let $\alpha(t)$ be a non-negative continuous function defined on $\mathbb{R}^n$ satisfying*

$$A_1|t|^2 \le \alpha(t) \le A_2|t|^2. \tag{4}$$

Then $\overline{Span}\{e_\lambda(t)\}$ is incomplete in C_α, where $\lambda \in \Lambda$ is defined in (2) *whose corresponding sequence $\{a^k\}$ satisfying $\rho = p = 2$, if and only if*

$$\limsup_{r\to\infty} \frac{n(r)}{r^2} < \infty, \tag{5}$$

and condition I is satisfied.

There are obvious ways in which our main result can be generalised: The example of Theorem 1.1 can be extended to much more general sets by using Lemma 2.1 in Section 2. We decided not to pursue elaborations; our aim is to present the essence of an interesting qualitative phenomenon, avoiding as far as possible obscuring technicalities.

2. Preliminaries. In this section, we shall collect some results on zeros of entire functions in $\mathbb{C}^n$. We are interested in entire functions with the distribution of zeros has the simplest geometric structure. Namely, in the sequel it is assumed everywhere that $f(z)$ has 'planar' zeros, i.e. the hyperplanes defined in (2). As Papush in **[9]** pointed out, this class of functions was firstly considered by L. Baumgartner (1914), and investigated by E. N. But (1974), L. I. Ronkin (1979) and A. B. Sekerin (1981). Here, we will present the main result of **[9]**, which is one of the main techniques in this paper.

Denote by $M_f(r) = \max_{|z|=r} |f(z)|$ and the order of growth of the function $f(z)$ is defined by

$$\rho_1 = \limsup_{r\to+\infty} \frac{\ln \ln M_f(r)}{\ln r}.$$

Assume further that the type of the function $f(z)$ is defined by

$$\sigma_f = \limsup_{r\to+\infty} \frac{\ln M_f(r)}{r^{\rho_1}}.$$

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Assuming that $\rho_1 < \infty$, we denote by

$$G(z) = \Pi_{k=1}^{\infty} B(\langle z, a^k|a^k|^{-2}\rangle, p), \tag{6}$$

where $B(u, p) = (1-u)\exp(u + u^2/2 + \ldots + u^p/p)$, p is a non-negative integer number defined in (3). For a canonical product $G(z)$, we denote by

$$\triangle_G = \limsup_{r\to+\infty} \frac{n(r)}{r^{\rho_1}}.$$

Then the description of zero sets of $G(z)$ is given by the following (see **[9]**).

LEMMA 2.1. *For the canonical product $G(z)$ of the form* (6)*, for $\rho = p$, we have the following implication: $\sigma_G < \infty$ if and only if both $\triangle_G < \infty$ and condition I are satisfied.*

REMARK 2.1. The canonical product $G(z)$ defined in (6) is similar to one-dimensional canonical product, which is called the Weierstrass products and the Hadamard representation of analytic functions of one variable, see **[7, 10]** for more details.

3. Proof of the main result. If $\overline{Span}\{e_\lambda(t)\}$, $\lambda \in \Lambda$ is incomplete in C_α, where Λ is defined in (2) for some sequence $\{a^k\}$ satisfying $\rho = p$, by the Hahn–Banach Theorem, there exists a nontrivial bounded linear functional T such that $\|T\| = 1$ and $T(e_\lambda(t)) = 0$ for $\lambda \in \Lambda$. So by the Riesz representation theorem, there exists a complex measure μ on $\mathbb{R}^n$ satisfying

$$\|\mu\| = \int_{\mathbb{R}^n} e^{\alpha(t)}|\mathrm{d}\mu(\mathrm{t})| = \|\mathrm{T}\|$$

and

$$T(h) = \int_{\mathbb{R}^n} h(t)\mathrm{d}\mu(\mathrm{t}), \quad h \in C_\alpha.$$

Define

$$f(z) = \int_{\mathbb{R}^n} e_z(t)\mathrm{d}\mu(\mathrm{t}),$$

then

$$|f(z)| \le \int_{\mathbb{R}^n} \exp\left(\left(\sum_{k=1}^{n} x_k t_k\right) - \alpha(t)\right) \exp(\alpha(t))|\mathrm{d}\mu(\mathrm{t})|,$$

thus, by (4), we have

$$\sup_{t\in\mathbb{R}^n}\left\{\left(\sum_{k=1}^{n} x_k t_k\right) - \alpha(t)\right\} \le \sum_{k=1}^{n} \sup_{t_k\in\mathbb{R}}\{x_k t_k - A_1 t_k^2\},$$

which yields

$$|f(z)| \le \|T\| \exp\{A|\Re z|^2\}$$

for all $z \in \mathbb{C}^n$. Thus, we have

$$\log M_f(r) \leq Ar^2$$

for all $z \in \mathbb{C}^n$. By Lemma 2.1, we can obtain (5). Actually, $f(z)$ has an expression of the kind

$$f(z) = \exp(P(z))\Pi_{k=1}^{\infty} G(z),$$

where $P(z)$ is a polynomial and $G(z)$ is a product defined in (6). The canonical representation of $f(z)$ is similar to the Hadamard representation and Weierstrass products of one complex variable analytical functions in **[7]** and **[10]**.

Conversely, if there exist real constants A_1, A_2 such that (4) and (5) hold, by Lemma 2.1, we know that there exists a canonical product $g(z)$ satisfying $g(\lambda) = 0$ for all $\lambda \in \Lambda$, which is defined in (2), and

$$|g(z)| \leq e^{A(|x|^2+|y|^2)}$$

holds for all $z \in \mathbb{C}^n$. Denote by e^k a vector which has its kth component equals to 1 and the others are 0, then

$$\exp(\langle z, e^k \rangle^2) = \exp(z_k^2).$$

For $z \in \mathbb{C}^n$, define

$$g_1(z) = g(z) \exp\left(A_3 \sum_{k=1}^{n} (\langle z, e^k \rangle)^2 \right), \tag{7}$$

where $A_3 > A$ is some positive constant satisfying

$$A_2 - \frac{1}{4(A + A_3)} < 0 \tag{8}$$

for constant A_2 defined in (4). Then we have the following estimate:

$$|g_1(z)| \leq \exp\left\{ (A + A_3) \left(\sum_{k=1}^{n} x_k^2 \right) - (A_3 - A) \left(\sum_{k=1}^{n} y_k^2 \right) \right\}, \ z \in \mathbb{C}^n. \tag{9}$$

Suppose that $g_1(z)$ is the entire function defined in (7), define

$$h_0(t) = \int_{\mathbb{R}^n} g_1(iy) e^{-\langle iy, t \rangle} \mathrm{d}m_n(y), \ t \in \mathbb{R}^n, \tag{10}$$

where m_n denotes the Lebesgue of $\mathbb{R}^n$. First note that $g_1(iy)$ is in $L^1(\mathbb{R}^n)$ by $A < A_3$ and (9), and $h_0(t)$ is continuous on $\mathbb{R}^n$.

Next, we claim that the integral

$$\int_{-\infty}^{+\infty} g_1(\zeta + i\eta, z_2, \ldots, z_n) \exp\{-(t_1(\zeta + i\eta) + t_2 z_2 + \cdots + t_n z_n)\} \mathrm{d}\eta \tag{11}$$

is independent of ζ for arbitrary real $t = (t_1, \ldots, t_n)$ and complex $z = (z_1, \ldots, z_n)$. To see this (see Figure 1), let Γ be a rectangular path in the $\zeta + i\eta$ -plane, with one edge

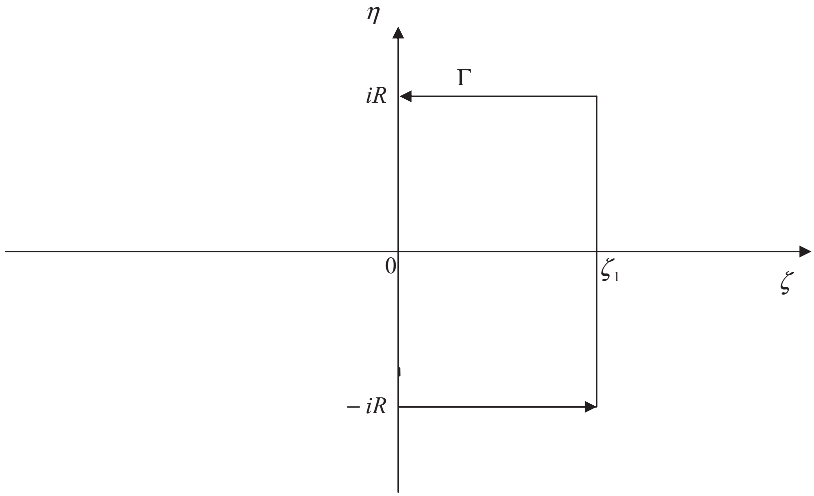

Figure 1. Paths of integration.

on the image axis, and another on the line $\zeta = \zeta_1$, whose horizontal edges move to infinity. By Cauchy's theorem, the integral of the integrand (11) over Γ is 0. From (9) we know that the contribution of horizontal edges to this integral is also 0. Thus, it follows that (11) is the same for $\zeta = \zeta_1$ as for $\zeta = 0$, which establishes our claim.

The same can be done for other coordinates. Hence, we conclude from (10) that

$$h_0(t) = \int_{\mathbb{R}^n} g_1(x+iy)e^{-\langle x+iy,t\rangle}\mathrm{d}m_n(y) \tag{12}$$

for every $x \in \mathbb{R}^n$. From (9) and (12), we have

$$|h_0(t)| \leq A_4 e^{(A+A_3)\left(\sum_{k=1}^n x_k^2 - x_k t_k\right)},$$

where $A_4 = \int_{\mathbb{R}^n} e^{-(A_3-A)|y|^2}\mathrm{d}m_n(y)$. Thus

$$|h_0(t)| \leq A_4 \prod_{k=1}^n e^{\inf\{(A+A_3)x_k^2 - x_k t_k :\ x_k \in \mathbb{R}\}},$$

direct calculation yields

$$|h_0(t)| \leq A_4 e^{-\frac{|t|^2}{4(A+A_3)}}. \tag{13}$$

From (13) we know that $h_0(t)$ is in $L^1(\mathbb{R}^n)$. Taking the inverse Fourier transform in (12), we obtain

$$g_1(z) = \int_{\mathbb{R}^n} h_0(t)e_z(t)\mathrm{d}m_n(t).$$

Therefore, from (4) and (13), if (8) holds, by properly choosing A_3, we obtain the bounded linear functional

$$T(h) = \int_{\mathbb{R}^n} h_0(t)h(t)\mathrm{d}m_n(t), \quad h \in C_\alpha$$

satisfying $T(e_\lambda) = 0$ for $\lambda \in \Lambda$, which is defined in (2) and

$$\|T\| = \int_{\mathbb{R}^n} e^{\alpha(t)}|h_0(t)\mathrm{d}m_n(t)| > 0.$$

4. An open problem. From [**1**, **2**, **4**], we know that if $\overline{Span}\{e^{\lambda t}\}$ is incomplete in C_α, which is the weighted Banach space of functions continuous on the real axis $\mathbb{R}$, then $\overline{Span}\{e^{\lambda t}\}$ can be characterised by the general Dirichlet series with frequency $\{\lambda\}$. It is well known that the zeros of entire functions of several variables can never be separated. Thus, the canonical product and residue approach that is employed in [**1**, **2**, **4**], characterising $\overline{Span}\{e^{i\lambda t}\}$ can never be applied to our situation. The following question seems interesting: *If $\overline{Span}\{e_\lambda(t)\}$, $\lambda \in \Lambda$ is incomplete in C_α, then how to characterise its closure?*

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