92.09 Explicit polynomial expressions for sums of powers of an arithmetic progression

Introduction

We shall derive the explicit polynomial expression in \( n \) for \( S_n^{(k)} = \sum_{i=1}^{n} [a + (i - 1)d]^k \) in terms of general Stirling numbers and also generalise the iterative method by integration for \( \sum_{i=1}^{n} i^k \) ([1]). So far only an evaluating approach ([2]) and implicit formulae have been derived ([3, 4]) in terms of binomial coefficients and Bernoulli numbers. Interested readers are encouraged to use the same method to explore \( \sum_{i=1}^{n} (a + b_{i-1}d)^k \) with \( b_i = \begin{pmatrix} i + 1 \\ 2 \end{pmatrix} \) or other sequences \( (b_i) \) with \( b_0 = 0 \).

We define the general permutation notation \( n_{P_r} = n(n - d)(n - 2d)\ldots[n - (r - 1)d] \) (with \( n_{P_1} = n \)) and \( s_{n-a} = (S_n - a)(S_n - a - d)(S_n - a - 2d)\ldots(S_n - a - (r - 1)d) \) so that, for example, \( s_{n-a}P_r \) denotes

\[
S_n^{(3)} - (3a + 3d)S_n^{(2)} + (3a^2 + 6ad + 2d^2)S_n^{(1)} - (a^3 + 3a^2d + 2ad^2)S_n^{(0)}.
\]

In such ‘polynomials’, \( S_n \) is a linear operator over any commutative ring; in particular, if \( S_n = S'_n + S''_n \), then \( s_{n-a}P_r = s_{n-a}P_{r-1} + s_{n-a}P_{r-1} \) since in three ‘polynomial’ expansions all the coefficients of the same ‘power’ are equal. Our method is based on the following theorem which can be proved by mathematical induction.

**Theorem 1:** \( d_{n}P_{r} = rs_{n-a}P_{r-1} \).

**Proof:**

\[
d_{n+1}P_{r} = d_{n}P_{r} + rd_{n}P_{r-1} = rd_{n-a}P_{r-1} + rd_{a+nd}P_{r-1} = rd_{a+nd}P_{r-1}.
\]

Note that only the inductive step will be displayed whenever mathematical induction is used throughout this paper, since the base steps are invariably trivial ones.

**General Stirling numbers**

For a sequence \( Q(a) = (a_n) \) define \( T_{n,k}^{Q(a)} \) to be the sum of all products of \( k \) numbers among its first \( n \) terms, with \( T_{n,0}^{Q(a)} = 1 \). Then

\[
T_{n,k}^{Q(a)} = a_n T_{n-1,k}^{Q(a)} + T_{n-1,k}^{Q(a)},
\]

since \( a_n T_{n-1,k}^{Q(a)} \) is the sum of all products of \( k \) numbers involving \( a_n \) and \( T_{n-1,k}^{Q(a)} \) is the sum of all products of \( k \) numbers not involving \( a_n \).
For simplicity, we shall write $T_{n,k}^{a,d}$ for $T_{n,k}^{(a + (n - 1)d)}$ and define

$$\binom{n}{k}_{a,d} = T_{n-1,n-k}^{a,d}.$$  Then from (1), we have

$$\binom{n}{k}_{a,d} = a_{n-1}T_{n-2,n-k-1}^{1,1} + T_{n-2,n-k}^{1,1} = (n-1)\binom{n-1}{k}_{1,1} + \binom{n-1}{k-1}_{1,1}$$

so that $\binom{n}{k}_{a,d}$ is indeed the usual Stirling number of the first kind $\binom{n}{k}$ (see [5]). In general, we shall call $\binom{n}{k}_{a,d}$ Stirling numbers of the first kind over the sequence $(a + (n - 1)d)$. In light of the inversion formula

$$\sum_{j=0}^{k} (-1)^{j} \binom{n + 1}{n - k + 1 + j} \binom{n - k + 1}{n - k + 1} = 0,$$

where $n \geq k \geq 1$ (see [5]), we define

$$V_{n,k}^{Q(a)} = \sum_{j=1}^{k} (-1)^{j-1} V_{n,k-j}^{Q(a)} T_{n-k+j,d}^{Q(a)}$$

for $n \geq k \geq 1$, (2)

and $V_{n,0}^{Q(a)} = 1$. Then from (1), we can obtain

$$V_{n,k}^{Q(a)} = a_{n-k+1} V_{n-1,k-1}^{Q(a)} + V_{n-1,k}^{Q(a)}$$

for $n \geq k \geq 1$, (3)

by using mathematical induction on $k$ for a fixed $n$ (for simplicity, the superscripts are omitted):

$$V_{n,k} = \sum_{j=1}^{k-1} (-1)^{j-1} V_{n,k-j} T_{n-k+j,j} + (-1)^{k-1} V_{n,0} T_{n,k}$$

$$= \sum_{j=1}^{k-1} (-1)^{j-1} (a_{n-k+j+1} V_{n-1,k-j-1} + V_{n-1,k-j}) T_{n-k+j,j} + (-1)^{k-1} T_{n,k}$$

$$= V_{n-1,k-1} T_{n-k+1,1} + \sum_{j=1}^{k-2} (-1)^{j-1} V_{n-1,k-j-1} (a_{n-k+j+1} T_{n-k+j,j} - T_{n-k+j+1,j+1})$$

$$+ (-1)^{k-2} a_{n-1} T_{n-1,k-1} + (-1)^{k-1} T_{n,k}$$

$$= V_{n-1,k-1} T_{n-k+1,1} + \sum_{j=1}^{k-2} (-1)^{j} V_{n-1,k-j-1} T_{n-k+j,j+1} + (-1)^{k-1} (-a_{n-1} T_{n-1,k-1} + T_{n,k})$$

$$= a_{n-k+1} V_{n-1,k-1} + V_{n-1,k-1} T_{n-k,1} + \sum_{j=1}^{k-2} (-1)^{j} V_{n-1,k-j-1} T_{n-k+j,j+1}$$

$$+ (-1)^{k-1} V_{n-1,0} T_{n-1,k}$$

$$= a_{n-k+1} V_{n-1,k-1} + \sum_{j=1}^{k} (-1)^{j-1} V_{n-1,k-j} T_{n-k+j-1,j}$$

$$= a_{n-k+1} V_{n-1,k-1} + V_{n-1,k-1}.$$
Similarly, we shall write $V_{n,k}^{ad}$ for $V_{n,k}^{(a+(n-1)d)}$ and call

\[ \left\{ \begin{array}{c} n \\ k \end{array} \right\}_{a,d} = V_{n-1,n-k}^{ad} \text{ Stirling numbers of the second kind over the sequence} \\
(a + (n - 1)d). \]

For example, we can tabulate $V_{n,k}^{2,3}$ using (3) as follows:

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>1</td>
<td>2</td>
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<td>4</td>
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<tr>
<td></td>
<td>2</td>
<td>7</td>
<td>15</td>
<td>26</td>
<td>40</td>
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<tr>
<td></td>
<td>3</td>
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<td>39</td>
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<td>445</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>26</td>
<td>203</td>
<td>1475</td>
<td>5597</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>40</td>
<td>445</td>
<td>1475</td>
<td>1031</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>52</td>
<td>595</td>
<td>595</td>
<td>32</td>
</tr>
</tbody>
</table>

**The explicit formulae**

Since $\frac{d_r P_{r,d}}{d_r - 1} = d_r n P_r$, it follows from Theorem 1 that

\[ s_n - a P_{r - 1,d} = \frac{d_r}{r} n P_r, \]

which can be used to derive the polynomial expression in $n$ for $S_n^{(k)}$ as follows.

\[ S_n^{(1)} = (S_n - a) + a S_n^{(0)} \]
\[ = s_n - a P_{1,d} + a s_n - a P_{0,d} \]
\[ = \frac{d_n P_2}{2} + a n P_1 \]
\[ = \frac{d}{2} n^2 + \left( a - \frac{d}{2} \right) n, \]
\[ S_n^{(2)} = (S_n - a)(S_n - a - d) + (2a + d)(S_n - a) + a^2 S_n^{(0)} \]
\[ = s_n - a P_{2,d} + (2a + d)s_n - a P_{1,d} + a^2 s_n - a P_{0,d} \]
\[ = \frac{d_n^2 P_3}{3} + \frac{(2a + d)d_n P_2}{2} + a^2 n P_1 \]
\[ = \frac{d^2}{3} n^3 + d \left( a - \frac{d}{2} \right) n^2 + \left( a^2 - ad + \frac{d^2}{6} \right) n, \]
\[ S_n^{(3)} = (S_n - a)(S_n - a - d)(S_n - a - 2d) + (3a + 3d)(S_n - a)(S_n - a - d) \]
\[ + (3a^2 + 3ad + d^2)(s_n - a) + a^3 s_n^{(0)} \]
\[ = s_n - a P_{3,d} + (3a + 3d)s_n - a P_{2,d} + (3a^2 + 3ad + d^2)s_n - a P_{1,d} + a^3 s_n - a P_{0,d} \]
\[ = \frac{d_n^3 P_4}{4} + \frac{(3a + 3d)d_n^2 P_3}{3} + \frac{(3a^2 + 3ad + d^2)d_n P_2}{2} + a^3 n P_1 \]
\[ = \frac{d^3}{4} n^4 + d^3 \left( a - \frac{d}{2} \right) n^3 + 3d \left( a^2 - ad + \frac{d^2}{6} \right) n^2 + a(a - d) \left( a - \frac{d}{2} \right) n, \]

which can also be derived iteratively by using the following theorem.
Theorem 2: Let \( S_n^{(k)} = \sum_{j=1}^{k} a_{k+1-j} n^{k+1-j} \), where \( a_{k+1-j} \) is a polynomial in \( a \) and \( d \). Then \( S_n^{(k+1)} = d(k+1) \int S_n^{(k)} dn + c_{k+1} n \), where \( c_{k+1} \) is a polynomial in \( a \) and \( d \) that can be determined by \( S_1^{(k+1)} = a^{k+1} \).

Proof: We use mathematical induction on \( n \):

\[
S_{n+1}^{(k+1)} = S_n^{(k+1)} + (a + nd)^{k+1}
\]

\[
= \left[ d(k+1) \int S_n^{(k)} dn + c_{k+1} n \right] + \left[ d(k+1) \int (a + nd)^{k} dn + a^{k+1} \right]
\]

\[
= d(k+1) \int S_n^{(k+1)} dn(n+1) + c_{k+1} n + a^{k+1}
\]

\[
= d(k+1) \int S_n^{(k)} dn(n+1) + c_{k+1} (n+1),
\]

where the last step is true, since \( a^{k+1} = S_1^{(k+1)} = c_{k+1} \) when \( n = 0 \).

If we look back, the last step of the derivation of each \( S_n^{(k)} \) involves Stirling numbers of the first kind and general Stirling numbers of the second kind. To be more specific, we can write

\[
S_n^{(1)} = \frac{d}{2} \begin{bmatrix} 2 \\ 2 \\ a_d \end{bmatrix} n^2 + \begin{bmatrix} 1 \\ 1 \\ a_d \end{bmatrix} - \frac{d}{2} \begin{bmatrix} 2 \\ 2 \\ a_d \end{bmatrix} n^2,
\]

\[
S_n^{(2)} = \frac{d^2}{3} \begin{bmatrix} 3 \\ 3 \\ a_d \end{bmatrix} n^3 + \frac{d}{2} \begin{bmatrix} 2 \\ 2 \\ a_d \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ a_d \end{bmatrix} n^2
\]

\[
+ \begin{bmatrix} 1 \\ 1 \\ a_d \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ a_d \end{bmatrix} - \frac{d}{2} \begin{bmatrix} 2 \\ 2 \\ a_d \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ a_d \end{bmatrix} + \frac{d^2}{3} \begin{bmatrix} 3 \\ 3 \\ a_d \end{bmatrix} n^2,
\]

\[
S_n^{(3)} = \frac{d^3}{4} \begin{bmatrix} 4 \\ 4 \\ a_d \end{bmatrix} n^4 + \frac{d^2}{3} \begin{bmatrix} 3 \\ 3 \\ a_d \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ a_d \end{bmatrix} n^3
\]

\[
+ \frac{d}{2} \begin{bmatrix} 2 \\ 2 \\ a_d \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ a_d \end{bmatrix} - \frac{d^2}{3} \begin{bmatrix} 3 \\ 3 \\ a_d \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ a_d \end{bmatrix} + \frac{d^3}{4} \begin{bmatrix} 4 \\ 4 \\ a_d \end{bmatrix} n^2
\]

\[
+ \begin{bmatrix} 1 \\ 1 \\ a_d \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ a_d \end{bmatrix} - \frac{d}{2} \begin{bmatrix} 2 \\ 2 \\ a_d \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ a_d \end{bmatrix} + \frac{d^2}{3} \begin{bmatrix} 3 \\ 3 \\ a_d \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ a_d \end{bmatrix} - \frac{d^3}{4} \begin{bmatrix} 4 \\ 4 \\ a_d \end{bmatrix} n^2,
\]

and in general

\[
S_n^{(k)} = \sum_{r=0}^{k} \sum_{j=k+1-r}^{k+1} (-1)^{j-k+1-r} d^{j-1} \int \begin{bmatrix} j \\ j \\ k+1 \\ k+1-r \end{bmatrix} \begin{bmatrix} k+1 \\ a_d \end{bmatrix} n^{k+1-r}. \tag{4}
\]
This may be proved by mathematical induction using properties of Stirling numbers via Theorem 2, or by way of formulae

\[ nPr = \frac{r}{d^{r-1}} \sum_{j=1}^{r} (-1)^{j-1} \left( \sum_{t=0}^{j-1} \binom{r-1}{j-1} \binom{r-1-t}{r-1-t} d^{r-1-t} \right) S_n^{(r-j)} \]

and

\[ S_n^{(k)} = \frac{k+1}{j} \binom{k+1}{j} n P_j. \]

To help readers verify the above, we first tabulate \( \binom{n}{k} \) below.

<table>
<thead>
<tr>
<th>( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
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<tbody>
<tr>
<td>( n )</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td></td>
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</tr>
<tr>
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<td>6</td>
<td>11</td>
<td>6</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>24</td>
<td>50</td>
<td>35</td>
<td>10</td>
<td>1</td>
</tr>
</tbody>
</table>

Then use (2) to calculate

\[ \binom{2}{1} = V_{1,1}^{a,d} = a, \quad \binom{3}{2} = V_{2,1}^{a,d} = 2a + d, \]

\[ \binom{3}{1} = V_{2,2}^{a,d} = a^2, \quad \binom{4}{3} = V_{3,1}^{a,d} = 3a + 3d, \quad \binom{4}{2} = V_{3,2}^{a,d} = 3a^2 + 3ad + d^2 \]

and

\[ \binom{4}{1} = V_{3,3}^{a,d} = a^3, \]

based on \( T_{1,1}^{a,d} = a, \quad T_{2,1}^{a,d} = 2a + d, \quad T_{2,2}^{a,d} = a(a + d), \quad T_{3,1}^{a,d} = 3a + 3d, \quad T_{3,2}^{a,d} = 3a^2 + 6ad + 2d^2 \) and \( T_{3,3}^{a,d} = a(a + d)(a + 2d). \)

For a given arithmetic progression, (4) can be directly used to derive the polynomial expression for the sum of any specific power. For example, we can use (4) and the tables in this and previous sections to obtain

\[
\sum_{i=1}^{n} [2 + 3(i - 1)]^3 = \frac{3^3}{4} \binom{4}{4} \binom{4}{2,3} n^4 + \frac{3^2}{3} \binom{3}{3} \binom{4}{3} \binom{4}{2,3} n^3 - \frac{3^3}{4} \binom{4}{4} \binom{4}{2,3} n^3 + \binom{3}{2} \binom{2}{2} \binom{4}{2,3} n^2 - \frac{3^2}{3} \binom{3}{3} \binom{4}{3} \binom{4}{2,3} n^2 + \binom{3}{2} \binom{2}{2} \binom{4}{2,3} n^2 - \frac{3^3}{4} \binom{4}{4} \binom{4}{2,3} n^3 + \binom{1}{1} \binom{1}{1} \binom{4}{2,3} n^2 - \frac{3}{2} \binom{2}{2} \binom{4}{2,3} n^2 + \binom{3}{3} \binom{4}{3} \binom{4}{2,3} n^2 - \frac{3^3}{4} \binom{4}{4} \binom{4}{2,3} n^3
\]
\[ \begin{align*}
\frac{3^3}{4} (1)(n^4) + \left[ \frac{3^2}{3} (1)(15) - \frac{3^3}{4} (6)(1) \right] n^3 + \left[ \frac{3}{2} (1)(39) - \frac{3^2}{3} (3)(15) + \frac{3^2}{4} (11)(1) \right] n^2 \\
+ \left[ (1)(8) - \frac{3}{2} (1)(39) + \frac{3^2}{3} (2)(15) - \frac{3^3}{4} (6)(1) \right] n \\
= \frac{27}{4} n^4 + \frac{9}{2} n^3 - \frac{9}{4} n^2 - n.
\end{align*} \]

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References


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92.10 Evaluating \( \sum_{n=1}^{N} (a + nd)^p \) again

In [1] Hirschhorn proves a formula for the sum

\[ S_p = \sum_{n=1}^{N} (a + nd)^p \]

involving Bernoulli numbers. When \( a = 0 \) and \( d = 1 \) the sum reduces to the celebrated sum \( \sum_{n=1}^{N} n^p \). In the book *Concrete Mathematics* [2], section 2.5 gives an illuminating comparison of various methods of evaluating the latter sum. Here we adapt one standard method, using exponential generating functions, for computing \( \sum_{n=1}^{N} n^p \) (described in [2, §7.6]) to calculate the sums \( S_p \).

A sequence \( a_0, a_1, a_2, \ldots \) of numbers has the ordinary generating function

\[ \sum_{p=0}^{\infty} a_p T^p \]

and the exponential generating function

\[ \sum_{p=0}^{\infty} \frac{a_p T^p}{p!} \].