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HYPERSURFACES IN $\mathbb{C}P^2$ AND $\mathbb{C}H^2$ WITH TWO DISTINCT PRINCIPAL CURVATURES

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Abstract. It is known that hypersurfaces in $\mathbb{C}P^n$ or $\mathbb{C}H^n$ for which the number g of distinct principal curvatures satisfies $g \leq 2$, must belong to a standard list of Hopf hypersurfaces with constant principal curvatures, provided that $n \geq 3$. In this paper, we construct a two-parameter family of non-Hopf hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ with $g = 2$ and show that every non-Hopf hypersurface with $g = 2$ is locally of this form.

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1. Introduction. It is known that hypersurfaces in $\mathbb{C}P^n$ or $\mathbb{C}H^n$ for which the number g of distinct principal curvatures satisfies $g \leq 2$ must be members of the Takagi/Montiel lists of homogeneous Hopf hypersurfaces, provided that $n \geq 3$. (See Theorems 4.6 and 4.7 of [**8**]). In particular, they must be Hopf. In this paper, we investigate the case $n = 2$.

We first show in Theorem 2 that Hopf hypersurfaces in $\mathbb{C}P^2$ or $\mathbb{C}H^2$ with $g \leq 2$ must be in the Takagi/Montiel lists. However, it turns out that there are also non-Hopf examples with $g \leq 2$ and the rest of the paper will be devoted to studying them.

REMARK 1. After the completion of this work we have learned of a preprint by Díaz-Ramos, Domínguez-Vazquez and Vidal-Castiñeira [**3**] where they classify hypersurfaces with two principal curvatures in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ using the notion of polar actions.

In what follows, all manifolds are assumed connected and all manifolds and maps are assumed smooth (C^∞) unless stated otherwise.

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2. Basic equations and results for hypersurfaces. We follow the notation and terminology of [**8**]. M^{2n-1} will be a hypersurface in a complex space form, either $\mathbb{C}P^n$ or $\mathbb{C}H^n$, of constant holomorphic sectional curvature $4c = \pm 4/r^2$. The locally defined field of unit normals is ξ, the structure vector field is $W = -J\xi$ and φ is the tangential projection of the complex structure J. The holomorphic distribution consisting of all tangent vectors orthogonal to W is denoted by $W^\perp$ and $\varphi^2\mathbf{v} = -\mathbf{v}$ for all $\mathbf{v} \in W^\perp$.

The shape operator A of M is defined by

$$A\mathbf{v} = -\widetilde{\nabla}_{\mathbf{v}}\xi,$$

where $\widetilde{\nabla}$ is the Levi-Civita connection of the ambient space and $\mathbf{v}$ is any tangent vector to M. (It follows that $\varphi AX = \nabla_X W$ for any tangent vector X.) The eigenvalues of A are the principal curvatures and the corresponding eigenvectors and eigenspaces are said to be principal vectors and principal spaces. The function $\langle AW, W\rangle$ is denoted by α. If W is a principal vector at all points of M (and so $AW = \alpha W$), we say that M is a *Hopf hypersurface* and α is called the Hopf principal curvature. For a Hopf hypersurface, the Hopf principal curvature is constant. We state the following fundamental facts (see Corollary 2.3 of [**8**]).

LEMMA 1. *Let M be a Hopf hypersurface and let $X \in W^\perp$ be a principal vector with associated principal curvature λ. Then*

(1) $(\lambda - \frac{\alpha}{2})A\varphi X = (\frac{\lambda\alpha}{2} + c)\varphi X$.
(2) *If $A\varphi X = \nu\varphi X$ for some scalar ν, then $\lambda\nu = \frac{\lambda+\nu}{2}\alpha + c$.*
(3) *If $\nu = \lambda$ in (2), then $\nu^2 = \alpha\nu + c$.*

2.1. Takagi's list and Montiel's list. There is a distinguished class of model hypersurfaces, which we list below. We use the established nomenclature (types A, B, C, D, E with subdivisions A_0, A_1, etc.) due to Takagi [**10**] and Montiel [**7**]. These lists consist precisely of the complete Hopf hypersurfaces with constant principal curvatures in their respective ambient spaces as determined by Kimura [**6**] and Berndt [**1**]. Equivalently, it is the list of homogeneous Hopf hypersurfaces, a fact which follows from the work of Takagi [**9**] and Berndt [**1**]. Non-Hopf homogeneous hypersurfaces exist in $\mathbb{C}\mathrm{H}^n$ but not in $\mathbb{C}\mathrm{P}^n$.

Takagi's list for $\mathbb{C}\mathrm{P}^n$

- (A_1) Geodesic spheres (which are also tubes over totally geodesic complex projective spaces $\mathbb{C}\mathrm{P}^{n-1}$).
- (A_2) Tubes over totally geodesic complex projective spaces $\mathbb{C}\mathrm{P}^k$, $1 \le k \le n-2$.
- (B) Tubes over complex quadrics (which are also tubes over totally geodesic real projective spaces $\mathbb{R}P^n$).
- (C) Tubes over the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^m$ where $2m+1 = n$ and $n \ge 5$.
- (D) Tubes over the Plücker embedding of the complex Grassmann manifold $G_{2,5}$ (which occur only for $n = 9$).
- (E) Tubes over the canonical embedding of the Hermitian symmetric space $SO(10)/U(5)$ (which occur only for $n = 15$).

Note that only types A_1 and B can occur when $n = 2$.

Montiel's list for $\mathbb{C}\mathrm{H}^n$

- (A_0) Horospheres.
- (A_1) Geodesic spheres and tubes over totally geodesic complex hyperbolic spaces $\mathbb{C}\mathrm{H}^{n-1}$.
- (A_2) Tubes over totally geodesic complex hyperbolic spaces $\mathbb{C}\mathrm{H}^k$, $1 \le k \le n-2$.
- (B) Tubes over totally geodesic real hyperbolic spaces $\mathbb{R}\mathrm{H}^n$.

Note that Type A_2 cannot occur when $n = 2$.

3. The Hopf case.

THEOREM 2. *Let M^3 be a Hopf hypersurface in $\mathbb{C}P^2$ or $\mathbb{C}H^2$ with $g \leq 2$ distinct principal curvatures at each point. Then M is an open subset of a hypersurface in the lists of Takagi and Montiel.*

Proof. It is well-known (see Theorem 1.5 of [**8**]) that umbilic hypersurfaces cannot occur in $\mathbb{C}P^n$ or $\mathbb{C}H^n$. In fact, Hopf hypersurfaces cannot have umbilic points, since by Lemma 1 the Hopf principal curvature α would have to satisfy $\alpha^2 = \alpha^2 + c$ at such points. Thus, when $n = 2$ the multiplicity of α as a principal curvature is either 1 or 2 at each point $p \in M$, and by continuity the multiplicity will be the same on an open set around p. Hence the set of points where α has multiplicity 2, and the set of points where α has multiplicity 1 (which coincides with the set of points where the holomorphic subspace $W^\perp$ is principal), are both open and closed in M. So, one set is empty and the other is all of M.

If α has multiplicity 2 on M, Lemma 1 shows that $\alpha\nu = \alpha^2 + 2c$ where ν is the other principal curvature. Thus, if $\alpha^2 + 2c \neq 0$ then ν must be a nonzero constant, while if $\alpha^2 + 2c = 0$ then ν must be identically zero. The classification of Hopf hypersurfaces with constant principal curvatures by Kimura [**6**] and Berndt [**1**] implies that M is an open subset of a hypersurface in the Takagi/Montiel lists. In fact, M must be a Type B hypersurface in $\mathbb{C}H^2$ (a tube around $\mathbb{R}H^2$) of radius ru with $\coth u = \sqrt{3}$.

The other possibility is that α has multiplicity 1 on M. Then the other principal curvature satisfies $\nu^2 = \alpha\nu + c$ and so is constant. Again, M must be an open subset of a hypersurface in the Takagi/Montiel list, in this case Type A_0 (a horosphere in $\mathbb{C}H^2$) or Type A_1 (a geodesic sphere in $\mathbb{C}P^2$ or $\mathbb{C}H^2$, or a tube over a totally geodesic $\mathbb{C}H^1$ in $\mathbb{C}H^2$). □

4. The non-Hopf case. Consider now a hypersurface M in the ambient space $\mathcal{X}$ (either $\mathbb{C}P^2$ or $\mathbb{C}H^2$). If M is not Hopf, then $AW \neq \alpha W$ on a nonempty open subset of M, and we can construct the *standard frame* (W, X, Y) as follows. First, choose the unit vector field X so that $AW = \alpha W + \beta X$ for a positive function β; then let $Y = \varphi X$. Then A is represented with respect to this frame by a matrix

$$\begin{pmatrix} \alpha & \beta & 0 \\ \beta & \lambda & \mu \\ 0 & \mu & \nu \end{pmatrix}, \tag{1}$$

where λ, μ, ν are also smooth functions.

PROPOSITION 3. *Let M^3 be a hypersurface in $\mathbb{C}P^2$ or $\mathbb{C}H^2$ and suppose that $AW \neq \alpha W$ on M. Then there are $g \leq 2$ distinct principal curvatures at each point if and only if $\mu = 0$ and*

$$\nu^2 - (\alpha + \lambda)\nu + (\lambda\alpha - \beta^2) = 0. \tag{2}$$

Proof. Since $AW \neq \alpha W$, the setup leading to (1) holds, and therefore $g \geq 2$ globally. Suppose now that $g = 2$ everywhere. We will construct the standard frame (W, X, Y) in a slightly different way.

First, note that $W^\perp$ intersects the two-dimensional principal space in a one-dimensional subspace. On any simply-connected domain in M, let $\widetilde{Y}$ be a unit principal

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vector field lying in $W^\perp$, corresponding to the principal curvature ν of multiplicity 2. Let $\widetilde{X} = -\varphi \widetilde{Y}$. Then $(W, \widetilde{X}, \widetilde{Y})$ is a local orthonormal frame.

Since the span of $\{W, \widetilde{X}\}$ is A-invariant, A is represented by a matrix of the form (1), with $\mu = 0$. (Although β was specified to be a positive function in (1), this can easily be arranged by changing the sign of Y if necessary.) Thus, we can drop the tildes on X and Y.

Furthermore, ν must be an eigenvalue of the upper-left 2×2 submatrix of A, from which the formula (2) follows. The converse is trivial. □

Proposition 3 implies that non-Hopf hypersurfaces with $g = 2$ are part of a class of hypersurfaces previously investigated by Díaz-Ramos and Domínguez-Vazquez [**2**] in the context of constant principal curvatures. Namely, one defines a distribution $\mathcal{H}$ to be the span of $\{W, AW, A^2W, \ldots\}$. For each $x \in M$, $\mathcal{H}_x \subset T_xM$ is the smallest subspace that contains W_x and is invariant under A. Díaz-Ramos and Domínguez-Vazquez study hypersurfaces where $\mathcal{H}$ has constant rank 2. (This is a generalization of the Hopf condition, under which $\mathcal{H}$ has constant rank one.) Since from Proposition 3 we have $\mu = 0$, it is clear that if M is non-Hopf with $g = 2$ then $\mathcal{H}$ has rank 2 for these hypersurfaces, but more is true:

Theorem 4. *Let M be a hypersurface in $\mathcal{X}$ with $AW \neq \alpha W$ and $g \leq 2$ principal curvatures at each point. Then $\mathcal{H}$ has rank 2, and is integrable. Furthermore, the derivatives of components α, β, λ, and ν are zero along directions tangent to $\mathcal{H}$, and they satisfy*

$$\begin{aligned} \frac{d\alpha}{ds} &= \beta(\alpha + \lambda - 3\nu) \\ \frac{d\beta}{ds} &= \beta^2 + \lambda^2 + \nu(\alpha - 2\lambda) + c \\ \frac{d\lambda}{ds} &= \frac{(\lambda - \nu)(\lambda^2 - \alpha\lambda - c)}{\beta} + \beta(2\lambda + \nu), \end{aligned} \tag{3}$$

where d/ds stands for the derivative with respect to Y.

We will postpone the proof of this theorem until Section 5.

Hypersurfaces in $\mathbb{C}P^2$ or $\mathbb{C}H^2$ with $\mathcal{H}$ of rank 2 and integrable are discussed in Section 6, where we prove the following existence result:

Theorem 5. *Suppose $\alpha(s), \beta(s), \lambda(s), \nu(s)$ comprise a smooth solution of the underdetermined system* (3)*, defined for s in an open interval $I \subset \mathbb{R}$, and such that $\beta(s)$ is nonvanishing. Then there exists a smooth immersion $\Phi : I \times \mathbb{R}^2 \to \mathcal{X}$ determining a hypersurface M, equipped with a standard frame (W, X, Y), such that Φ maps the $\mathbb{R}^2$-factors onto leaves of $\mathcal{H}$. The components of the shape operator are constant along these leaves and they coincide with the given solution. Furthermore, the leaves are homogeneous and have Gauss curvature zero.*

Corollary 1. *Suppose $\alpha(s), \beta(s), \lambda(s), \nu(s)$ satisfy the system* (3) *and the algebraic condition* (2)*. Then the hypersurface constructed by Theorem 5 is a non-Hopf hypersurface with two distinct principal curvatures. Conversely, every such hypersurface is locally of this form.*

Proof. The first statement follows immediately from Theorem 5 and the 'if' part of Proposition 3. The second statement follows from the 'only if' part of the proposition and Theorem 4. □

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This last result shows that Theorems 4.6 and 4.7 of [**8**], quoted at the beginning of Section 1, do not extend to $n = 2$. For, given initial values $\alpha_0, \beta_0, \lambda_0$ such that $\beta_0 \neq 0$, we can define a function $F(\alpha, \beta, \lambda)$ on a neighbourhood of this point in $\mathbb{R}^3$ such that $\nu = F(\alpha, \beta, \lambda)$ satisfies (2) identically. Then substituting this for ν in the system (3) gives a determined system. Applying standard existence theory for systems of ODE, and using our initial values at $s = 0$, will yield a solution $\alpha(s)$, $\beta(s)$, $\lambda(s)$ of (3) with $\nu(s) = F(\alpha(s), \beta(s), \lambda(s))$ (so that (2) is satisfied), and which is defined for s on an open interval I containing zero. Because the system is autonomous, using the values $(\alpha(s_1), \beta(s_1), \lambda(s_1))$ for any nonzero $s_1 \in I$ as initial conditions for the system will recover the same solution.

To summarize, there is a two-parameter family of solution trajectories T for (3) which satisfy (2) with β non-vanishing. Each of these determines a non-Hopf hypersurface M_T with $g = 2$, up to rigid motions. Conversely, given any non-Hopf hypersurface M' with $g = 2$ and $p_0 \in M'$, we may use the components of the shape operator of M' at p_0 as initial conditions to determine an M_T which is congruent to an open subset of M' containing p_0.

5. Moving frames calculations. In this section, we will prove Theorem 4 using the techniques of moving frames and exterior differential systems. Background material in this subject may be found in the textbook [**5**]. We begin by reviewing the geometric structure of the frame bundle that we will use.

An orthonormal frame (e_1, e_2, e_3, e_4) at a point in $\mathcal{X}$ is defined to be unitary if $Je_1 = e_2$ and $Je_3 = e_4$. We let $\mathcal{F}$ be the bundle of *unitary frames* on $\mathcal{X}$. On $\mathcal{F}$ there are canonical forms ω^i and connection forms ω^i_j for $1 \leq i, j \leq 4$. These have the property that if (e_1, e_2, e_3, e_4) is any local unitary frame field on $\mathcal{X}$ and f is the corresponding local section of $\mathcal{F}$, then the $f^*\omega^i$ comprise the dual coframe field, and

$$\langle e_i, \widetilde{\nabla}_{\mathbf{v}} e_j \rangle = \mathbf{v} \lrcorner f^* \omega^i_j,$$

where $\widetilde{\nabla}$ is the Levi-Civita connection on $\mathcal{X}$ and $\mathbf{v}$ is tangent to $\mathcal{X}$. The connection forms satisfy $\omega^i_j = -\omega^j_i$, but also the *structure equations*

$$d\omega^i = -\omega^i_j \wedge \omega^j, \quad d\omega^i_j = -\omega^i_k \wedge \omega^k_j + \Omega^i_j,$$

where Ω^i_j are the curvature 2-forms. The latter encode the curvature tensor of $\mathcal{X}$, because for any local section,

$$f^*\Omega^i_j = \tfrac{1}{2} R^i_{jk\ell} f^*(\omega^k \wedge \omega^\ell), \quad R^i_{jk\ell} = \langle e_i, R(e_k, e_\ell) e_j \rangle.$$

(The structure equations and their relation to the curvature tensor hold on the orthonormal frame bundle of any Riemannian manifold; see Section 2.6 in [**5**].) Moreover, because $\mathcal{X}$ is a Kähler manifold and hence J is parallel with respect to $\widetilde{\nabla}$, we have

$$\omega^3_1 = \omega^4_2, \quad \omega^3_2 = -\omega^4_1,$$

with similar relationships holding among the curvature 2-forms. We will use 1-forms $\omega^1, \dots, \omega^4, \omega^2_1, \omega^4_1, \omega^4_2, \omega^4_3$ as a (globally defined) coframe on $\mathcal{F}$. In order to compute the exterior derivatives of these 1-forms, we will need to know the curvature 2-forms. Using the fact that $\mathcal{X}$ is a space of constant holomorphic sectional curvature $4c$, Theorem 1.1 in [**8**] implies that

$$\Omega^2_1 = -c(4\omega^1 \wedge \omega^2 + 2\omega^3 \wedge \omega^4), \qquad \Omega^4_1 = -c(\omega^1 \wedge \omega^4 - \omega^2 \wedge \omega^3),$$
$$\Omega^4_3 = -c(2\omega^1 \wedge \omega^2 + 4\omega^3 \wedge \omega^4), \qquad \Omega^4_2 = -c(\omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^4).$$

Our method for studying and constructing (framed) hypersurfaces will be to treat the images of the sections f as integral submanifolds of a Pfaffian exterior differential system. Briefly, a Pfaffian system $\mathcal{I}$ on a manifold B is a graded ideal inside the algebra $\Omega^* B$ of differential forms on B, which near any point is generated algebraically by a finite set of 1-forms and their exterior derivatives. A submanifold $N \subset B$ is an integral of $\mathcal{I}$ if and only if $i^*\psi = 0$ for all differential forms ψ in $\mathcal{I}$, where i: $N \to B$ is the inclusion map. (We will often abbreviate this by saying that $\psi = 0$ *along* N.)

We will next show that a frame for a hypersurface M, adapted as in Section 4, corresponds to an integral submanifold of a certain Pfaffian system. Given a standard frame (W, X, Y) on M satisfying $AW \neq \alpha W$, we can define a local section f: $M \to \mathcal{F}|_M$ by letting

$$e_3 = W, \quad e_4 = JW, \quad e_2 = X, \quad e_1 = -Y. \tag{4}$$

Then the pullbacks of the 1-forms on $\mathcal{F}$ satisfy

$$f^*\omega^4 = 0, \quad f^* \begin{bmatrix} \omega^4_3 \\ \omega^4_2 \\ \omega^4_1 \end{bmatrix} = \begin{pmatrix} \alpha & \beta & 0 \\ \beta & \lambda & -\mu \\ 0 & -\mu & \nu \end{pmatrix} f^* \begin{bmatrix} \omega^3 \\ \omega^2 \\ \omega^1 \end{bmatrix}, \tag{5}$$

where $\alpha, \beta, \lambda, \mu, \nu$ are the functions on M giving the components of the shape operator (1).

To see that these conditions are equivalent to the vanishing of certain 1-forms along a submanifold, we need to introduce the shape operator components as extra variables. In particular, we let $\alpha, \beta, \lambda, \mu, \nu$ be coordinates on $\mathbb{R}^5$, and we define following 1-forms on $\mathcal{F} \times \mathbb{R}^5$:

$$\begin{aligned} \theta_0 &:= \omega^4 \\ \theta_1 &:= \omega^4_1 - \nu\omega^1 + \mu\omega^2 \\ \theta_2 &:= \omega^4_2 + \mu\omega^1 - \lambda\omega^2 - \beta\omega^3 \\ \theta_3 &:= \omega^4_3 - \beta\omega^2 - \alpha\omega^3. \end{aligned}$$

(The ω^i and ω^i_j are pulled back from $\mathcal{F}$ to $\mathcal{F} \times \mathbb{R}^5$.) Given a standard frame on M, the fibred product of f with the graphs of $\alpha, \beta, \lambda, \mu, \nu$ gives a mapping $\widehat{f}$: $M \to \mathcal{F} \times \mathbb{R}^5$ whose image is a three-dimensional integral manifold of the Pfaffian system generated by 1-forms $\theta_0, \ldots, \theta_3$. Moreover, this submanifold satisfies the *independence condition* $\widehat{f}^*(\omega^1 \wedge \omega^2 \wedge \omega^3) \neq 0$. Conversely, every three-dimensional integral manifold of the differential forms $\theta_0, \ldots, \theta_3$ that satisfies the independence condition arises in exactly this way, from a standard adapted frame along a hypersurface in $\mathcal{X}$.

Proof of Theorem 4. Let $\mathcal{U} \subset \mathbb{R}^5$ be defined by $\beta \neq 0$, $\mu = 0$ and

$$(\nu - \alpha)(\nu - \lambda) = \beta^2. \tag{6}$$

(This is just (2) rewritten.) Because $\beta \neq 0$, we have $\alpha - \nu \neq 0$ at each point of $\mathcal{U}$, and $\mathcal{U}$ is a smooth three-dimensional submanifold of $\mathbb{R}^5$. We will use β, ν and $\tau = (\nu - \alpha)/\beta$ as coordinates on $\mathcal{U}$, in terms of which

$$\alpha = \nu - \beta\tau, \quad \lambda = \nu - \beta/\tau. \tag{7}$$

Note that τ is always nonzero. The geometric meaning of τ is that, if we write the unit ν-eigenvector in the span of $\{W, X\}$ as $\cos\phi\, W + \sin\phi\, X$, then $\tau = \tan\phi$.

Now let M be a non-Hopf hypersurface with $g \leq 2$ distinct principal curvatures. In the proof of Proposition 3 we developed a (local) standard frame on M for which the components of the shape operator satisfy $\mu = 0$ and (6). Let f be the corresponding section of $\mathcal{F}|_M$. Then the image of $\widehat{f}$ is an integral of $\theta_0, \ldots, \theta_3$ which lies in $\mathcal{F} \times \mathcal{U}$. Accordingly, we pull back the forms from $\mathcal{F} \times \mathbb{R}^5$ to $\mathcal{F} \times \mathcal{U}$, giving

$$\begin{aligned}
\theta_1 &= \omega_1^4 - \nu\omega^1 \\
\theta_2 &= \omega_2^4 - (\nu - \beta/\tau)\omega^2 - \beta\omega^3 \\
\theta_3 &= \omega_3^4 - \beta\omega^2 - (\nu - \beta\tau)\omega^3,
\end{aligned}$$

and let $\mathcal{I}$ be the Pfaffian exterior differential system on $\mathcal{F} \times \mathcal{U}$ generated by these re-defined 1-forms.

As an algebraic ideal, $\mathcal{I}$ is generated by these 1-forms and their exterior derivatives. We may simplify the latter by omitting wedge products involving the $\theta_0, \ldots, \theta_3$ as factors. For example, we compute

$$-d\theta_0 = \theta_1 \wedge \omega^1 + \theta_2 \wedge \omega^2 + \theta_3 \wedge \omega^3,$$

so that $d\theta_0$ adds no new algebraic generators for the ideal; we express this fact by writing $d\theta_0 \equiv 0 \mod \theta_0, \ldots, \theta_3$. Similarly, we compute

$$\left.\begin{aligned}
-d\theta_1 &\equiv \pi_1 \wedge \omega^1 + \pi_4 \wedge (\omega^2 - \tau\omega^3), \\
-d\theta_2 &\equiv \pi_4 \wedge \omega^1 + \pi_3 \wedge \omega^2 + \pi_2 \wedge \omega^3, \\
-d\theta_3 &\equiv -\tau\pi_4 \wedge \omega^1 + \pi_2 \wedge \omega^2 + \big((1+\tau^2)\pi_1 - 2\tau\pi_2 - \tau^2\pi_3\big) \wedge \omega^3
\end{aligned}\right\} \mod \theta_0, \ldots, \theta_3, \tag{8}$$

where

$$\begin{aligned}
\pi_1 &:= d\nu + \frac{3c\tau}{1+\tau^2}\omega^1, \\
\pi_2 &:= d\beta + \left(\beta^2 - 2\beta\nu\tau + 2c + \frac{\beta\nu}{\tau}\right)\omega^1, \\
\pi_3 &:= d\left(\nu - \frac{\beta}{\tau}\right) + \left(3\beta\nu - \frac{\beta^2}{\tau}\right)\omega^1, \\
\pi_4 &:= \frac{\beta}{\tau}\omega_1^2 + (c - \beta\nu\tau)\omega^3.
\end{aligned} \tag{9}$$

The 1-forms $\pi_1, \ldots, \pi_4$, along with $\omega^1, \omega^2, \omega^3$ and $\theta_0, \ldots, \theta_3$, complete a coframe on $\mathcal{F} \times \mathcal{U}$ which is adapted to $\mathcal{I}$ in the sense that the generator 2-forms of $\mathcal{I}$ are most simply expressed in terms of this coframe.

Suppose that Σ is an integral 3-fold of $\mathcal{I}$ satisfying the independence condition. Let $\Theta_1, \Theta_2, \Theta_3$ be the 2-forms on the right-hand side of (8), which must vanish along

Σ. The vanishing of Θ_1 implies (using Cartan's lemma) that

$$\pi_1 = m\,\omega^1 + p\,\widetilde{\omega}^2, \quad \pi_4 = p\,\omega^1 + q\,\widetilde{\omega}^2 \tag{10}$$

for some functions m, p, q along Σ. (For convenience, we will let $\widetilde{\omega}^2$ denote $\omega^2 - \tau\omega^3$ from now on.) On the other hand

$$\Theta_3 + \tau\Theta_2 = (\pi_2 + \tau\pi_3) \wedge \widetilde{\omega}^2 + (1+\tau^2)\pi_1 \wedge \omega^3,$$

and applying Cartan's lemma to the vanishing of this 2-form yields

$$\pi_2 + \tau\pi_3 = s\,\widetilde{\omega}^2 + u\,\omega^3, \quad (1+\tau^2)\pi_1 = u\,\widetilde{\omega}^2 + v\,\omega^3 \tag{11}$$

for some functions s, u, v along Σ. Comparing (10) and (11) shows that $u = p(1+\tau^2)$ and $m = v = 0$; hence

$$\pi_1 = p\,\widetilde{\omega}^2, \quad \pi_4 = p\,\omega^1 + q\,\widetilde{\omega}^2, \quad \pi_2 + \tau\pi_3 = p(1+\tau^2)\omega^3 + s\,\widetilde{\omega}^2,$$

along Σ. Substituting these into the equation $\Theta_2 = 0$ implies that

$$\pi_3 = q\,\omega^1 + t\,\widetilde{\omega}^2 + s\,\omega^3$$

for an additional function t along Σ.

Substituting these values into the definitions (9) of π_1 through π_4 lets us determine the values of the exterior derivatives

$$\begin{aligned}
d\nu &= -\frac{3c\tau}{1+\tau^2}\omega^1 + p\,\widetilde{\omega}^2,\\
d\beta &= \left(2\beta\nu\tau - \beta^2 - \frac{\beta\nu}{\tau} - 2c - \tau q\right)\omega^1 + (s - \tau t)\widetilde{\omega}^2 + (p(1+\tau^2) - \tau s)\omega^3,\\
d\lambda &= d\left(\nu - \frac{\beta}{\tau}\right) = \left(\frac{\beta^2}{\tau} - 3\beta\nu + q\right)\omega^1 + t\,\widetilde{\omega}^2 + s\,\omega^3
\end{aligned} \tag{12}$$

along Σ, as well as

$$\omega_1^2 = \frac{\tau}{\beta}(p\,\omega^1 + q\,\widetilde{\omega}^2 + (\beta\nu\tau - c)\omega^3).$$

Of course, the 1-forms on the right in (12) must be closed along Σ. Computing the exterior derivatives of these, modulo $\theta_0, \ldots, \theta_3$ and using the above values for $d\nu$, $d\beta$, $d\lambda$ and ω_1^2, gives algebraic conditions that p, q, s, t must satisfy. For example, we compute that

$$0 = d^2\nu \wedge \widetilde{\omega}^2 = \frac{4c\tau(2\tau^2 - 1)}{\beta(1+\tau^2)} p\,\omega^1 \wedge \omega^2 \wedge \omega^3$$

along Σ. The vanishing of this 3-form implies that at each point of Σ, either $p = 0$ or $\tau^2 = \frac{1}{2}$. So, if $p \neq 0$ at some point of Σ, then $\tau^2 = \frac{1}{2}$ on an open set around that point. On the other hand, the equations (12) imply that

$$d\tau = -\left(\nu(1+\tau^2) + \frac{c\tau(2-\tau^2)}{\beta(1+\tau^2)}\right)\omega^1 - \frac{\tau(p\tau - s)}{\beta}\widetilde{\omega}^2 + \frac{p\tau(1+\tau^2)}{\beta}\omega^3, \tag{13}$$

so that if τ is constant on an open set in Σ then $p = 0$ on that set. Thus, we conclude that p vanishes identically on Σ.

With this conclusion taken into account, we have

$$0 = d^2\nu = \frac{3c(1-\tau^2)}{\beta(1+\tau^2)^2} s\,\omega^1 \wedge \widetilde{\omega}^2 + \frac{3c}{\beta(1+\tau^2)}\big((\beta\nu - q)\tau^3 - c\tau^2 + \beta^2 - \beta\nu\tau\big)\,\omega^2 \wedge \omega^3$$

along Σ. From the first term, at each point either $s = 0$ or $\tau^2 = 1$, and (13) (with $p = 0$) lets us conclude that $s = 0$ whenever τ is locally constant. Hence, s vanishes identically on Σ, and from the second term

$$q = \frac{\beta\nu\tau^3 - c\tau^2 + \beta^2 - \beta\nu\tau}{\tau^3}.$$

With these values for p, q, s taken into account, we have

$$0 = d^2\tau = \frac{c\tau^2(2-\tau^2)}{\beta^2(1+\tau^2)} t\,\omega^1 \wedge \widetilde{\omega}^2$$

along Σ, which implies that at each point either $t = 0$ or $\tau^2 = 2$. If the latter happens on an open set, then (13) implies that $\nu = 0$ on that same open set; in that case, substituting $\nu = 0$ and $d\tau = 0$ into the system 2-forms gives

$$\Theta_3 + \tau\Theta_2 = -c(2\omega^2 + \tau\omega^3) \wedge \omega^1,$$

so that the vanishing of these 2-forms is incompatible with the independence condition. From this contradiction we conclude that $t = 0$ identically on Σ.

Using the values deduced for p, q, s, t above, we conclude that the following 1-forms vanish along Σ:

$$\begin{aligned}
\theta_4 &:= d\nu + \frac{3c\tau}{1+\tau^2}\omega^1, \\
\theta_5 &:= d\beta + \left(\beta^2 + \frac{\beta^2}{\tau^2} - \beta\nu\tau + c\right)\omega^1, \\
\theta_6 &:= d\lambda + \frac{c\tau^2 - \beta^2(1+\tau^2) + 2\beta\nu\tau^3 + \beta\nu\tau}{\tau^3}\omega^1, \\
\theta_7 &:= \frac{\beta}{\tau}\omega_1^2 + \frac{c\tau^2 - \beta^2 - \beta\nu\tau^3 + \beta\nu\tau}{\tau^3}\omega^2 + \frac{\beta^2 - \beta\nu\tau}{\tau^2}\omega^3.
\end{aligned}$$

Now we can establish the assertions in the theorem. Note first that $\widehat{f}^*\omega^1$ annihilates the distribution $\mathcal{H}$ on M, so to show that $\mathcal{H}$ is integrable we compute

$$\begin{aligned}
d\omega^1 &\equiv -\omega_1^2 \wedge \omega^2 - \omega_1^4 \wedge \omega^3 \quad \mod \theta_0 \\
&= \lambda\omega^2 \wedge \omega^3 - \lambda\omega^2 \wedge \omega^3 \quad \mod \theta_7 \\
&= 0.
\end{aligned}$$

Next, the vanishing of $\theta_4, \theta_5, \theta_6$ implies that ν, β, λ (and hence α) are constant along the leaves of $\mathcal{H}$. Since $Y = -e_1$, the vanishing of these 1-forms also gives us the

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Y-derivatives of these variables:

$$\begin{aligned} Y\nu &= \frac{3c\tau}{1+\tau^2}, \\ Y\beta &= \beta^2 + \frac{\beta^2}{\tau^2} - \beta\nu\tau + c = \beta^2 + (\lambda-\nu)^2 + \nu(\alpha-\nu) + c, \qquad (14) \\ Y\lambda = Y(\nu - \beta/\tau) &= \frac{c\tau^2 - \beta^2(1+\tau^2) + 2\beta\nu\tau^3 + \beta\nu\tau}{\tau^3}. \end{aligned}$$

Using $\alpha = \nu - \beta\tau$ we also get

$$Y\alpha = -\beta(\nu + \beta\tau + \beta/\tau) = -\beta(3\nu - \alpha - \lambda).$$

Thus, α, β, λ and ν satisfy the underdetermined system of differential equations (3). (We leave it to the interested reader to check that the right-hand side of $Y\lambda$ in (14) coincides with the third equation in (3), once the substitutions (7) are made.) □

REMARK 2. In the preceding proof, the possible values of the 1-forms $\omega_1^2, d\beta, d\nu, d\tau$ (as well as $\omega^4, \omega_1^4, \omega_2^4, \omega_3^4$) on an integral submanifold Σ satisfying the independence condition are completely determined at each point of $\mathcal{F} \times \mathcal{U}$. In other words, Σ must be tangent to the rank 3 distribution on $\mathcal{F} \times \mathcal{U}$ whose tangent planes are annihilated by $\theta_0, \ldots, \theta_7$ at each point. Of course, three-dimensional integral submanifolds of such a distribution only exist at points where the Frobenius condition is satisfied. This means we must check that the exterior derivatives of $\theta_0, \ldots, \theta_7$ are zero modulo these same 1-forms. Fortunately, this condition holds identically, and there exists a unique local integral submanifold Σ through each point of $\mathcal{F} \times \mathcal{U}$. Global existence will follow from Theorem 5.

6. 2-Hopf hypersurfaces. As remarked earlier, one can view the condition that $\mathcal{H}$ is of rank 2 as a weakening of the Hopf hypersurface condition. By itself, this condition is too weak: for example, one can show using exterior differential systems that three-dimensional hypersurfaces in $\mathcal{X}$ for which $\mathcal{H}$ has rank 2 are abundant, at least locally, since examples can be constructed using the Cartan–Kähler theory depending on a choice of two functions of two variables. Thus, we impose the additional condition that $\mathcal{H}$ is an integrable distribution. Such hypersurfaces are still quite flexible, and depend locally on a choice of five functions of one variable; for example, there exists such a hypersurface through any given curve Γ in $\mathcal{X}$, with the distribution $\mathcal{H}$ prescribed along Γ (provided that the prescribed 2-plane is neither tangent nor perpendicular to the curve at any point). Motivated by these considerations, we make the following definition:

DEFINITION 1. A hypersurface in $\mathbb{C}P^n$ or $\mathbb{C}H^n$ is said to be k-Hopf if $\mathcal{H}$ is integrable and of rank k.

When $k = 1$, the integrability condition is vacuous and this reduces to the usual notion of a Hopf hypersurface. Because of Theorem 4, we are interested in the 2-Hopf condition when $n = 2$.

PROPOSITION 6. *Let $M^3 \subset \mathcal{X}$ be a 2-Hopf hypersurface. Then the $\mathcal{H}$-leaves are flat within M.*

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Proof. Let (W, X, Y) be a standard (local) frame on M, and let $f : M \to \mathcal{F}$ be the associated unitary frame, as defined by (4). With respect to the (W, X, Y) basis, the shape operator has the form (1) with $\mu = 0$. So, $f^*\omega_1^4 = \nu f^*\omega^1$. By hypothesis, $f^*\omega^1$ is integrable (i.e., its exterior derivative is zero modulo itself), so the same is true of $f^*\omega_1^4$. Because ω^2 and ω^3 restrict to be an orthonormal coframe along an $\mathcal{H}$-leaf, we compute the Gauss curvature K using the equation

$$d\omega_3^2 \equiv K\omega^2 \wedge \omega^3 \qquad \text{mod } \omega^1.$$

(All forms here are understood to be pulled back via f.) Since $d\omega_3^2 = -d\omega_1^4 \equiv 0$ modulo ω^1, then $K = 0$. □

REMARK 3. It is easy to see that the 2-Hopf condition on M^3 requires that $\nabla_W X = \lambda Y$. In fact, for a hypersurface such that $\mathcal{H}$ has rank 2, this is equivalent to integrability of $\mathcal{H}$. To explain this, note that $\nabla_X W = \varphi AX = \lambda Y$. Then

$$\langle \nabla_W X, W\rangle = -\langle X, \nabla_W W\rangle = -\langle X, \varphi AW\rangle = 0.$$

Then $[X, W] = \nabla_X W - \nabla_W X = \lambda Y - \langle \nabla_W X, Y\rangle Y$. For integrability, $\langle [X, W], Y\rangle$ must be zero. In addition, integrability implies that $\mathcal{H}$-components of $\nabla_W X$ and $\nabla_W W$ (as well as those of $\nabla_X X$ and $\nabla_X W$) vanish, which explains why the curvature tensor of each leaf must vanish.

We now turn to the more specialized hypersurfaces of Theorem 5, which can be characterized as follows:

PROPOSITION 7. *Let M^3 be as in Proposition 6. If $\alpha = \langle AW, W\rangle$ is constant along the $\mathcal{H}$-leaves, then all the other components of the shape operator (with respect to a standard basis) are also constant along these leaves, and satisfy the differential equations* (3).

Proof. As in Section 5, we will set up a Pfaffian exterior differential system whose solutions are framed hypersurfaces of the type under consideration. Here, the system will encode the conditions that $\mathcal{H}$ is rank 2 and integrable. If $f : M \to \mathcal{F}$ is a unitary frame derived from a standard basis as in (4), then the pullbacks of the 1-forms on $\mathcal{F}$ satisfy (5) with $\mu = 0$. So, we define 1-forms

$$\begin{aligned}
\theta_0 &:= \omega^4 \\
\theta_1 &:= \omega_1^4 - \nu\omega^1 \\
\theta_2 &:= \omega_2^4 - \lambda\omega^2 - \beta\omega^3 \\
\theta_3 &:= \omega_3^4 - \beta\omega^2 - \alpha\omega^3,
\end{aligned}$$

with $\alpha, \beta, \lambda, \nu$ as extra variables.

Since $\mathcal{H}$ is annihilated by the pullback of ω^1, it is integrable if and only if $d\omega^1$ is a multiple of ω^1. We compute

$$d\omega^1 \equiv (\omega_1^2 - \lambda\omega^3) \wedge \omega^2 \qquad \text{mod } \theta_0, \theta_1, \theta_2, \theta_3. \tag{15}$$

Thus, $\mathcal{H}$ is integrable if and only if $\omega_1^2 - \lambda\omega^3$ equals some linear combination of ω^1 and ω^2. This is equivalent to the vanishing of

$$\theta_4 := \omega_1^2 - \rho\omega^1 - \delta\omega^2 - \lambda\omega^3$$

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for some functions δ and ρ along M. (As with the shape operator components, we will introduce δ and ρ as new variables.) We compute

$$d\theta_1 \equiv (\beta\delta - \beta^2 - \lambda^2 + \alpha\lambda + c)\omega^2 \wedge \omega^3 \qquad \text{mod } \theta_0, \dots, \theta_4, \omega^1,$$

and thus we must have

$$\delta = \frac{\beta^2 + \lambda^2 - \alpha\lambda - c}{\beta} \tag{16}$$

in order to satisfy the independence condition.

Accordingly, let $\mathcal{W} \subset \mathbb{R}^5$ be the open set with coordinates $\alpha, \beta, \lambda, \nu$ and ρ, with $\beta \neq 0$, and let $\mathcal{I}$ be the Pfaffian system on $\mathcal{F} \times \mathcal{W}$ generated by $\theta_0, \dots, \theta_4$ (with δ given by (16)). Differentiating these 1-forms modulo their span gives the generator 2-forms

$$\left.\begin{aligned} -d\theta_1 &\equiv \pi_1 \wedge \omega^1, \\ -d\theta_2 &\equiv \pi_2 \wedge \omega^2 + \pi_3 \wedge \omega^3, \\ -d\theta_3 &\equiv \pi_3 \wedge \omega^2 + \pi_4 \wedge \omega^3, \\ -d\theta_4 &\equiv \pi_5 \wedge \omega^1 \\ &+ \left(\frac{2\lambda - \alpha}{\beta}\pi_2 + \frac{\beta^2 - \lambda^2 + \alpha\lambda + c}{\beta^2}\pi_3 - \frac{\lambda}{\beta}\pi_4\right) \wedge \omega^2 + \pi_2 \wedge \omega^3. \end{aligned}\right\} \text{mod } \theta_0, \dots, \theta_4, \tag{17}$$

where

$$\begin{aligned} \pi_1 &= d\nu + \rho(\lambda - \nu)\omega^2 + \beta\rho\,\omega^3, \\ \pi_2 &= d\lambda + \left(\frac{(\lambda - \nu)(\lambda^2 - \alpha\lambda - c)}{\beta} + \beta(2\lambda + \nu)\right)\omega^1, \\ \pi_3 &= d\beta + (\beta^2 + \lambda^2 + \nu(\alpha - 2\lambda) + c)\omega^1, \\ \pi_4 &= d\alpha + \beta(\alpha + \lambda - 3\nu)\omega^1, \\ \pi_5 &= d\rho - \rho^2\omega^2. \end{aligned}$$

REMARK 4. Integral submanifolds of $\mathcal{I}$ are in 1-to-1 correspondence with 2-Hopf hypersurfaces in $\mathcal{X}$, so it is of interest to know how large the set of such surfaces is. The system $\mathcal{I}$ is not involutive. However, it is easy to see from the 2-form generators (17) that $\pi_5 \wedge \omega^1$ must vanish along any integral manifold satisfying the independence condition. When this 2-form is adjoined, the resulting ideal is involutive, and Cartan's test indicates that solutions depend on five functions of one variable. For example, given any curve γ in $\mathcal{X}$ and a 2-plane field E along γ which is transverse to the J-invariant subspace containing $T_p\gamma$ for every $p \in \gamma$, there is a 2-Hopf hypersurface containing γ with $\mathcal{H} = E$ along γ.

The set of 2-Hopf hypersurfaces M satisfying the additional hypothesis that α is constant along $\mathcal{H}$-leaves is considerably smaller. For, let Σ be the integral manifold of $\mathcal{I}$ corresponding to such a hypersurface. Because $\mathcal{H}$ is annihilated by the pullbacks to M of ω^2 and ω^3, then the constancy of α along the $\mathcal{H}$-leaves implies that π_4 must be a multiple of ω^1 along Σ. On the other hand, the vanishing of the 2-form $d\theta_3$ implies, by Cartan's lemma, that π_4 must be a linear combination of ω^2 and ω^3. Thus, $\pi_4 = 0$ and $\pi_3 \wedge \omega^2 = 0$ along Σ.

Substituting these into $d\theta_4$ and applying Cartan's lemma implies that π_5 and π_2 must be linear combinations of ω^1 and $\omega^3 + ((2\lambda - \alpha)/\beta)\,\omega^2$ along Σ. When we compare this with result of Cartan's lemma applied to the vanishing of $d\theta_2$, we see that there are scalars r, t such that

$$\pi_2 = r(\omega^3 + ((2\lambda - \alpha)/\beta)\omega^2), \quad \pi_3 = r\omega^2, \quad \pi_5 = t\omega^1,$$

along Σ.

Because π_4 vanishes along Σ, the same is true of its exterior derivative. We compute

$$\begin{aligned} d\pi_4 \wedge \omega^2 \equiv \beta(-3\pi_1 + \pi_2 + \pi_4 + 3\beta\rho\,\omega^3) \wedge \omega^1 \wedge \omega^2 \\ + (\alpha + \lambda - 3\nu)\pi_3 \wedge \omega^1 \wedge \omega^2 \quad \text{mod } \theta_0, \ldots, \theta_4. \end{aligned}$$

From the top line of (17), we see that $\pi_1 \wedge \omega^1 = 0$ along Σ; substituting this and the values for π_2, π_3, π_5 along Σ into the above 3-form, we conclude that $r = -3\beta\rho$. Hence, the 1-form

$$\pi_2 + 3\rho(\beta\omega^3 + (2\lambda - \alpha)\omega^2)$$

must vanish along Σ. Taking an exterior derivative of this form modulo $\theta_0, \ldots, \theta_4$, wedging with ω^1, and substituting the known values for the π's, we conclude that $\rho = 0$ identically.

Because π_2 and π_3 must vanish along Σ, and π_1 must be a multiple of ω^1, we respectively conclude that λ, β and ν are constant along the $\mathcal{H}$-leaves. Comparing the form of π_2 and π_3 with (3), we can verify that α, β, λ satisfy the correct differential equations. □

Proof of Theorem 5. Given a solution $\alpha(s)$, $\beta(s)$, $\lambda(s)$, $\nu(s)$ of the system (3) defined on interval I, we define a Pfaffian system $\mathcal{J}$ on $I \times \mathcal{F}$ which is generated (in part) by substituting each solution component for the corresponding variable in the generator 1-forms of the system $\mathcal{I}$ used in the proof of Proposition 7 (with the values for δ and ρ deduced there):

$$\begin{aligned} \vartheta_0 &:= \omega^4 \\ \vartheta_1 &:= \omega^4_1 - \nu(s)\omega^1 \\ \vartheta_2 &:= \omega^4_2 - \lambda(s)\omega^2 - \beta(s)\omega^3 \qquad \delta(s) := \frac{\beta(s)^2 + \lambda(s)^2 - \alpha(s)\lambda(s) - c}{\beta(s)}. \\ \vartheta_3 &:= \omega^4_3 - \beta(s)\omega^2 - \alpha(s)\omega^3, \\ \vartheta_4 &:= \omega^2_1 - \delta(s)\omega^2 - \lambda(s)\omega^3, \end{aligned}$$

To these, we add one more generator 1-form

$$\vartheta_5 := \omega^1 + ds,$$

the vanishing of which ensures that d/ds is the Y-derivative.

Our system $\mathcal{J}$ is a Frobenius system, so again we might invoke the Frobenius theorem (see, e.g., [**11**]) to obtain a unique connected maximal integral 3-fold of $\mathcal{J}$ through any point. Although the image of this under the fibration $\mathcal{F} \to M$ would be one of the desired hypersurfaces, we would not get any information about whether

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the hypersurface is complete or compact. In what follows, we give a more explicit construction.

Recall that $\mathcal{X} = Q/S^1$, where $Q \subset \mathbb{C}^3$ is the sphere $S^5(r)$ or anti-de Sitter space $H^5_1(r)$ defined by

$$\langle \mathbf{z}, \mathbf{z} \rangle = \epsilon r^2,$$

and $\langle\ ,\ \rangle$ is the Hermitian inner product on $\mathbb{C}^3$ with signature $+++$ (when we take $\epsilon = 1$ and get $\mathcal{X} = \mathbb{C}\mathrm{P}^2$), or $-++$ (when we take $\epsilon = -1$ and get $\mathcal{X} = \mathbb{C}\mathrm{H}^2$). The S^1 action multiplies the coordinates by a unit modulus scalar, so that the quotient map $\pi : Q \to \mathcal{X}$ is just the restriction of complex projectivization. The metric on the orthogonal complement to the fibres of π descends to the quotient $\mathcal{X}$, and has constant holomorphic sectional curvature $4c$ where $c = \epsilon/r^2$. (For more detail, see **[8]**, pp. 235–237.)

Recall from **[4]** that we may lift the S^1-quotient to a quotient map $\Pi : G \to \mathcal{F}$, where G is the group of matrices that are unitary with respect to the inner product (i.e., $G = U(3)$ for $\epsilon = 1$ and $G = U(1, 2)$ for $\epsilon = -1$). In more detail, we define the submersion $\Pi : G \to \mathcal{F}$ as follows: given an element $g \in G$ with columns (E_0, E_1, E_2), Π takes g to the unitary frame

$$e_1 = \pi_* E_1, \quad e_2 = \pi_* \mathrm{i} E_1, \quad e_3 = \pi_* E_2, \quad e_4 = \pi_* \mathrm{i} E_2$$

at basepoint $\pi(E_0)$. In fact, we can identify G itself with the unitary frame bundle of Q, using $\mathbf{z} = rE_0$ as the basepoint map and E_1, E_2 generating the frame as above. Then the S^1 action on G that is equivariant with respect to this basepoint map is simply left-multiplication by elements of the one-parameter subgroup $\exp(\mathrm{i}tI)$.

We will construct solutions to the system $\mathcal{J}$ by pulling the system back via Π to $I \times G$, constructing solutions there, and projecting back down. For this purpose we need to express the pullbacks of 1-forms ω^i and ω^i_j in terms of the Maurer–Cartan forms of G. The latter are complex-valued 1-forms ψ^a_b defined by

$$dE_a = E_b \psi^b_a, \quad 0 \le a, b \le 2,$$

where we regard the columns E_a as vector-valued functions on G. These columns satisfy

$$\langle E_0, E_0 \rangle = \epsilon, \quad \langle E_0, E_i \rangle = 0, \quad \langle E_i, E_j \rangle = \delta_{ij}, \quad 1 \le i, j \le 2,$$

so we have

$$\psi^b_a = \begin{cases} -\epsilon \overline{\psi^a_b} & \text{if exactly one of } a, b \text{ is equal to zero,} \\ -\overline{\psi^a_b} & \text{otherwise.} \end{cases}$$

Then, using formulas developed in **[4]**, we have

$$\begin{aligned} \Pi^*(\omega^1 + \mathrm{i}\omega^2) &= \psi^1_0, \quad \Pi^*(\mathrm{i}\omega^2_1) = \psi^1_1 - \psi^0_0, \\ \Pi^*(\omega^3 + \mathrm{i}\omega^4) &= \psi^2_0, \quad \Pi^*(\mathrm{i}\omega^4_3) = \psi^2_2 - \psi^0_0, \\ \Pi^*(\omega^3_1 + \mathrm{i}\omega^4_1) &= \psi^2_1. \end{aligned} \tag{18}$$

For the sake of brevity, we will write $\eta_1 := \mathrm{Re}\,\psi^1_0$, $\eta_2 := \mathrm{Im}\,\psi^1_0$ and $\eta_3 := \mathrm{Re}\,\psi^2_0$ in what follows.

Let $\widehat{\mathcal{J}}$ be the Frobenius system on $I \times G$ generated by the pullbacks of the 1-forms of $\mathcal{J}$. Using (18), we see that $\widehat{\mathcal{J}}$ is generated by

$$\begin{aligned}
\Pi^*\vartheta_0 &= \operatorname{Im}\psi_0^2, \\
\Pi^*\vartheta_1 &= \operatorname{Im}\psi_1^2 - \nu(s)\eta_1, \\
\Pi^*\vartheta_2 &= \operatorname{Re}\psi_1^2 - \lambda(s)\eta_2 - \beta(s)\eta_3, \\
\Pi^*\vartheta_3 &= \operatorname{Im}(\psi_2^2 - \psi_0^0) - \beta(s)\eta_2 - \alpha(s)\eta_3, \\
\Pi^*\vartheta_4 &= \operatorname{Im}(\psi_1^1 - \psi_0^0) - \delta(s)\eta_2 - \lambda(s)\eta_3, \\
\Pi^*\vartheta_5 &= \eta_1 + ds.
\end{aligned}$$

Since this system is of rank 6 on the 10-dimensional manifold $I \times G$, its maximal integral manifolds are four-dimensional; in particular, they are foliated by orbits of the S^1-action. We specify three-dimensional slices to this action by adding the extra 1-form $\operatorname{Im}(\psi_0^0 + \psi_1^1 + \psi_2^2)$, whose exterior derivative is zero modulo the 1-forms of $\widehat{\mathcal{J}}$. (In fact, this 1-form is closed on G, and its maximal integral manifolds are the cosets of the subgroup of special unitary matrices.) Let $\widehat{\mathcal{K}}$ denote the resulting rank 7 Frobenius system.

We will obtain integral 3-manifolds of $\widehat{\mathcal{K}}$ as follows. We will first construct a curve which is an integral of $\widehat{\mathcal{K}}$ but along which, in addition, $\eta_2 = \eta_3 = 0$. Then we will take a union of integral surfaces of $\widehat{\mathcal{K}}$ transverse to the curve. (The Frobenius condition guarantees that this union is an integral 3-fold of $\widehat{\mathcal{K}}$.) Fix a value $s_0 \in I$ and a point $u_0 \in G$; then the curve takes the form $(s, U(s))$ where $U(s)$ is a G-valued matrix satisfying the initial value problem

$$\frac{dU}{ds} = -U\begin{pmatrix} 0 & \epsilon & 0 \\ 1 & 0 & \mathrm{i}\nu(s) \\ 0 & \mathrm{i}\nu(s) & 0 \end{pmatrix}, \qquad U(s_0) = u_0.$$

This is a system of linear ODE, so its solution is smooth and defined for all $s \in I$.

Next, for each fixed $s \in I$ we construct an integral surface of $\widehat{\mathcal{K}}$ passing through $U(s)$. Along such a surface, η_2 and η_3 are closed, so there are local coordinates x, w such that $\eta_2 = dx$ and $\eta_3 = dw$. Then the surface must be given by a G-valued function V satisfying the simultaneous initial value problems

$$\frac{\partial V}{\partial x} = VA(s), \qquad \frac{\partial V}{\partial w} = VB(s), \qquad V(0,0) = U(s),$$

where

$$A(s) = \begin{pmatrix} -\frac{\mathrm{i}}{3}(\beta+\delta) & \epsilon\mathrm{i} & 0 \\ \mathrm{i} & \frac{\mathrm{i}}{3}(2\delta(s) - \beta(s)) & -\lambda(s) \\ 0 & \lambda(s) & \frac{\mathrm{i}}{3}(2\beta(s) - \delta(s)) \end{pmatrix},$$

$$B(s) = \begin{pmatrix} -\frac{\mathrm{i}}{3}(\alpha+\lambda) & 0 & -\epsilon \\ 0 & \frac{\mathrm{i}}{3}(2\lambda(s) - \alpha(s)) & -\beta(s) \\ 1 & \beta(s) & \frac{\mathrm{i}}{3}(2\alpha(s) - \lambda(s)) \end{pmatrix}.$$

It is easy to check that A and B satisfy the solvability condition $[A(s), B(s)] = 0$. Thus, V is defined for all $x, w \in \mathbb{R}^2$; in fact the solution is given by

$$V = U(s) \exp(xA(s) + wB(s)),$$

and thus for a fixed s, V sweeps out a left coset of a two-dimensional abelian subgroup of G. By varying s, we obtain a smooth matrix-valued function $V(s, x, w)$ such that $(s, V(s, x, w))$ is an integral 3-fold of $\widehat{\mathcal{K}}$. Then we set $\Phi(s, x, w) = b \circ \Pi(V(s, x, w))$, where $b : \mathcal{F} \to \mathcal{X}$ is the basepoint map, to obtain the desired immersion. Moreover, for fixed s each leaf $\Phi(s, x, w)$ is the orbit of a two-dimensional abelian subgroup of G, acting by isometries on $\mathcal{X}$. □

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