

A CHARACTERIZATION OF ODD ORDER EXTENSIONS OF THE FINITE SIMPLE GROUPS $\text{PSp}(4, q)$, $\text{G}_2(q)$, $\text{D}_4^2(q)$

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Let p denote an odd prime integer and let $q = p^f$ where f is a positive integer. Let \mathcal{H} denote the projective symplectic group $\text{PSp}(4, q)$, the Dickson group $\text{G}_2(q)$, or the Steinberg “trialeity twisted” group $\text{D}_4^2(q)$ over a field F_q of q elements. Then \mathcal{H} is simple and the Sylow 2-subgroups of \mathcal{H} have centers of order 2 so that involutions which centralize a Sylow 2-subgroup of \mathcal{H} form a single conjugacy class of \mathcal{H} .

Let σ denote an automorphism of F_q . Then σ induces, in the natural way, an automorphism of \mathcal{H} (cf. [2]) which fixes an involution in the center of a Sylow 2-subgroup of \mathcal{H} . In fact, $\langle \sigma \rangle$, the cyclic subgroup of $\text{Aut}(F_q)$ generated by σ , acts faithfully on \mathcal{H} and we may form the natural semi-direct product $\langle \sigma \rangle \mathcal{H}$. If σ is an odd ordered automorphism of F_q , then $\langle \sigma \rangle \mathcal{H}$ is an odd order extension of \mathcal{H} with trivial 2-core. In fact, any odd order extension of \mathcal{H} with trivial 2-core is of this form (cf. [2]).

Let j be an involution in the center of a Sylow 2-subgroup of \mathcal{H} such that j is fixed by σ . Then the centralizer $C(j)$ of j in $\langle \sigma \rangle \mathcal{H}$ is a semi-direct product $\langle \sigma \rangle C_{\mathcal{H}}(j)$ with trivial 2-core.

For each of the 3 possibilities for \mathcal{H} , $C_{\mathcal{H}}(j)$ has a subgroup \mathcal{Y} of index 2 containing subgroups L_1, L_2 such that $L_1 \cong \text{SL}(2, q_1)$, $L_2 \cong \text{SL}(2, q_2)$ (where q_1, q_2 are prime powers), $[L_1, L_2] = \{1\}$, $L_1 \cap L_2 = \langle j \rangle$ and $\mathcal{Y} = L_1 L_2$.

It has been shown in [4], [5], and [9] that if a finite group G contains an involution j such that $C_G(j)$ has a subgroup \mathcal{Y} of index 2 of the above type, then $G = C_G(j) O(G)$ ($O(G)$ denotes the 2-core of G ; i.e., the largest normal subgroup of odd order in G) or $G \cong \text{PSp}(4, q)$ or $G \cong \text{G}_2(q)$ or $G \cong \text{D}_4^2(q)$ for some odd prime power q . However, for example, in classifying finite groups by the structure of their Sylow 2-subgroups, one may arrive at a situation

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in which the centralizer $C_G(j)$ of an involution j in a group G has trivial 2-core and has a normal subgroup \mathcal{H} of odd index such that \mathcal{H} has a subgroup \mathcal{Y} of index 2 of the above type. This is, of course, the case with the groups $\langle \sigma \rangle \mathcal{H}$ above where \mathcal{H} is $PSp(4, q)$, $G_2(q)$ or $D_4^2(q)$, q is an odd prime power and σ is an odd ordered automorphism of F_q . To handle this situation we prove the following more general result:

THEOREM. *If G is a finite group with an involution j such that*

a) $O(C_G(j)) = \{1\}$ and

b) $C_G(j)$ contains a normal subgroup \mathcal{Y} of index 2δ with δ odd or a normal subgroup \mathcal{H} of index δ with δ odd such that \mathcal{H} contains a subgroup \mathcal{Y} of index 2 where in either case \mathcal{Y} contains subgroups L_1, L_2 such that $L_1 \cong SL(2, q_1)$, $L_2 \cong SL(2, q_2)$ (where q_1, q_2 are prime powers), $[L_1, L_2] = \{1\}$, $L_1 \cap L_2 = \langle j \rangle$ and $\mathcal{Y} = L_1 L_2$, then j is in the center of some Sylow 2-subgroup of G , q_1 and q_2 are both odd and one of the following holds:

(i) $G = C_G(j)O(G)$.

(ii) $q_1 = q_2$, L_1 and L_2 are not normal in $C_G(j)$ and $G \cong \langle \sigma \rangle PSp(4, q)$ where σ is an automorphism of order δ of a field of $q = q_1 = q_2$ elements.

(iii) $q_1 = q_2$, $L_1 \triangleleft C_G(j)$, $L_2 \triangleleft C_G(j)$ and $G \cong \langle \sigma \rangle G_2(q)$ where σ is an automorphism of order δ of a field of $q = q_1 = q_2$ elements.

(iv) one of the numbers q_1, q_2 is the cube of the other, $L_1 \triangleleft C_G(j)$, $L_2 \triangleleft C_G(j)$ and $G \cong \langle \sigma \rangle D_4^2(q)$ where σ is an automorphism of order δ of a field of $q = \min\{q_1, q_2\}$ elements.

Thus, for the rest of the paper we assume that the theorem is false. Hence we assume that G is a finite group with an involution j such that $C_G(j)$ satisfies the hypotheses of the theorem and that G does not satisfy the conclusion of the theorem and we shall arrive at a contradiction. By induction, we may assume that all groups of order less than $|G|$ satisfy the theorem and that δ is minimal among all groups of order $|G|$ contradicting the theorem.

If $\delta = 1$, then the theorem follows from [4], [5], and [9]. Thus we have $\delta > 1$.

Note that $j \in Z(L_1) \cap Z(L_2)$ so both q_1 and q_2 are odd prime powers.

Our notation is fairly standard. If X is a finite group, then $O(X)$ denotes the 2-core of X ; i.e., the largest odd order normal subgroup of G .

If $x^y = y^{-1}xy = z$, we write $y : x \rightarrow z$. If $y : x \rightarrow z$ and $y : z \rightarrow x$, then we write $y : x \leftrightarrow z$. If $y : x \rightarrow x^{-1}$, then we say that y inverts x . If p is a prime, then an S_p -subgroup of a group X is a Sylow p -subgroup of X .

Let $q_1 = p_1^{n_1}$, $q_2 = p_2^{n_2}$ where p_1, p_2 are odd prime integers and n_1, n_2 are positive integers. Then:

$$q_i - \varepsilon_i = 2^{\alpha_i} u_i, \quad q_i + \varepsilon_i = 2v_i$$

where $\varepsilon_i = \pm 1$, $\alpha_i \geq 2$ and u_i, v_i are odd for $i = 1, 2$.

Also let F_1, F_2 denote fields of q_1, q_2 elements respectively and view $SL(2, q_i)$ as the group of 2×2 matrices with coefficients in F_i of determinant 1 for $i = 1, 2$. As is well known, $\text{Aut}(F_i)$ acts faithfully in the natural way on $GL(2, q_i)$ and $SL(2, q_i)$ as follows:

if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, q_i)$ or $SL(2, q_i)$ where

$a, b, c, d \in F_i$ and if $\sigma \in \text{Aut}(F_i)$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\sigma = \begin{pmatrix} a^\sigma & b^\sigma \\ c^\sigma & d^\sigma \end{pmatrix} \text{ for } i = 1, 2.$$

Finally fix isomorphisms

$$\phi_i : SL(2, q_i) \rightarrow L_i \quad \text{for } i = 1, 2.$$

Clearly $\phi_i \left\{ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} = j$ for $i = 1, 2$.

The paper is organized as follows. In §1, we study $C_G(j)$ to obtain various properties of G and to factorize $C_G(j)$ into a semi-direct product $C_G(j) = \mathcal{K}A$ where $|A| = \delta$ and A acts like ‘‘field automorphisms’’ on \mathcal{K} . In §2, we examine $C_G(A)$ to show, among other facts, that $p_1 = p_2$. Then, in §3 and 4, we construct a semi-direct product subgroup $\tilde{G}A$ of G such that $C_G(j) \leq \tilde{G}A$ and such that $\tilde{G}A$ is strongly embedded in G . Using [1], it is then easy to obtain a contradiction to prove the theorem.

§1. In this section we examine the structure of $C_G(j)$ and prove, among other facts, that $\alpha_1 = \alpha_2$, $O(G) = \{1\}$ and that G has only one conjugacy class of involutions.

LEMMA 1.1. $\mathcal{Y} = L_1 L_2 \triangleleft C_G(j)$.

Proof. If $\mathcal{K} \triangleleft C_G(j)$ and $|\mathcal{K} : \mathcal{Y}| = 2$, then $O^2(\mathcal{K}) = \mathcal{Y}$ since L_1 and L_2 are generated by their elements of odd order and thus $\mathcal{Y} \triangleleft C_G(j)$.

The proof of [4, (2B)] yields:

LEMMA 1.2. *If $H \leq G$ and T is an S_2 -subgroup of $C_H(j)$ such that j is characteristic in T , then T is an S_2 -subgroup of H . In particular, an S_2 -subgroup of $C_G(j)$ is an S_2 -subgroup of G so that j is in the center of an S_2 -subgroup of G .*

Clearly:

LEMMA 1.3. *$Z(\mathcal{Y}) = \langle j \rangle$ and all involutions of $\mathcal{Y} - \langle j \rangle = L_1 L_2 - \langle j \rangle$ are conjugate in \mathcal{Y} .*

Since $|C_G(j)/\mathcal{Y}| = 2\delta$, there exists a unique subgroup \mathcal{L} of $C_G(j)$ such that $|C_G(j) : \mathcal{L}| = 2$ and $\mathcal{L} > \mathcal{Y}$.

LEMMA 1.4. *$\{L_1, L_2\}$ is invariant in $C_G(j)$ and $\mathcal{L} \leq N_G(L_1) = N_G(L_2)$.*

Proof. The first part follows easily from the Krull-Schmidt theorem applied to the group $\mathcal{Y}/\langle j \rangle \cong PSL(2, q_1) \times PSL(2, q_2)$. Since $|\mathcal{L} : \mathcal{Y}| = \delta$ and $|C_G(j) : \mathcal{L}| = 2$, the lemma follows.

LEMMA 1.5. *$C_{\mathcal{L}}(\mathcal{Y}) = \langle j \rangle$.*

Proof. Since $|\mathcal{L} : \mathcal{Y}| = \delta$ and $C_{\mathcal{L}}(\mathcal{Y}) \cap \mathcal{Y} = \langle j \rangle$, $|C_{\mathcal{L}}(\mathcal{Y})| = 2d$ where $d|\delta$. But $C_{\mathcal{L}}(\mathcal{Y}) \triangleleft C_G(j)$ and $0(C_G(j)) = \{1\}$ so that $d = 1$.

LEMMA 1.6. *There exists a subgroup A of \mathcal{L} of order δ and homomorphisms $\beta_i : A \rightarrow \text{Aut}(F_i)$ for $i = 1, 2$ such that: if $a \in A$ and $k_i \in SL(2, q_i)$, then*

$$(1.1) \quad \phi_i(k_i)^a = \phi_i(k_i^{\beta_i(a)}) \quad \text{for } i = 1, 2.$$

Moreover, $\mathcal{L} = \mathcal{Y}A$, $\mathcal{Y} \cap A = \{1\}$, $\text{Ker}(\beta_1) \cap \text{Ker}(\beta_2) = \{1\}$ and A is abelian on at most 2 generators.

Proof. Clearly $C_{\mathcal{Y}}(L_1) = L_2 \triangleleft C_{\mathcal{L}}(L_1) \triangleleft \mathcal{L}$ and $|\mathcal{L} : L_1 C_{\mathcal{L}}(L_1)|$ divides δ . It follows from the structure of $\text{Aut}(L_1)$ that there exists a subgroup A_1 of \mathcal{L} and a homomorphism $\beta_1 : A \rightarrow \text{Aut}(F_1)$ such that $A_1 \geq C_{\mathcal{L}}(L_1)$, $\mathcal{L} = L_1 A_1$, $L_1 \cap A_1 \leq C_{\mathcal{L}}(L_1)$ and such that $\phi_1(k_1)^a = \phi_1(k_1^{\beta_1(a)})$ for all $a \in A_1$ and $k_1 \in SL(2, q_1)$. Hence $L_1 \cap A_1 = C_{\mathcal{L}}(L_1) \cap L_1 = \langle j \rangle$, $\mathcal{Y} \cap A_1 = L_2 \triangleleft A_1$ and $|A_1/L_2|$ divides δ . Again it follows that there exists a subgroup A_2 of A_1 and a homomorphism $\beta_2 : A_2 \rightarrow \text{Aut}(F_2)$ such that $A_2 \geq C_{A_1}(L_2)$, $A_1 = L_2 A_2$, $L_2 \cap A_2 \leq C_{A_1}(L_2)$ and such that $\phi_2(k_2)^a = \phi_2(k_2^{\beta_2(a)})$ for all $a \in A_2$ and $k_2 \in SL(2, q_2)$. Hence $A_2 \cap \mathcal{Y} = A_2 \cap A_1 \cap \mathcal{Y} = A_2 \cap L_2 = C_{A_1}(L_2) \cap L_2 = \langle j \rangle$ so that $\langle j \rangle$ is an S_2 -subgroup of A_2 . Hence A_2 has a normal 2-complement A . Then $\mathcal{L} = \mathcal{Y}A$, $\mathcal{Y} \cap A = \{1\}$ and the restrictions of β_1, β_2 to A give the desired homomorphisms.

isms. Also $\text{Ker}(\beta_1) \cap \text{Ker}(\beta_2) = \{1\}$ follows from Lemma 1.5. Now it follows that conjugation induces a monomorphism of A into $O^{2'}((\text{Aut}(L_1)/\text{Inn}(L_1)) \times (\text{Aut}(L_2)/\text{Inn}(L_2)))$ so that A is abelian on at most 2 generators as required.

Let $|\text{Im}(\beta_i)| = \delta_i$; then $\delta_i | \delta$ and $\delta_i | n_i$ so that $n_i = \delta_i f_i$ where f_i is a positive integer for $i = 1, 2$. Hence if n_i is a 2-power, then $\delta_i = 1$, A is cyclic, A centralizes L_i and is faithful on L_j where $\{i, j\} = \{1, 2\}$. So that if both n_1, n_2 are 2-powers, then $\delta = 1$ which is not the case. Thus we have:

(1.2) n_1 and n_2 are not both 2-powers.

Let $\sigma_i \in \text{Aut}(F_i)$ be such that $\sigma_i : x \rightarrow x^{p_i^{f_i}}$ for all $x \in F_i$; then $\text{Im}(\beta_i) = \langle \sigma_i \rangle$ for $i = 1, 2$. Let F_i^* denote the fixed subfield of σ_i , and let $|F_i^*| = q_i^*$; then $q_i^* = p_i^{f_i}$ for $i = 1, 2$. Let γ_i be a primitive root of F_i for $i = 1, 2$. If $\varepsilon_i = 1$, then $-\gamma_i^{u_i} \in F_i^*$ and $-\gamma_i^{u_i}$ is a non-square in F_i and we can choose $\lambda_i, \mu_i \in F_i$ such that $\lambda_i + \mu_i \sqrt{-\gamma_i^{u_i}}$ is a generator for the group of elements in the field $F_i(\sqrt{-\gamma_i^{u_i}})$ of F_i - norm 1. In this case, set:

$$\rho_i = \begin{bmatrix} \gamma_i & 0 \\ 0 & \gamma_i^{-1} \end{bmatrix}, \quad \kappa_i = \begin{bmatrix} \lambda_i & \mu_i \\ -\gamma_i^{u_i} \mu_i & \lambda_i \end{bmatrix}, \quad b_i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

If $\varepsilon_i = -1$, choose $\lambda_i, \mu_i \in F_i$ such that $\lambda_i + \mu_i \sqrt{-1}$ is a generator for the group of elements in the field $F_i(\sqrt{-1})$ of F_i - norm 1 and choose η_i, ζ_i in F_i^* such that $\eta_i^2 + \zeta_i^2 = 1$. In this case, set:

$$\rho_i = \begin{bmatrix} \lambda_i & \mu_i \\ -\mu_i & \lambda_i \end{bmatrix}, \quad \kappa_i = \begin{bmatrix} \gamma_i & 0 \\ 0 & \gamma_i^{-1} \end{bmatrix}, \quad b_i = \begin{bmatrix} \eta_i & \zeta_i \\ \zeta_i & \eta_i \end{bmatrix}.$$

Then we always have:

$$(1.3) \quad b_i^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \rho_i^{b_i} = \rho_i^{-1}, \quad o(\rho_i) = q_i - \varepsilon_i, \quad o(\kappa_i) = q_i + \varepsilon_i.$$

Let:

$$(1.4) \quad a_i = \rho_i^{u_i}, \quad t_i = a_i^{2^{\alpha_i - 2}} \quad \text{and} \quad Q_i = \langle a_i, b_i \rangle.$$

Then:

$$(1.5) \quad \kappa_i^{t_i} = \kappa_i^{-1} \quad \text{and} \quad Q_i \text{ is an } S_2\text{-subgroup of } SL(2, q_i).$$

Moreover,

$$(1.6) \quad b_i^{\sigma_i} = b_i, \rho_i^{\sigma_i} = \rho_i^{\epsilon_i \nu^{\sigma_i}}, a_i^{\sigma_i} = a_i \quad \text{and} \quad \kappa_i^{\sigma_i} = \kappa_i^{-\epsilon_i \nu^{\sigma_i}}.$$

In order to simplify the notation, we shall identify elements of $SL(2, q_i)$ with their ϕ_i -images in L_i and we shall suppress the homomorphism β_i in the action of the elements of A on L_i for $i = 1, 2$. Thus we shall utilize Lemma 1.6 with this in mind.

Set:

$$(1.7) \quad x = t_1 t_2 \quad \text{and} \quad y = b_1 b_2.$$

Then x and y are involutions of $\mathcal{Z} - \langle j \rangle$.

A slight modification of the argument of [4, (2D)] yields:

LEMMA 1.7. *If $L_i \ntriangleleft C_G(j)$ for $i = 1$ or $i = 2$, then there exists an element $n \in C_G(j) - \mathcal{Z}$ such that $n^2 \in \langle j \rangle$ and $L_1^n = L_2$.*

Note that if there is an involution $n \in C_G(j) - \mathcal{Z}$ such that $L_1^n = L_2$ then [7] implies that G satisfies conclusion (ii) of our theorem. However, we also have:

LEMMA 1.8. *If $C_G(j)$ contains an element n such that $n^2 = j$ and $L_1^n = L_2$, then $G = C_G(j)O(G)$.*

Proof. Since $[A, Q_1] = 1$, by conjugating n by an element of L_2 , we may assume that $Q_2 = Q_1^n$. Now $C_{\mathcal{Z}}(Q_1 Q_2) = \langle j \rangle \times A$ so that n normalizes A . A slight modification of the proof of [4, (2E)] yields that $C_G(j) - \mathcal{Z}$ contains no involutions and then the remainder of the proof of [4, (2E)] applies directly to yield the lemma.

Thus we may henceforth assume:

$$(1.8) \quad L_i \triangleleft C_G(j) \quad \text{for} \quad i = 1 \quad \text{and} \quad i = 2.$$

LEMMA 1.9. *$C_G(j)$ contains a unique normal subgroup \mathcal{H} of index δ containing \mathcal{Z} such that $C_G(j) = \mathcal{H}A$ and $\mathcal{H} \cap A = \{1\}$.*

Proof. Conjugation induces a homomorphism $\theta : C_G(j) \rightarrow \text{Aut}(L_1) \times \text{Aut}(L_2)$. By Lemma 1.5, $\text{Ker}(\theta) \cap \mathcal{Z} = \langle j \rangle$. Thus an S_2 -subgroup of $\text{Ker}(\theta)$ has order 2 or 4. However, $\langle j \rangle \leq Z(\text{Ker}(\theta))$ so that $\text{Ker}(\theta)$ has a normal 2-complement which must be $\{1\}$ since $O(C_G(j)) = \{1\}$. Thus $|\text{Ker}(\theta)| = 2$ or 4. If $|\text{Ker}(\theta)| = 4$, then $\mathcal{Z} \text{Ker}(\theta)$ is a normal subgroup of $C_G(j)$ of index δ . If $\text{Ker}(\theta) = \langle j \rangle$, consider the natural homomorphism

$$\beta : \text{Aut}(L_1) \times (\text{Aut}(L_2) \rightarrow (\text{Aut}(L_1)/\text{Inn}(L_1)) \times (\text{Aut}(L_2)/\text{Inn}(L_2)).$$

Then $\beta \cdot \theta$ has kernel \mathcal{Z} so that $C_G(j)/\mathcal{Z}$ is abelian; hence, there always exists a normal subgroup \mathcal{K} of $C_G(j)$ of index δ such that $\mathcal{Z} \leq \mathcal{K}$ and the rest readily follows.

Observe that \mathcal{K} satisfies the hypotheses on the structure of the centralizer of an involution in [4] and that if $H \leq G$, then all S_2 -subgroups of $C_H(j)$ lie in $\mathcal{K} \cap H$.

We now can prove:

LEMMA 1.10.

- (i) G has only one conjugacy class of involutions.
- (ii) There exists an involution $n \in \mathcal{K} - \mathcal{Z}$ such that n acts by conjugation on L_i as:

$$(1.9) \quad \begin{bmatrix} 0 & 1 \\ -\gamma_i^{n_i} & 0 \end{bmatrix} \text{ if } \varepsilon_i = 1 \text{ and as } \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ if } \varepsilon_i = -1$$

for $i = 1, 2$.

- (iii) $\alpha_1 = \alpha_2$
- (iv) $\mathcal{K} = \mathcal{Z} \langle n \rangle$

Proof. The proof of [4, (2F)] yields (i), (iii), and the fact that there exists an involution $n \in \mathcal{K} - \mathcal{Z}$ which acts on $L_i = SL(2, q_i)$ as an involution in $PGL(2, q_i) - SL(2, q_i)$ for $i = 1, 2$. Then, by conjugating n by an element of $\mathcal{Z} = L_1 L_2$, we arrive at (ii).

Henceforth, let $\alpha = \alpha_1 = \alpha_2$.

We now have:

$$(1.10) \quad \rho_i^n = \rho_i^{-1}, \quad a_i^n = a_i^{-1}, \quad \kappa_i^n = \kappa_i \quad \text{for } i = 1, 2.$$

When $\varepsilon_i = 1$, we also have $b_i^n = b_i a_i$. However, if $\varepsilon_i = -1$, then $b_i^{-1} b_i^n \in \langle \rho_i \rangle$ and $b_i^{-1} b_i^n$ is fixed by σ_i . Then, as in [4, p. 146], there is an integer m such that ρ_i^m is fixed by σ_i and such that $(b_i \rho_i^m)^n = (b_i \rho_i^m) a_i$. Replacing b_i by $b_i \rho_i^m$ in this case, we have:

$$(1.11) \quad b_i^n = b_i a_i \quad \text{for } i = 1, 2.$$

COROLLARY 1.10.1. $O(G) = \{1\}$.

Proof. Clearly $\langle n, j \rangle$ is a 4-subgroup of G whose three involutions are conjugate in G . Since $C_G(t) \cap O(G) \leq O(C_G(t)) = \{1\}$ for any involution t of $\langle n, j \rangle$ and $O(G) = \langle C_G(t) \cap O(G) \mid t \in \langle n, j \rangle^* \rangle$, the corollary follows.

LEMMA 1.11.

- (i) n normalizes Q_i for $i = 1$ and 2 so that $Q_1Q_2\langle n \rangle$ is an S_2 -subgroup of G .
- (ii) $[n, A] = \{1\}$.
- (iii) $C_G(\mathcal{Z}) = \langle j \rangle$.

Proof. Clearly (i) holds. Since n normalizes $C_{\mathcal{Z}}(Q_1Q_2) = \langle j \rangle \times A$, n normalizes A so that $[n, A] \leq A \cap \mathcal{X} = \{1\}$ and (ii) holds. Finally, if θ denotes the homomorphism defined in the proof of Lemma 1.9, then $\langle j \rangle \leq \text{Ker}(\theta) = C_G(\mathcal{Z}) \leq C_{\mathcal{X}}(\mathcal{Z}) = \langle j \rangle$ and (iii) follows.

LEMMA 1.12. $O^{2'}(C_G(j)) = \mathcal{X}$.

Proof. If $q_i > 3$, then L_i is generated by 2-elements and if $q_i = 3$, then $L_i\langle n \rangle \cong GL(2, 3)$ is also generated by 2-elements. Hence $\mathcal{X} = L_1L_2\langle n \rangle \leq O^{2'}(C_G(j)) \leq \mathcal{X}$ and we are done.

COROLLARY 1.12.1. *If an involution t of $C_G(j)$ inverts an odd order subgroup Q of $C_G(j)$, then $Q \leq \mathcal{X}$.*

For future reference, we have:

$$(1.12) \quad C_{\mathcal{X}}(x, j) = \langle \rho_1, \rho_2, y, n \rangle \text{ and } C_G(x, j) = C_{\mathcal{X}}(x, j)A.$$

$$(1.13) \quad C_{\mathcal{X}}(n, j) = \langle \kappa_1, \kappa_2, x, n \rangle \text{ and } C_G(n, j) = C_{\mathcal{X}}(n, j)A.$$

Since n inverts ρ_1, ρ_2 and x inverts κ_1, κ_2 we have:

$$(1.14) \quad O^{2'}(C_G(x, j)) = C_{\mathcal{X}}(x, j).$$

$$(1.15) \quad O^{2'}(C_G(n, j)) = C_{\mathcal{X}}(n, j).$$

§ 2. In this section, we obtain information about $C_G(A)$ and show, among other facts, that $p_1 = p_2$

The proof of [4, (3A)] yields:

LEMMA 2.1. *If D is a 4-subgroup of G , then D is conjugate in G to $\langle x, j \rangle$ or $\langle n, j \rangle$ and $N_G(D)/C_G(D) \cong S_3$, the symmetric group on 3 symbols.*

From this lemma, we can demonstrate:

LEMMA 2.2. *If $E = \langle n, j \rangle$, then $E \leq C_G(A)$ and $(N_G(E) \cap C_G(A))/(C_G(E) \cap C_G(A)) \cong S_3$, the symmetric group on 3-symbols.*

Proof. Let $V_i = \langle \kappa_i^2 \rangle$ for $i = 1, 2$ and $V = V_1 \times V_2$; then $|V_i| = v_i$ for $i = 1, 2$ and $|V| = v_1 v_2$ is odd. By (1.15), $O^2(C_G(E)) = (V \times E) \langle x \rangle$ and the Frattini argument yields $N_G(E) = C_G(E) (N_G(E) \cap N_G(E \times \langle x \rangle))$. Hence there exists a 3-element $\zeta \in N_G(E) \cap N_G(E \times \langle x \rangle)$ such that $\zeta : j \rightarrow n \rightarrow nj \rightarrow j$. Since ζ normalizes $C_G(E \times \langle x \rangle) = E \times \langle x \rangle \times A$, it follows that $\zeta \in N_G(A)$. On the other hand, t_1 fixes j , $t_1 : n \leftrightarrow nj$ and $[t_1, A] = \{1\}$. Consequently t_1^3 fixes n , $t_1^3 : nj \leftrightarrow j$ and $[t_1^3, A] = \{1\}$. Thus $\langle t_1, t_1^3 \rangle \leq N_G(E) \cap C_G(A)$ and the lemma follows.

LEMMA 2.3.

(i) $C_{\mathcal{Z}}(A) = L_1^* L_2^*$ where $L_i^* = C_{L_i}(A) \cong SL(2, q_i^*)$ for $i = 1, 2$ and $[L_1^*, L_2^*] = \{1\}$ and $L_1^* \cap L_2^* = \langle j \rangle$.

(ii) $C_G(A, j) = C_{\mathcal{X}}(A) \times A$ where $C_{\mathcal{X}}(A) = C_{\mathcal{Z}}(A) \langle n \rangle$.

Proof. If $l_i \in L_i$ for $i = 1, 2$ and $(l_1 l_2)^a = l_1 l_2$ for $a \in A$, then $l_1^{-1} l_1^a \in L_1 \cap L_2 = \langle j \rangle$. Since a is of odd order, $l_1^a = l_1$, then $l_2^a = l_2$ and the lemma follows easily.

LEMMA 2.4.

(i) $p_1 = p_2$.

(ii) $C_G(A)/A$ is isomorphic to $G_2(q^*)$ or $D_4^2(q^*)$ where $q^* = \min\{q_1^*, q_2^*\}$.

Proof. Let $M = C_G(A)$; then $C_{M/A}(jA) = C_M(j)A/A = C_G(A, j)/A \cong C_{\mathcal{X}}(A)$. But $L_i^* = L_i \cap C_G(A) \triangleleft C_G(A, j)$ for $i = 1, 2$ and j is conjugate to n in M . Hence $M/A \neq C_{M/A}(jA)O(M/A)$ and [4], [5] yield the result.

§3. In this section, we lay the groundwork for the construction of a strongly embedded subgroup of G and we prove that an S_p -subgroup of $C_G(j)$ is not an S_p -subgroup of G .

Let $p = p_1 = p_2$. Since $\alpha_1 = \alpha_2 = \alpha$, we have $\varepsilon_1 = \varepsilon_2$, so let $\varepsilon = \varepsilon_1 = \varepsilon_2$.

We now introduce the following notation: let the images of

$$\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix}, \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

under the isomorphism $\phi_i : SL(2, q_i) \rightarrow L_i$ be denoted by $x_i(\alpha)$, $x_{-i}(\alpha)$, $h_i(\alpha)$, ω_i respectively and let X_i , X_{-i} , H_i be the subgroups of L_i generated by elements of the form $x_i(\alpha)$, $x_{-i}(\alpha)$, $h_i(\alpha)$ respectively for $i = 1, 2$. We have:

$$L_i = X_i H_i \cup X_i H_i \omega_i X_i \quad \text{for } i = 1, 2.$$

Set:

$$d_i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{if } \varepsilon = 1 \text{ and } d_i = 1 \text{ if } \varepsilon = -1 \text{ for } i = 1, 2 \text{ and let:}$$

$$(3.1) \quad h_0 = n d_1 d_2.$$

Then $\mathcal{X} = \langle L_1 L_2, h_0 \rangle$, $h_0 \in C_G(H_1 H_2)$ and $h_0^2 \in H_1 H_2$. Let:

$$(3.2) \quad H = \langle H_1 H_2, h_0 \rangle.$$

Then $|H| = (q_1 - 1)(q_2 - 1)$ and, as on [4, p. 163], H is abelian and is the direct product of two cyclic subgroups of orders $q_1 - 1$ and $q_2 - 1$.

Let D denote the 4-subgroup contained in H and denote the involutions in D by:

$$j = j_0, j_1, j_2.$$

Clearly $D = \langle x, j \rangle$ if $\varepsilon = 1$ and $D = \langle n, j \rangle$ if $\varepsilon = -1$. The involution $\omega_1 \omega_2$ inverts H and $|C_{\mathcal{X}}(D)| = 2(q_1 - 1)(q_2 - 1)$; hence:

$$(3.3) \quad C_G(D) = C_{\mathcal{X}}(D)A \text{ where } C_{\mathcal{X}}(D) = H \langle \omega_1 \omega_2 \rangle.$$

$$(3.4) \quad O^{2'}(C_G(D)) = C_{\mathcal{X}}(D).$$

The case $q_1 = q_2 = 3$ has been excluded so that $D \neq H$ and H is the unique subgroup of its isomorphism type in $O^{2'}(C_G(D))$ so that $N_G(D) = N_G(H)$.

By Lemma 2.1, there exists a 3-element $\eta \in N_G(H) = N_G(D)$ such that $\eta : j_0 \rightarrow j_1 \rightarrow j_2 \rightarrow j_0$. Since ω_1 fixes $j = j_0$ and $\omega_1 : j_1 \leftrightarrow j_2$, $N_G(D) = \langle C_G(D), \omega_1, \omega_1^2 \rangle$ by Lemma 2.1. Also ω_1 centralizes $C_G(D)/O^{2'}(C_G(D)) = O^{2'}(C_G(D))A/O^{2'}(C_G(D))$ so that ω_1^2 also centralizes $C_G(D)/O^{2'}(C_G(D))$ and hence $\langle O^{2'}(C_G(D)), \omega_1, \omega_1^2 \rangle \trianglelefteq N_G(D)$. Thus $O^{2'}(N_G(D)) = \langle O^{2'}(C_G(D)), \omega_1, \omega_1^2 \rangle$ and $C_G(D)/O^{2'}(C_G(D)) \leq Z(N_G(D)/O^{2'}(C_G(D)))$. By Lemma 2.1, there exists a unique normal subgroup M of $N_G(D)$ such that $|N_G(D) : M| = 2$, $C_G(D) \leq M$, $\omega_1 \notin M$ and $|M : C_G(D)| = 3$. Then $M/O^{2'}(C_G(D))$ is abelian of order 3δ and ω_1 acts non-trivially on this abelian group. It follows that M contains a normal subgroup M^* containing $O^{2'}(C_G(D))$ such that ω_1 normalizes M^* , ω_1 inverts $M^*/O^{2'}(C_G(D))$, $|M^*/O^{2'}(C_G(D))| = 3$, $M = M^*C_G(D)$ and $M^* \cap C_G(D) = O^{2'}(C_G(D))$. Moreover, $M^* \langle \omega_1 \rangle \leq O^{2'}(N_G(D))$, $M = M^*A$, $M^* \cap A = M^* \cap C_G(D) \cap A = C_{\mathcal{X}}(D) \cap A = \{1\}$ and $M^* \langle \omega_1 \rangle \triangleleft N_G(D) = M^* \langle \omega_1 \rangle A$. Thus $O^{2'}(N_G(D)) = M^* \langle \omega_1 \rangle$ and $O^{2'}(N_G(D)) \cap A = M^* \cap A = \{1\}$ so that $N_G(D) =$

$O^{2'}(N_G(D))A$, $O^{2'}(N_G(D)) \cap A = \{1\}$, $O^{2'}(N_G(D)) \cap C_G(D) = O^{2'}(C_G(D))$ and $O^{2'}(N_G(D))/O^{2'}(N_G(D)) \cap C_G(D) \cong S_3$. We may now assume that the 3-element η lies in $O^{2'}(N_G(D))$. Let T denote the S_2 -subgroup of H ; we have $[A, T \cap (H_1H_2)] = \{1\}$ and $[A, h_0] = \{1\}$ so that $[A, T] = \{1\}$ since $T = (T \cap (H_1H_2))h_0$. Consequently, $[A, T \langle \omega_1\omega_2 \rangle] = \{1\}$ where $T \langle \omega_1\omega_2 \rangle$ is an S_2 -subgroup of $O^{2'}(C_G(D))$. The Frattini argument implies that we may assume that $\eta \in N_G(T \langle \omega_1\omega_2 \rangle)$ also. Then [8, Kapitel I, Satz 18.6] implies that η centralizes an element $r \in T \langle \omega_1\omega_2 \rangle - T$ which must be an involution. Note that $C_G(D) = H \langle \omega_1\omega_2 \rangle \times A$ has a normal 2-complement $O(H)A$. However, $C_{O(H)A}(r) = A$ which implies that $\eta \in N_G(A)$ and hence $[\eta, A] \leq O^{2'}(N_G(D)) \cap A = \{1\}$. Since $C_A(H_1H_2) = \{1\}$ and $O^{2'}(N_G(D))/H$ is clearly dihedral of order 12, we have proved:

LEMMA 3.1. *Let D be the 4-subgroup of H . Then $D \neq H$, $N_G(H) = N_G(D) = O^{2'}(N_G(D))A$, $O^{2'}(N_G(D)) \cap A = \{1\}$, $H \leq O^{2'}(N_G(D))$ and $W = O^{2'}(N_G(D))/H$ is dihedral of order 12. Moreover, $C_G(H) = H$, there exists a 3-element $\eta \in O^{2'}(N_G(D)) \cap C_G(A)$ such that $O^{2'}(N_G(D)) = \langle H, \omega_1, \omega_2, \eta \rangle$ and $O^{2'}(N_G(D)) \cap C_G(j) = O^{2'}(C_G(D)) = H \langle \omega_1\omega_2 \rangle$.*

Combining Corollary 1.12.1 and the proof of [4, (4B)], we have:

LEMMA 3.2. *If P is a p -subgroup of $C_G(j)$ which is inverted by j_1 or j_2 , then $P \leq X_a X_b$ where $a \in \{1, -1\}$ and $b \in \{2, -2\}$.*

As in [4, (4C)], since H normalizes X_i and X_{-i} for $i = 1$ and 2 , we have:

LEMMA 3.3. *Let $\pi \in (X_a X_b)^\#$ with $a \in \{1, -1\}$ and $b \in \{2, -2\}$. Then:*

- (i) *If $\pi \in X_a \cup X_b$, then π has $\frac{1}{2}(q_1 - 1)(q_2 - 1)$ conjugates under H , all of which belong to $X_a X_b - X_a - X_b$.*
- (ii) *If $\pi \in X_a^\#$ or $X_b^\#$, then the conjugates of π under H consist of $X_a^\#$ or $X_b^\#$, respectively.*

Let $X = X_1 \times X_2$. Then:

$$(3.5) \quad N_{\mathcal{X}}(X) = XH \triangleleft N_G(X) \cap C_G(j) = N_{\mathcal{X}}(X)A.$$

$$(3.6) \quad C_G(X, j) = X \times \langle j \rangle$$

We can now demonstrate:

LEMMA 3.4. *An S_p -subgroup of $C_G(j)$ is not an S_p -subgroup of G .*

Proof. Assume that the lemma is false. If $p \nmid \delta = |A|$ then X is an S_p -subgroup of G which contradicts Lemma 2.4 (ii) since an S_p -subgroup of

$G_2(q^*)$ or of $D_4^2(q^*)$ is not elementary abelian; hence $p \nmid \delta$. Let A_p denote the S_p -subgroup of A . Then $T = XA_p$ is an S_p -subgroup of G . We claim that X is the unique elementary abelian subgroup of T of order q_1q_2 . For, let U be an elementary abelian subgroup of T such that $U \not\leq X$. Since A_p is abelian on at most 2-generators, $|U/U \cap X| \leq p^2$. If $|U/U \cap X| = p$, let $u = za \in U - U \cap X$ where $z \in X$ and $a \in A_p^*$. But $|C_X(u)| = |C_X(a)|$ and a acts as a non-trivial “field automorphism” of order p on X_1 or X_2 . It follows that $|C_X(a)| < p < q_1q_2$ and hence $|U| = p|U \cap X| \leq p|C_X(u)| = p|C_X(a)| < q_1q_2$. If $|U/U \cap X| = p^2$, then the “projection” of U into A_p is a subgroup Y of A_p which is elementary abelian of type (p, p) . Since Y is not cyclic, $[Y, X_i] \neq \{1\}$ for $i = 1$ and 2 and hence $|C_X(Y)| = q_1^{\frac{1}{p}} q_2^{\frac{1}{p}}$. But $X \cap U \leq C_X(U) = C_X(Y)$ so that $|U| = p^2|U \cap X| \leq p^2 q_1^{\frac{1}{p}} q_2^{\frac{1}{p}} < q_1q_2$. Thus X is the unique elementary abelian subgroup of T of order q_1q_2 and X is weakly closed in T with respect to G .

Now Lemma 1.2 and (3.6) imply that $\langle j \rangle$ is an S_2 -subgroup of $C_G(X)$. Hence $C_G(X)$ has a normal 2-complement and the Frattini argument gives:

$$(3.7) \quad N_G(X) = O(C_G(X)) (N_G(X) \cap C_G(j)) = O(C_G(X)) HA \text{ where } O(C_G(X)) \cap (HA) = O(C_G(X)) \cap C_G(X) \cap (HA) = O(C_G(X)) \cap \langle j \rangle = \{1\}.$$

Thus $(O(C_G(X))H) \cap A = (O(C_G(X)H) \cap (HA) \cap A) = H \cap A = \{1\}$, $O(C_G(X))H \triangleleft N_G(X)$ and $N_G(X)/O(C_G(X))H \cong A$. Since $p \nmid |A|$, $N_G(X)$ has a normal subgroup of index p .

Applying [6, Theorem 14.4.2] to the weakly closed subgroup X of T with respect to G , we conclude that G has a normal subgroup J of index p . Now Lemma 1.12 implies that $C_J(j) \geq \mathcal{K}$ so that $C_J(j) = \mathcal{K}(A \cap J) \trianglelefteq C_G(j)$ and hence $O(C_J(j)) = \{1\}$. Also Lemma 2.1 implies that there is a 2-element $w \in N_G(\langle n, j \rangle)$ such that $w : j \leftrightarrow n$. Since $w \in J$, $J \neq C_J(j)O(J)$; then, since $|J| < |G|$, we conclude that J satisfies conclusions (iii) or (iv) of the theorem. Hence $J' \triangleleft G$, $|G/J'|$ is odd and either $q_1 = q_2 = q$ and $J' \cong G_2(q)$ or one of the numbers q_1, q_2 is the cube of the other and $J' \cong D_4^2(q)$ where $q = \min \{q_1, q_2\}$. However, $O(G) = \{1\}$ by Corollary 1.10.1 so that G satisfies conclusions (iii) or (iv) of the theorem. But this is false and the lemma follows.

§ 4. In this section we construct a strongly embedded subgroup and use it to obtain a final contradiction.

Let $\{a, b\} = \{1, 2\}$ and let U be a non-trivial subgroup of X_b . Then:

$$(4.1) \quad C_{\mathcal{X}}(U) = X_b \times L_a \leq C_G(U, j).$$

Thus Q_b is an S_2 -subgroup of $C_G(U, j)$ and Lemma 1.2 and the proof of [4, (3C)] yields:

LEMMA 4.1. *If $\{a, b\} = \{1, 2\}$ and U is a non-trivial subgroup of X_b , then:*

- (i) $C_G(U) = O(C_G(U))C_G(U, j)$ and $X_b \leq O(C_G(U))$.
- (ii) $N_G(U) = O(C_G(U))(C_G(j) \cap N_G(U))$.
- (iii) Q_a is an S_2 -subgroup of $C_G(U)$.

For $U = X_b$, we have:

$$(4.2) \quad N_{\mathcal{X}}(X_b) = X_b L_a H \triangleleft N_G(X_b) \cap C_G(j) = N_{\mathcal{X}}(X_b)A.$$

In this case, we also have:

LEMMA 4.2.

- (i) $O(C_G(X_b)) \cap C_G(j) = X_b$.
- (ii) $O(C_G(X_b))/X_b$ is abelian and inverted by j .
- (iii) $O(C_G(X_b))$ is nilpotent.

Proof. Let $Y = O(C_G(X_b)) \cap C_G(j)$; clearly $X_b \leq Y$ and $[Y, L_a] \leq L_a \cap O(C_G(X_b)) = \{1\}$. Hence $Y \leq C_G(L_a)$ so that $Y \leq O(C_G(X_b)) \cap C_G(X_a) \cap C_G(L_a) \leq X_b$ which implies (i). Since (ii), (iii) follow immediately from (i), we are done.

LEMMA 4.3. *Let $\{a, b\} = \{1, 2\}$ and let M denote the S_p -subgroup of $O(C_G(X_b))$. Then $M \neq X_b$.*

Proof. Assume that $M = X_b$ and let Q be an S_p -subgroup of $C_G(j)$ such that $X = X_1 \times X_2 \leq Q$. By (3.6), $\langle j \rangle$ is an S_2 -subgroup of $C_G(Q, j)$ so that $\langle j \rangle$ is an S_2 -subgroup of $C_G(Q)$. Hence $C_G(Q) = L \langle j \rangle$ where $L = O(C_G(Q))$. Since $Z(Q) \leq L$, we may choose an S_p -subgroup Q^* of $C_L(j)$ such that $Q^* \geq Q$. Then Q^*Q is a p -subgroup of $C_G(j)$ and hence $Q^* = Z(Q)$. By Lemma 3.4, Q is not an S_p -subgroup of $N_G(Q)$. Since $N_G(Q) = L(N_G(Q) \cap C_G(j))$, we have:

$$\frac{|N_G(Q)|_p}{|Q|} = \frac{|L|_p}{|C_L(j)|_p} \geq p.$$

Now let Q_1 be an S_p -subgroup of L normalized by j . Since $Z(Q)$ is the S_p -subgroup of $C_L(j)$, $Q_1 > Z(Q) = L \cap Q$. Hence there exists an element

$z \in Q_1^*$ inverted by j . But $Q_1 \leq C_G(X_b) = O(C_G(X_b))C_G(X_b, j)$ so that $z \in O(C_G(X_b))$ and $z \notin O(C_G(X_b)) \cap C_G(j) = X_b$ which implies that $M \neq X_b$, proving the lemma.

Again let $\{a, b\} = \{1, 2\}$ and let M denote the S_p -subgroup of $O(C_G(X_b))$. Then D , the 4-subgroup contained in H , acts on M . Since $D^\# = \{j_0, j_1, j_2\}$, letting

$$(4.3) \quad M_i = M \cap C_G(j_i) \quad \text{for } i = 0, 1, 2,$$

we have:

$$(4.4) \quad H \text{ normalizes } M_i \text{ for } i = 0, 1, 2.$$

$$(4.5) \quad M_0 = X_b \text{ and } M = M_0M_1M_2.$$

$$(4.6) \quad M_1^{q_2} = M_2 \text{ and } M_2^{q_1} = M_1.$$

Since j inverts M_1 and M_2 , we have:

$$(4.7) \quad M_i \leq O^{2'}(C_G(j_i)) \text{ and } M_i \text{ is elementary abelian for } i = 1, 2.$$

$$(4.8) \quad |M_1| = |M_2| \leq q_1q_2 \text{ and } |M| = q_b|M_1|^2.$$

Hence $1 < |M/X_b| \leq q_1^2q_2^2$, M/X_b is an elementary abelian p -group which admits L_a and M/X_b has L_a -composition factors which are faithful irreducible L_a -modules over the field of p elements (since j inverts M/X_b).

Also if $\eta \in O^{2'}(N_G(D))$ is the 3-element of Lemma 3.1, then (4.7) and Lemma 3.2 yield:

$$(4.9) \quad M_1 \leq (X_aX_b)^\eta \text{ and } M_2 \leq (X_cX_d)^\eta \text{ where } a, c \in \{1, -1\} \text{ and } b, d \in \{2, -2\}.$$

Arrange notation so that $q_1 \geq q_2$ and set $a = 1$ and $b = 2$ in the above. Since $1 < |M/X_2| \leq q_1^2q_2^2 \leq q_1^4$, we have the following 3 possibilities by [4, (1E)]:

(I) $|M/X_2| = q_1^4$, (II) $|M/X_2| = q_1^{8/3}$, and (III) $|M/X_2| = q_1^2$.

If (I), then $q_1 = q_2$, $|M_1| = |M_2| = q_1q_2$, equality holds in (4.9), and $|M/X_2| = q_1^2q_2^2$. If (II), then $|M_1| = |M_2| = q_1^{4/3} > q_1 \geq q_2$, and, as on [4, p. 165], we conclude that $q_1 = q_2^3$, equality holds in (4.9) and that $|M/X_2| = q_1^2q_2^2$.

However, in order to treat possibility (III), we shall need a deeper analysis. To this end, let $P = MX_1$, let \tilde{M} denote the S_p -subgroup of $O(C_G(X_1))$, let $\tilde{P} = \tilde{M}X_2$, let $\tilde{M}_0 = \tilde{M} \cap C_G(j) = X_1$, and let $\tilde{M}_i = \tilde{M} \cap C_G(j_i)$ for $i = 1, 2$.

LEMMA 4.4. *If $|M/X_2| = q_1^2$, then $|\tilde{M}/X_1| = q_1^2q_2^2$.*

Proof. Assume $|M/X_2| = q_1^2$, then $|M_1| = |M_2| = q_1 > 3$ and it follows from the discussion in [4, p. 166] that:

- (i) $M_1 = X_a^\gamma$ or $q_1 = q_2$ and $M_1 = X_b^\gamma$ and
- (ii) $M_2 = X_c^{\gamma^2}$ or $q_1 = q_2$ and $M_2 = X_d^{\gamma^2}$. Taking into account (4.6), we have one of the following 4 cases:

- (1) $M = X_2 X_{-1}^\gamma X_1^{\gamma^2}$
- (2) $M = X_2 X_1^\gamma X_{-1}^{\gamma^2}$
- (3) $M = X_2 X_2^\gamma X_2^{\gamma^2}$ and $q_1 = q_2$
- (4) $M = X_2 X_{-2}^\gamma X_{-2}^{\gamma^2}$ and $q_1 = q_2$.

If we have case (1), set $\omega = \omega_1 \omega_2 \gamma \in N_G(H)$; then, as in [5, (6C)], $P \cap P^\omega = X_{-1}^\gamma X_2^{\gamma^2}$. The proof of [5, (6C)] yields that M is elementary abelian and $X_{-1}^\gamma \leq Z(P)$. Hence $P^{\omega_1 \gamma} = X_{-1} X_2 X_{-1}^{\gamma^2}$ centralizes X_1 ; since j inverts $X_{-1}^\gamma, X_2^\gamma$ and $X_{-1}^{\gamma^2}$, Lemma 4.1 implies that $P^{\omega_1 \gamma} \leq \tilde{M}$. Thus $(X_{-1} X_2)^\gamma \leq \tilde{M}$ and $|\tilde{M}_1|^2 = |M/X_1| \geq q_1^2 q_2^2$ so that $|\tilde{M}/X_1| = q_1^2 q_2^2$. A similar argument yields the result for case (2). If we have case (3), then γ normalizes M and $X_2 \leq Z(M)$ so that M is elementary abelian. Then, as in the proof of [5, (6C)], $X_2^{\gamma^i} \leq C_G(X_1)$ for $i = 1$ or 2 . Since j inverts $X_2^{\gamma^i}$, we have $X_2^{\gamma^i} \leq \tilde{M}$. Thus, if the conclusion of the lemma is false, then we would have case (1) or (2) for \tilde{M} which implies that $|M/X_2| = q_1^2 q_2^2$ and the lemma holds in case (3). Finally, assume case (4), then, as in the proof of [5, (6C)],

$$[X_{-2}^\gamma, X_{-2}^{\gamma^2}] = X_2, Z(M) = X_2 \text{ and } Z(P) = X_2.$$

Let A_p denote the S_p -subgroup of A (possibly $A_p = \{1\}$). Then $P^* = PA_p$ is an S_p -subgroup of $N_G(X_2)$ by Lemma 4.1. Let R be an S_p -subgroup of G containing P^* ; clearly $Z(R) \leq R \cap C_G(X_2) \cap C_G(X_1) = (P(A_p \cap C_G(X_2)) \cap C_G(X_1))$. However, $A_p \cap C_G(X_2)$ acts faithfully on $P/M \cong X_1$ so $Z(R) \leq P \cap Z(P) = Z(P) = X_2$. Now Lemma 4.1 implies that Q_1 is an S_2 -subgroup of $C_G(Z(R))$. However, if $|M/X_1| \neq q_1^2 q_2^2$, then we must have case (4) for \tilde{M} . Then there would exist an S_p -subgroup \tilde{R} of G such that $Z(\tilde{R}) \leq X_1$ and Q_2 is an S_2 -subgroup of $C_G(Z(\tilde{R}))$. It follows that Q_1 and Q_2 are conjugate in G which implies that Q_1 and Q_2 are conjugate to each other in $C_G(j)$ which is impossible so the lemma also holds in case (4).

We now have:

LEMMA 4.5. *There is a choice for α, β with $\{\alpha, \beta\} = \{1, 2\}$ such that $|O(C_G(X_\beta))/X_\beta|_p = q_1^2 q_2^2$.*

We now assume that α, β are chosen so as to satisfy this lemma.

The proof of [4, (4H)] now yields:

LEMMA 4.6. *Let M be the S_p -subgroup of $O(C_G(X_\beta))$ and let $P = X_\alpha M$. Then:*

- (i) M/X_β is elementary abelian of order $q_1^2 q_2^2$.
- (ii) With a suitable choice of notation, we have:

$$(a) \quad P = X_\alpha X_\beta (X_{-\alpha} X_\beta)^\gamma (X_\alpha X_\beta)^{\gamma^2}$$

(4.10) or

$$(b) \quad P = X_\alpha X_\beta (X_{-\alpha} X_{-\beta})^\gamma (X_\alpha X_{-\beta})^{\gamma^2}.$$

Let $\mathcal{N} = O^{2'}(N_G(H)) = O^{2'}(N_G(D)) = \langle H, \omega_1, \omega_2, \eta \rangle$ and let $W = \mathcal{N}/H$. As we have seen in Lemma 3.1, W is dihedral of order 12.

As in [4, §6] it follows that:

LEMMA 4.7. *With suitable notation, we may assume that (4.10) (b) holds.*

Since $H \leq N_G(P)$ and $H \cap P = \{1\}$, if we set

$$(4.1) \quad \mathfrak{B} = HP$$

then \mathfrak{B} is a subgroup of G of order $(q_1 - 1)(q_2 - 1)q_1^2 q_2^2$. Set

$$(4.12) \quad \tilde{G} = \mathfrak{B} \mathcal{N} \mathfrak{B}.$$

Then [4, (6D) and (6E)] yield:

LEMMA 4.8. \tilde{G} is a subgroup of G .

We can now show:

LEMMA 4.9. *Let $\omega : W \rightarrow \mathcal{N}$ be a transversal; then:*

- (i) \tilde{G} is the disjoint union of the 12 double cosets $\mathfrak{B}\omega(w)\mathfrak{B}$ for $w \in W$.
- (ii) $A \leq N_G(\tilde{G})$ and $\tilde{G} \cap A = \{1\}$.
- (iii) $C_{\tilde{G}}(j) = \mathfrak{B}$
- (iv) Either $q_1 = q_2$ and $\tilde{G} \cong G_2(q)$ where $q = q_1 = q_2$ or one of q_1, q_2 is the cube of the other and $\tilde{G} \cong D_4^2(q)$ where $q = \min \{q_1, q_2\}$.

Proof. Clearly \tilde{G} is the union of the double cosets $\mathfrak{B}\omega(w)\mathfrak{B}$ for $w \in W$ and $\mathfrak{B} \cap \mathcal{N} = H(P \cap \mathcal{N})$. Let $\pi \in P \cap \mathcal{N}$, then $[\pi, H] \leq H \cap P = \{1\}$ so $\pi \in C_G(H) \cap \mathcal{N} \cap P = H \cap P = \{1\}$. Now (i) follows. Since $A \leq N_G(P) \cap N_G(H)$, $A \leq N_G(\tilde{G})$ and $Y = \tilde{G}A$ is a subgroup of G . Clearly $N_Y(P) = N_{\tilde{G}}(P)A = \mathfrak{B}A$; let $a \in \tilde{G} \cap A$, then $a \in N_{\tilde{G}}(P) = \mathfrak{B}$ and $a \in PH \cap C_G(j) \cap A = (X_\alpha X_\beta H) \cap A \leq \mathcal{X} \cap A = \{1\}$, so (ii) holds. Now $L_\alpha = \langle X_\alpha, X_\alpha^{a^\alpha} \rangle \leq \tilde{G}$; similarly $L_\beta \leq \tilde{G}$, so that $\mathcal{X} = L_1 L_2 H \leq G$ and (iii) follows. But then (iv) follows from [4] and [5] and the fact that $\eta : j \rightarrow j_1$. Q.E.D.

Using the subgroup $\tilde{G}A$ we can arrive at a final contradiction. Since $\tilde{G}A$ satisfies conclusions (iii) or (iv) of the theorem, $G \neq \tilde{G}A$. Assume that G has a proper normal subgroup N ; then $|N|$ is even by Corollary 1.10.1. But $Q_1 Q_2 \langle n \rangle \leq \tilde{G}$ and $Q_1 Q_2 \langle n \rangle$ is an S_2 -subgroup of G , hence $N \cap \tilde{G} \neq \{1\}$. Since \tilde{G} is simple by Lemma 4.10 (iv), $\tilde{G} \leq N$ and $C_N(j) = \mathcal{X}(A \cap N) \leq C_G(j)$. Thus $O(C_N(j)) = \{1\}$ and the theorem holds for N by our choice of G . But then either $q_1 = q_2$ and $N' \cong G_2(q)$ where $q = q_1 = q_2$ or one of q_1, q_2 is the cube of the other and $N' \cong D_4^2(q)$ where $q = \min\{q_1, q_2\}$. Since $N' \triangleleft G$ and $O(G) = \{1\}$, the theorem holds for G , which is false. Thus G is simple and $\tilde{G}A$ cannot contain all elements of G of even order. However, $\tilde{G}A$ has only one conjugacy class of involutions and $C_G(j) \leq \tilde{G}A$. Thus, in the language of [1], $\tilde{G}A$ is strongly embedded in G . Then [1, Satz 4] implies that $G \cong PSL(2, q)$ or $Sz(q)$ or $PSU(3, q)$ where $q = 2^m \geq 4$. Hence an S_2 -subgroup of G has center of order $q \geq 4$ (cf. [1, §4]). However, we know that $Q_1 Q_2 \langle n \rangle$ is an S_2 -subgroup of G and $Z(Q_1 Q_2 \langle n \rangle) = \langle j \rangle$ which is a contradiction and the theorem follows.

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