ON THE REPRESENTATION OF STRICTLY CONTINUOUS LINEAR FUNCTIONALS

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1. Introduction

Let $X$ be a topological space, $E$ a real or complex topological vector space, and $C(X, E)$ the vector space of all bounded continuous $E$-valued functions on $X$; when $E$ is the real or complex field this space will be denoted by $C(X)$. The notion of the strict topology on $C(X, E)$ was first introduced by Buck (1) in 1958 in the case of $X$ locally compact and $E$ a locally convex space. In recent years a large number of papers have appeared in the literature concerned with extending the results contained in Buck's paper. In particular, a number of these have considered the problem of characterising the strictly continuous linear functionals on $C(X, E)$; see, for example, (2), (3), (4) and (5). In this paper we suppose that $X$ is a completely regular Hausdorff space and that $E$ is a Hausdorff topological vector space with a non-trivial dual $E'$. The main result established is Theorem 3.2, where we prove a representation theorem for the strictly continuous linear functionals on the subspace $C^*(X, E)$ which consists of those functions $f$ in $C(X, E)$ such that $f(X)$ is totally bounded.

Throughout, we use the notation and terminology introduced in (5).

2. Preliminaries

Let $\mathcal{B}$ be the $\sigma$-algebra of Borel subsets of $X$, and $M(X)$ the Banach space of all bounded regular Borel measures on $X$. The topology $\tau$ of $E$ may be determined by a family $\mathcal{C}$ of $\mathcal{F}$-semi-norms, $\{v_i : i \in I\}$ say (see (7), p. 2)), and without loss of generality we can assume that $\mathcal{C}$ is full in the sense that, if $v_{i_1}, \ldots, v_{i_m}$ is any finite collection of members of $\mathcal{C}$, then $\max_{1 \leq k \leq m} v_{i_k}$ is also in $\mathcal{C}$, and $\lambda v \in \mathcal{C}$ for all $\lambda > 0$ and $v \in \mathcal{C}$. For each $i \in I$, let $M_i(X, E')$ denote the set of all finitely additive $E'$-valued set functions $\mu$ on $\mathcal{B}$ which have the following properties:

(i) for each $a \neq 0$ in $E$, $\mu(a)(F) = \mu(F)(a)(F \in \mathcal{B})$ defines an element $\mu_a$ of $M(X)$;
(ii) there exists a constant $k$ such that $|\mu|_i(X) \leq k$, where, for each $F \in \mathcal{B}$, we define $|\mu|_i$ by

$$|\mu|_i(F) = \sup \left| \sum_{i} \mu_{a_i}(F_i) \right|,$$

the supremum being taken over all finite partitions $\{F_i\}$ of $F$ into members of $\mathcal{B}$ (henceforth referred to as a $\mathcal{B}$-partition) and all finite collections $\{a_i\}$ of points in $E$ such that $\nu_i(a_i) \leq 1$. 

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Let \( M(X, E') = \bigcup_{i \in I} M_i(X, E') \). We now suppose that \( m \in M_i(X, E') \), \( F \in \mathcal{B} \), and \( f \in C_{ub}(X, E) \). For each \( F \in \mathcal{B} \), let \( \mathcal{D}_F \) be the collection of all \( \alpha = \{F_1, \ldots, F_n; x_1, \ldots, x_n\} \), where \( \{F_j\} (j = 1, \ldots, n) \) is a \( \mathcal{B} \)-partition of \( F \) and \( x_j \in F_j \).

If \( \alpha_1, \alpha_2 \in \mathcal{D}_F \), define \( \alpha_1 \subseteq \alpha_2 \) if and only if each set which appears in \( \alpha_1 \) is contained in some set in \( \alpha_2 \). In this way \( \mathcal{D}_F \) becomes an indexing set. Let \( \omega_\alpha = \sum_{j=1}^{n} m(F_j)(f(x_j)) \). We then have the following

**Lemma 2.1.** \( \{\omega_\alpha\} (\alpha \in \mathcal{D}_F) \) is a Cauchy net.

**Proof.** Let \( \varepsilon > 0 \) (and without loss of generality suppose that \( \varepsilon < 1/4 \)). Then the set \( V = \{x \in E: \nu_i(x) \leq \varepsilon\} \) is a \( \tau \)-neighbourhood of 0 in \( E \). \( f(X) \) is totally bounded and so there exist points \( y_1, \ldots, y_n \) in \( X \) such that \( f(X) \subseteq \bigcup_{j=1}^{n} (f(y_j) + V) \). Let \( V_j = \{x \in X: f(x) - f(y_j) \in V\} \). Each \( V_j \) is closed, and so is in \( \mathcal{B} \). Let \( F_j = V_j \cap F \) (\( 1 \leq j \leq n \)) and define \( G_1 = F_1, G_j = F_j \setminus \bigcup_{k=1}^{j-1} F_k \) (\( 2 \leq j \leq n \)). By keeping those \( G_j \)'s which are non-empty we get a \( \mathcal{B} \)-partition, \( \{G_1, \ldots, G_n\} \) say, of \( F \). Choose \( x_1 \in G_i \) and let \( \alpha_0 = \{G_1, \ldots, G_n, x_1, \ldots, x_n\} \). Note that \( \nu_i(f(x) - f(y)) \leq 2\varepsilon \) if \( x, y \) are in the same \( G_j \).

Then for \( \alpha_1, \alpha_2 \supseteq \alpha_0 \), we have

\[
|\omega_{\alpha_1} - \omega_{\alpha_2}| \leq |\omega_{\alpha_1} - \omega_{\alpha_0}| + |\omega_{\alpha_0} - \omega_{\alpha_2}|
\]

Now

\[
|\omega_{\alpha_1} - \omega_{\alpha_2}| = \left| \sum_{k} m(F_k)f(y_k) - \sum_{j=1}^{n} m(G_j)f(x_j) \right| = \left| \sum_{j=1}^{n} \left( \sum_{F_k \subseteq G_j} m(F_k)f(y_k) - \sum_{F_k \subseteq G_j} m(F_k)f(x_j) \right) \right|
\]

\[
= \left| \sum_{j=1}^{n} \sum_{F_k \subseteq G_j} m(F_k)(f(y_k) - f(x_j)) \right|
\]

Note that

\[
\nu_i\left(\left[\frac{1}{2\varepsilon}\right]\left(f(y_k) - f(x_j)\right)\right) \leq \left[\frac{1}{2\varepsilon}\right] \nu_i(f(y_k) - f(x_j)) \leq \frac{1}{2\varepsilon} \nu_i(f(y_k) - f(x_j)) \leq 1,
\]

where \([t]\) denotes the integer part of \( t \). It follows that

\[
\left| \sum_{j=1}^{n} \sum_{F_k \subseteq G_j} m(F_k)(f(y_k) - f(x_j)) \right| \leq |m_i|(F),
\]

and so

\[
|\omega_{\alpha_1} - \omega_{\alpha_2}| \leq \frac{1}{2\varepsilon} \left| m_i \right|(F) < 4\varepsilon \left| m_i \right|(F)
\]

since \( 0 < \varepsilon < 1/4 \).
Similarly we can prove that $|\omega_{a_1} - \omega_{a_2}| < 4\varepsilon |m|_i (F)$. Thus $|\omega_{a_1} - \omega_{a_2}| < 8\varepsilon |m|_i (F)$, and since $\varepsilon$ is arbitrary the result follows.

In view of the above lemma, we can now make the following

**Definition 2.2.** Let $\mu \in M(X, E')$ and let $f \in C_{rb}(X, E)$. The integral of $f$ with respect to $\mu$ is defined by

$$\int_X d\mu f = \lim_\alpha w_{\alpha},$$

where the limit is taken over the indexing set $\mathcal{D}_X$.

Let $C(X) \otimes E$ denote the vector space spanned by the set of all functions of the form $\phi \otimes a$, where $\phi \in C(X)$, $a \in E$, and $(\phi \otimes a)(x) = \phi(x)a$ ($x \in X$). It is straightforward to show that, if $\phi \in C(X)$ and $a \in E$, then $\int_X d\mu (\phi \otimes a) = \int_X \phi d\mu_a$. Also it is easy to show that the equation

$$\Phi(f) = \int_X d\mu f \quad (f \in C_{rb}(X, E))$$

defines a linear functional $\Phi$ on $C_{rb}(X, E)$.

Every topological vector space has a base of closed, balanced, shrinkable neighbourhoods of $0$ (6). (A neighbourhood $W$ of 0 in a TVS is said to be shrinkable if $\lambda W \subseteq \text{int} \, W$ for $0 \leq \lambda \leq 1$.)

If $W$ is a base of closed, balanced, shrinkable $\tau$-neighbourhoods of 0 in $E$, then the Minkowski functional $\rho_W$ of each $W \in W$ is continuous (6, Theorem 5). We also note that, for each $W \in W$, $W = \{x \in E: \rho_W(x) \leq 1\}$, and that $\rho_W$ is positive homogeneous.

**Lemma 2.3.** Let $m \in M_i(X, E')$. Then

(a) $|m|_i \in M(X)$;

(b) there exists a $W_i \in W$ such that

$$\left| \int_X dmf \right| \leq \int_X (\rho_{W_i} \circ f) d|m|_i \leq \|f\|_i \|m|_i (X) \quad (f \in C_{rb}(X, E)),$$

where $\|f\|_i = \sup_{x \in X} \rho_{W_i}(f(x))$.

**Proof.** (a) It follows immediately from the definition that $|m|_i$ is a bounded non-negative-valued set function on $X$. We show that $|m|_i$ is countably additive, as follows.

It is straightforward to show that $|m|_i$ is finitely additive. Let $\{A_k\} (k = 1, 2, \ldots)$ be a sequence of disjoining sets in $\mathcal{B}$ and suppose that $\bigcup_{k=1}^\infty A_k = A$. For any positive integer $n$,

$$|m|_i (A) \geq |m|_i \left( \bigcup_{k=1}^n A_k \right) = \sum_{k=1}^n |m|_i (A_k),$$

and so

$$|m|_i (A) \geq \sum_{k=1}^\infty |m|_i (A_k). \quad (1)$$
Let $\epsilon > 0$. Then there exist a $\mathcal{B}$-partition $\{F_j\}$ $(1 \leq j \leq m)$ of $A$ and a collection of points $\{a_j\} (1 \leq j \leq m)$ with $\nu_i(a_j) \leq 1$ such that

$$|m_i(A)| \leq \left| \sum_{j=1}^{m} m_{a_j}(F_j) \right| + \epsilon.$$

Since each $m_{a_j}$ is countably additive and $\{F_j \cap A_k : k = 1, 2, \ldots\}$ is a partition of $F_j$, we have $m_{a_j}(F_j) = \sum_{k=1}^{\infty} m_{a_j}(F_j \cap A_k) (1 \leq j \leq m)$. Hence

$$|m_i(A)| \leq \left| \sum_{j=1}^{m} \sum_{k=1}^{\infty} m_{a_j}(F_j \cap A_k) \right| + \epsilon \leq \sum_{k=1}^{\infty} |m_i(A_k)| + \epsilon. \tag{2}$$

Since $\epsilon$ is arbitrary, it follows from (1) and (2) that $|m_i|$ is countably additive.

To complete the proof of (a) we show that $|m_i|$ is regular. Let $\epsilon > 0$ and $F \in \mathcal{B}$. There exist a $\mathcal{B}$-partition $\{F_j\}$ $(1 \leq j \leq m)$ of $F$ and a collection $\{a_j\} (1 \leq j \leq m)$ of points with $\nu_i(a_j) \leq 1$ such that

$$|m_i(F)| \leq \sum_{j=1}^{m} |m_{a_j}(F_j)| + \epsilon.$$

Since each $m_{a_j}$ is regular, there exist compact sets $K_j (j = 1, \ldots, m)$ such that $K_j \subseteq F_j$ and $|m_{a_j}(F_j)| < |m_{a_j}(K_j)| + \epsilon/2^i$. Let $K = \bigcup_{j=1}^{m} K_j$. Then $K \subseteq F$ and

$$|m_i(F)| \leq \sum_{j=1}^{m} |m_{a_j}(K_j)| + 2\epsilon.$$

Moreover, for each $j = 1, \ldots, m$, there exists a $\mathcal{B}$-partition of $K_j$, $\{G_{j,1}, \ldots, G_{j,q_j}\}$ say, such that

$$|m_{a_j}(K_j)| < \sum_{l=1}^{q_j} |m_{a_j}(G_{j,l})| + \epsilon/2^j. \tag{*}$$

If $m_{a_j}(G_{j,l}) \neq 0$, we can write $|m_{a_j}(G_{j,l})| = m(G_{j,l})(a_{j,l})$, where

$$a_{j,l} = \frac{m_{a_j}(G_{j,l})}{m_{a_j}(G_{j,l})} a_j,$$

and we note that $\nu_i(a_{j,l}) \leq 1$ for all $j$ and $l$.

If $m_{a_j}(G_{j,l}) = 0$ for some $j$ and $l$, then the contribution of such terms to the summation in (*) is zero, and so we define $a_{j,l} = 0$ for these terms.

Thus

$$|m_i(F)| < \sum_{j=1}^{m} \sum_{l=1}^{q_j} m(G_{j,l})(a_{j,l}) + 3\epsilon \leq |m_i|K + 3\epsilon.$$

Since $\epsilon$ is arbitrary it follows that $|m_i(F)| = \sup_{K \subseteq F} |m_i|(K)$, where $K$ is compact. Similarly we can prove that $|m_i(F)| = \inf_{G \subseteq F} |m_i|(G)$, where $G$ is open. Thus $|m_i|$ is regular, and so is an element of $M(X)$. 

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(b) Let $W_i$ be a closed, balanced, shrinkable $\tau$-neighbourhood of 0 in $E$ such that $\{x \in E: \nu_i(x) \leq 1\} \supseteq W_i = \{x \in E: \rho_{W_i}(x) \leq 1\}$. For any $\epsilon > 0$, there exist a $\mathcal{B}$-partition, $\{F_j: 1 \leq j \leq m\}$ say, of $X$, and points $x_j \in F_j$ such that

$$\left| \int_X dm f \right| \leq \left| \sum_{j=1}^m m(F_j)f(x_j) \right| + \epsilon$$

and

$$\left| \sum_{j=1}^m (\rho_{W_i \circ f})(x_j) \left| m_i \right| (F_j) \right| \leq \left| \int_X (\rho_{W_i \circ f})(x) \right| d \left| m_i \right| + \epsilon.$$

Let $H_1$ (resp. $H_2$) be the set of $j \in \{1, \ldots, m\}$ such that $\rho_{W_i}(f(x_j)) \neq 0(\rho_{W_i}(f(x_j)) = 0)$. We note that, if $j \in H_2$, then $\nu_i(t_0f(x_j)) \leq 1$ for all $t > 0$. Then

$$\left| \int_X dm f \right| \leq \sum_{j \in H_1} (\rho_{W_i \circ f})(x_j) \left| m_i \right| (F_j) \left( \frac{f(x_j)}{\rho_{W_i \circ f}(x_j)} \right)$$

$$+ \sum_{j \in H_2} \frac{\epsilon}{\left| m_i \right| (X)} \left| m_i \right| (X) \left( \frac{f(x_j)}{\rho_{W_i \circ f}(x_j)} \right) + \epsilon$$

$$\leq \sum_{j \in H_1} (\rho_{W_i \circ f})(x_j) \left| m_i \right| (F_j) \left( \frac{f(x_j)}{\rho_{W_i \circ f}(x_j)} \right) + 2\epsilon.$$

We note that, if $\left| m_i \right| (X) = 0$, then the inequality we are seeking to establish holds trivially.

It follows that

$$\left| \int_X dm f \right| \leq \sum_{j \in H_1} (\rho_{W_i \circ f})(x_j) \left| m_i \right| (F_j) + 2\epsilon$$

$$\leq \int_X (\rho_{W_i \circ f})(x) \ d \left| m_i \right| + 3\epsilon,$$

and so, since $\epsilon$ is arbitrary,

$$\left| \int_X dm f \right| \leq \int_X (\rho_{W_i \circ f}) \ d \left| m_i \right|.$$

The other inequality is straightforward to prove.

### 3. The representation theorem

**Definition 3.1.** The pair $(X, E)$ is said to have the $\beta$-density property if $C(X) \otimes E$ is $\beta$-dense in $C(X, E)$.

It has been proved in (5) that $C(X) \otimes E$ has the $\beta$-density property in each of the following cases:

(a) if $X$ is a completely regular Hausdorff space of finite covering dimension and $E$ is any topological vector space;

(b) if $X$ is any completely regular Hausdorff space and $E$ is a locally convex space.

In the sequel we shall assume that $X$ is a completely regular Hausdorff space and that $(X, E)$ has the $\beta$-density property.
Theorem 3.2. For each \( \mu \in M(X, E') \), the equation

\[
\Phi(f) = \int_X d\mu f \quad (f \in C_b(X, E))
\]

defines a \( \beta \)-continuous linear functional \( \Phi \) on \( C_b(X, E) \). Conversely, if \( \Phi \) is a \( \beta \)-continuous linear functional on \( C_b(X, E) \), then there exists a unique \( \mu \in M(X, E') \) such that \( \Phi \) is given by (3).

Proof. Let \( \mu \in M(X, E') \) and suppose that \( \Phi \) is the linear functional on \( C_b(X, E) \) defined by (3). Now \( \mu \in M_i(X, E') \) for some \( i \in I \), and so, by Lemma 2.3(a), \( |\mu|_i \in M(X) \). It follows from (3, Lemma 4.2) that the equation

\[
\Phi_i(\phi) = \int_X \phi d |\mu|_i \quad (\phi \in C(X))
\]

defines a \( \beta \)-continuous linear functional \( \Phi_i \) on \( C(X) \). Thus, using the notation of (5), there exists a function \( \psi \) in \( B_0(X) \), \( 0 \leq \psi \leq 1 \), such that \( |\Phi_i(\phi)| \leq 1 \) whenever \( \phi \in C(X) \) and \( \|\psi \phi\| \leq 1 \). Let \( W_i \) be a closed, balanced, shrinkable \( \tau \)-neighbourhood of 0 defined as in the proof of Lemma 2.3(b) and let \( f \in U(\psi, W_i) \). Then, since \( \|\psi(\rho_w \circ f)\| = \|\rho_w(\psi f)\| \leq 1 \), it follows from Lemma 2.3(b) that

\[
|\Phi(f)| \leq \int_X (\rho_w \circ f) d |\mu|_i \leq 1.
\]

Thus \( \Phi \) is \( \beta \)-continuous.

Conversely, let \( \Phi \) be a \( \beta \)-continuous linear functional on \( C_b(X, E) \). Then there exist a \( \nu_i \in C \) and a \( \psi \in B_0(X) \) such that \( |\Phi(f)| \leq 1 \) for all \( f \in U(\psi, V_i) \), where \( V_i = \{x \in E : \nu_i(x) \leq 1\} \). For each \( a \neq 0 \) in \( E \), let \( \Phi_a(\phi) = \Phi(\phi \otimes a)(\phi \in C(X)) \). It is straightforward to prove that \( \Phi_a \) is a \( \beta \)-continuous linear functional on \( C(X) \), and so, by (3, Lemma 4.5), there exists a unique \( \mu_a \in M(X) \) such that

\[
\Phi_a(\phi) = \int_X \phi d\mu_a \quad (\phi \in C(X)).
\]

For each \( F \in \mathfrak{B} \), the functional \( \mu(F) \), defined by

\[
(\mu(F))(a) = \mu_a(F) \quad (a \in E),
\]

is an element of \( E' \), as follows. It is straightforward to show that \( \mu(F) \) is linear. Since \( \Phi \) is \( \beta \)-continuous it is continuous with respect to the uniform topology on \( C(X, E) \) and so there exists a closed, balanced, shrinkable \( \tau \)-neighbourhood \( W \) of 0 in \( E \), such that \( |\Phi(\phi)| \leq 1 \) whenever \( \phi \in U(1, W) \). Consider \( h \in C(X) \), with \( 0 \leq h \leq 1 \). Then \( \rho_w(h(x)a) = h(x)\rho_w(a) \leq \rho_w(a) \) for all \( x \in X \), and so \( h \otimes a \in U(1, W) \) whenever \( \rho_w(a) \leq 1 \). Thus \( |\Phi_a(h)| = |\Phi(h \otimes a)| \leq 1 \) whenever \( \rho_w(a) \leq 1 \), which implies that \( |\Phi_a(h)| \leq \rho_w(a) \) for all \( a \in E \). If \( h \in C(X) \) and \( \|h\| \leq 1 \), then \( |\Phi_a(h)| \leq 4\rho_w(a) \). Thus \( \|\Phi_a\| \leq 4\rho_w(a) \), and so from the inequalities

\[
|\mu(F)(a)| = |\mu_a(F)| \leq \|\mu_a\| = \|\Phi_a\| \leq 4\rho_w(a),
\]

the continuity of \( \mu(F) \) follows.
Thus \( \mu : \mathcal{B} \to E' \), defined by

\[
(\mu(F))(x) = \mu_x(F) \quad (F \in \mathcal{B}, \ x \in E),
\]

is a finitely additive \( E' \)-valued set function on \( \mathcal{B} \) with property (i). Moreover \( |\mu|_t(X) \) is finite for some \( t \in I \), as we now show.

There exists an \( \mathcal{F} \)-semi-norm \( v_t \) in \( E' \) such that

\[
\{ x \in E : v_t(x) \leq 1 \} \subseteq W = \{ x \in E : \rho_v(x) \leq 1 \}.
\]

Let \( \{ F_j \} (1 \leq j \leq m) \) be a \( \mathcal{B} \)-partition of \( X \) and let \( \{ a_j \} \) be any collection of points in \( E \) such that \( v_t(a_j) \leq 1 \) (1 \( \leq j \leq m \)). We now proceed by using the same argument as the one given in (8, Lemma 4). Let \( \epsilon > 0 \). Each \( \mu_{a_j} \) is regular and so there exist compact sets \( K_j \subseteq F_j \) such that \( |\mu_{a_j}|(F_j \setminus K_j) < \epsilon/2m \), and open sets \( V_j \supseteq K_j \) such that \( |\mu_{a_j}|(V_j \setminus K_j) < \epsilon/2m \) for \( j = 1, \ldots, m \); since the \( K_j \)'s are disjoint compact sets and \( X \) is completely regular, the \( V_j \)'s may be chosen so that \( V_j \cap V_{j'} = \emptyset \) \( (j \neq j') \). Choose functions \( g_j \) \((1 \leq j \leq m)\) in \( C(X) \), \( 0 \leq g_j \leq 1 \), such that \( g_j(x) = 1 \) for \( x \in K_j \) and \( \text{supp } g_j \subseteq V_j \). Let

\[
h = \sum_{j=1}^{m} g_j \otimes a_j.
\]

Then \( h \in C(X, E) \) and \( v_t(h(x)) \leq 1 \) for all \( x \in X \), and so \( |\Phi(h)| \leq 1 \). By using the above inequalities as in the proof of (8, Lemma 4) we have that

\[
\left| \sum_{j=1}^{m} \mu(F_j)a_j \right| < \epsilon + |\Phi(h)| \leq \epsilon + 1.
\]

Since \( \epsilon \) is arbitrary, it follows that \( \mu \) satisfies condition (ii).

Let \( g \) be any function in \( C(X) \otimes E \). Then \( g = \sum_{j=1}^{P} \phi_j \otimes b_j \), where \( \phi_j \in C(X) \), and \( b_j \in E \), and so

\[
\Phi(g) = \sum_{j=1}^{P} \Phi(\phi_j \otimes b_j) = \sum_{j=1}^{P} \int_{X} \phi_j \, d\mu b_j = \sum_{j=1}^{P} \int_{X} d\mu(\phi_j \otimes b_j) = \int_{X} d\mu g.
\]

Since \( C(X) \otimes E \) is \( \beta \)-dense in \( C_{\text{b}}(X, E) \), it follows from the above that \( \Phi(f) = \int_{X} d\mu f \) for all \( f \in C_{\text{b}}(X, E) \).

Finally, \( \mu \) is unique, as we now show. Suppose that there is an \( m \) in \( M(X, E') \) such that \( \Phi(f) = \int_{X} d\mu f \) for all \( f \in C_{\text{b}}(X, E) \). In particular, for any \( \phi \in C(X) \) and \( x \in E \),

\[
\int_{X} d\mu(\phi \otimes x) = \int_{X} dm(\phi \otimes x).
\]

Hence \( \int_{X} d\mu_{\phi} = \int_{X} dm_{\phi} \) for all \( \phi \in C(X) \), and so, by (3, Lemma 4.5), \( \mu_{\phi} = m_{\phi} \). Thus, for any Borel set \( F \) and any \( x \in E \), \( \mu(F)(x) = \mu_x(F) = m_x(F) = m(F)(x) \). It follows that \( \mu(F) = m(F) \), and so \( \mu = m \), as required.

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