On γ-matrices and their application to the binomial series.

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The known methods of "summing" divergent series, e.g. the means of Cesàro, Riesz, Borel, Lindelöf, Mittag-Leffler are particular cases of the transformation of a *sequence* (formed from the partial sums) by a *T-matrix*. An equivalent method is that of the transformation of the *series* by a *γ-matrix*, the fundamental properties of which have been proved by Carmichael, Perron and Bosanquet. [1] The employment of γ-matrices has several advantages, namely:

(*a*) γ-matrices are defined by two conditions,[1] whereas T-matrices are defined by three; [2]

(*b*) γ-matrices are applied to the terms of the series, while the application of T-matrices requires the formation of the partial sums;

(*c*) γ-matrices, as proved by Dienes, are more general, since to every T-matrix corresponds an equivalent γ-matrix, while there are γ-matrices having no equivalent T-matrix.[3]

A disadvantage of γ-matrices is that the matrix product of two γ-matrices may not exist, or may not be a γ-matrix.[4]

The paper presented here outlines a possible pseudo-algebra of γ-matrices by introducing the λ-mean and the term-product. Further topics treated are: operations on γ-matrices yielding another γ-matrix; applications to the binomial series; the connection between semi-regularity and "right" value; the increase of the effective range by contracting the series; γ-matrices efficient at isolated points.

1. *Definition and formal properties of γ-matrices.*

An infinite matrix $G \equiv (g_{n,k})$ is a γ-matrix if it satisfies the following two conditions.[1]

(1.1) $$\sum_{k=1}^{\infty} \left| g_{n,k} - g_{n,k+1} \right| \leqq M \qquad \text{for every } n \geqq 1,$$

(1.2) $$g_{n,k} \to 1 \text{ as } n \to \infty \qquad \text{for every fixed } k.$$

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[1] P. Dienes, *The Taylor Series* (Oxford, 1931), 396-397. This book will be referred to as *T.S.*

[2] *T.S.* 389. [3] *T.S.* 399.

[4] P. Vermes, "Product of a T-matrix and a γ-matrix." *Journal London Math. Soc.* 21 (1946), 129-134 (129).

[1.I] *The elements of a γ-matrix are bounded.*

For by (1.1) and (1.2)

$$(1.3) \qquad |g_{n,k}| \leqq |g_{n,k}-g_{n,1}| + |g_{n,1}| \leqq |g_{n,1}| + M \leqq K.$$

[1.II] *If $G^{(i)}$ are γ-matrices $(i=0, 1, 2, ..p)$ and $l = \sum_{i=0}^{p} \lambda_i \neq 0$, then the matrix $H \equiv \frac{1}{l} \sum_{i=0}^{p} \lambda_i G^{(i)}$ is a γ-matrix.*

Proof: By hypothesis $\sum_{k=1}^{\infty} |g_{n,k}^{(i)} - g_{n,k+1}^{(i)}| \leqq M_i$; hence

$$\sum_{k=1}^{\infty} |h_{n,k} - h_{n,k+1}| \leqq \frac{1}{|l|} \sum_{i=0}^{p} |\lambda_i| \sum_{k=1}^{\infty} |g_{n,k}^{(i)} - g_{n,k+1}^{(i)}| \leqq \frac{1}{|l|} \sum_{i=0}^{p} |\lambda_i| M_i.$$

Thus H satisfies (1.1) and obviously also (1.2).

(1.4) *Definition.* We shall call the matrix H the *λ-mean* of the matrices $G^{(i)}$.

[1.III] *The λ-mean of an infinity of γ-matrices is a γ-matrix provided that*

(*a*) $|g_{n,k}^{(i)}| \leqq K$ *for every i, n, and k;*

$\sum_{k=1}^{\infty} |g_{n,k}^{(i)} - g_{n,k+1}^{(i)}| \leqq M$ *for every i and n;*

(*b*) $\sum_{i=0}^{\infty} |\lambda_i| = L$ *exists and is finite, and* $\sum_{i=0}^{\infty} \lambda_i = l \neq 0.$

Proof:

H exists since

$$|l| \cdot |h_{n,k}| \leqq \sum_{i=0}^{\infty} |\lambda_i| \cdot |g_{n,k}^{(i)}| \leqq K L.$$

Also

$$|l| \sum_{k=1}^{\infty} |h_{n,k} - h_{n,k+1}| \leqq \sum_{i=0}^{\infty} |\lambda_i| \sum_{k=0}^{\infty} |g_{n,k}^{(i)} - g_{n,k+1}^{(i)}| \leqq L M \text{ for every } n.$$

Again the series $\sum_{i=0}^{\infty} \lambda_i g_{n,k}^{(i)}$ converge uniformly for every n by (*a*) and (*b*); hence for every k

$$\lim_{n\to\infty} h_{n,k} = \frac{1}{l} \lim_{n\to\infty} \sum_{i=0}^{\infty} \lambda_i g_{n,k}^{(i)} = \frac{1}{l} \sum_{i=0}^{\infty} \lambda_i \lim_{n\to\infty} g_{n,k}^{(i)} = 1.$$

(1.5) *Definition.* The matrix $C \equiv (c_{n,k}) = (a_{n,k}\, b_{n,k})$ is the *term-product* of $A \equiv (a_{n,k})$ and $B \equiv (b_{n,k})$.

[1.IV] *The term product of γ-matrices is a γ-matrix.*

The proof follows from the identity

$$c_{n,k} - c_{n,k+1} = a_{n,k}\,(b_{n,k} - b_{n,k+1}) + b_{n,k+1}\,(a_{n,k} - a_{n,k+1}),$$

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whence by (1.1) and (1.3)

$$\sum_{k=1}^{\infty} | c_{n,k} - c_{n,k+1} | \leqq K_1 M_2 + K_2 M_1.$$

Also $c_{n,k} \to 1$ as $n \to \infty$ by (1.2).

[1.V] *If A is a γ-matrix, the matrix $(1/a_{n,k})$ is a γ-matrix if and only if* $|a_{n,k}| \geqq L > 0$.

The sufficiency of the condition follows from:

$$\sum_{k=1}^{\infty} \left| \frac{1}{a_{n,k}} - \frac{1}{a_{n,k+1}} \right| = \sum_{k=1}^{\infty} \left| \frac{a_{n,k} - a_{n,k+1}}{a_{n,k}\, a_{n,k+1}} \right| \leqq \frac{M}{L^2} \text{ and } \lim_{n\to\infty} \frac{1}{a_{n,k}} = 1.$$

The necessity follows from (1.3) for the matrix $(1/a_{n,k})$.[1]

NOTE: It appears from section 1 that a pseudo-algebra of γ-matrices could be formulated in which λ-mean and term-product would represent sum and product, and the γ-matrix $u_{n,k} = 1$ for every n and k would replace the unit matrix.

2. *Definitions of γ-sums, consistency, regularity, semi-regularity.*

Definition. The *generalized sum* of the series Σc_k by the matrix G is

$$s = \lim_{n\to\infty} \sum_{k=1}^{\infty} g_{n,k}\, c_k$$

provided that the infinite series on the right-hand side is convergent for every n and the limit of its sum, as $n \to \infty$, exists.

G sums every convergent series to its correct sum if and only if

(2.1) *G is a γ-matrix.*[2]

We say that *B is consistent with A* if[3]

$$\text{(2.2)} \qquad \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k}\, c_k \rightrightarrows \lim_{n\to\infty} \sum_{k=1}^{\infty} b_{n,k}\, c_k$$

where the symbol ($\rightrightarrows$) indicates that the existence of the left-hand side implies that of the right-hand side and the equality of the two limits.

If the existence of either side implies that of the other and the equality of the limits, we write

$$\text{(2.3)} \qquad \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k}\, c_k \rightleftarrows \lim_{n\to\infty} \sum_{k=1}^{\infty} b_{n,k}\, c_k,$$

and A and B are said to be *mutually consistent.*

The matrix G is *regular*[4] if

$$\text{(2.4)} \qquad \lim_{n\to\infty} \sum_{k=1}^{\infty} g_{n,k}\, c_k \rightleftarrows \lim_{n\to\infty} \sum_{k=1}^{\infty} g_{n,k+1}\, c_k.$$

G is *semi-regular*[5] if

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[1] This remark is due to Mr H. Kestelman.

[2] *T.S.* 393, 396-397. [3] *T.S.* 411-412. [4] *T.S.* 418. [5] *T.S.* 420.

(2.5) $$\lim_{n\to\infty}\sum_{k=1}^{\infty} g_{n,k}\, c_k \rightleftarrows \lim_{n\to\infty}\sum_{k=1}^{\infty} g_{n,k+1}\, c_k.$$

A matrix may not be regular or semi-regular *in general* as defined above, but may be regular or semi-regular *with respect to a particular class of series.*[1]

3. *Operations on γ-matrices.*

The suppression of a finite number of columns, or the addition of certain columns, yields another γ-matrix. These changes may affect the existence and value of the γ-sums.

(3.1) *Definition.* Removing the first p columns of a matrix A, we obtain the matrix $A^{(p)} \equiv (a_{n,k+p})$. It will be called the *p-th diminutive of A.*

[3.I] *The diminutive of a γ-matrix G is a γ-matrix. If G is regular or semi-regular, so is $G^{(p)}$.*

Obviously (1.1) and (1.2) are satisfied by $G^{(p)}$. Regularity or semi-regularity follows from the identity

$$\sum_{k=1}^{\infty} g_{n,k+p}\, c_k = \sum_{k=1+p}^{\infty} g_{n,k}\, c_{k-p}$$

and from (2.4) or (2.5).

(3.2) *Definition.* Adding p new columns $a_{n,i}$ $(i=0,-1,-2,\ldots)$ to the matrix A on the left, we obtain the matrix $A^{+p}\equiv(a_{n,k-p})$. It will be called the *p-th extension of A.*

(3.3) If in addition in the new columns $a_{n,i}\to 1$ as $n\to\infty$, we call A^{+p} a *proper extension* of A.

[3.II] *The proper extension of a γ-matrix G is a γ-matrix. If G is regular or semi-regular, so is G^{+p}.*

The matrix G^{+p} satisfies (1.1) and (1.2) since the elements in the new columns are bounded if they satisfy (3.3).

If G is regular, we have from (2.4) and (3.3) for the series $0+c_1+c_2+\ldots$

$$\lim_{n\to\infty}\sum_{k=0}^{\infty} g_{n,k}\, c_k \rightleftarrows \lim_{n\to\infty}\sum_{k=0}^{\infty} g_{n,k+1}\, c_k$$

Hence G^{+1} is regular.

By repeating the argument, we prove that G^{+p} is regular. The case when G is semi-regular can be proved similarly.

(3.4) *Definition.* The matrix $A^{p\times}$, obtained from the matrix A by repeating each column p times, will be called a *p-fold stretched matrix.*

[1] See for example [3.IV] of this paper.

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[3.III] *A stretched γ-matrix is a γ-matrix. Stretching may destroy regularity or semi-regularity.*

If A satisfies (1.1) and (1.2), $A^{p\times}$ obviously does. As an example of regularity being destroyed, we consider Borel's γ-matrix [1]

$$(3.5)\qquad g_{n,k}=\frac{1}{k!}\int_0^n e^{-t}t^k\,dt=1-e^{-n}\left(1+n+\frac{n^2}{2!}+\ldots+\frac{n^k}{k!}\right)\ (k,\,n\geqq 0)$$

which sums the series $1-1+1-\ldots\ldots$ to the value $\frac{1}{2}$.

Here applying the matrix $G^{2\times}$ we obtain

$$\sigma_n=\sum_k g^{2\times}_{n,k}(-1)^k=g_{n,0}-g_{n,0}+g_{n,1}-g_{n,1}+\ldots=0,$$

and

$$\sigma_n^1=\Sigma\, g^{2\times}_{n,k+1}(-1)^k=g_{n,0}-g_{n,1}+g_{n,1}-g_{n,2}+\ldots=g_{n,0}.$$

Thus $\sigma_n\to 0$, $\sigma_n^1\to 1$, showing that $G^{2\times}$ is not semi-regular, while G is semi-regular.[2] The example also shows that $G^{2\times}$ is not consistent with G.

A related problem of contracting the series will be discussed later in [5.VI].

[3.IV] *The λ-mean H, formed from a γ-matrix G, its first p proper extensions and its first q diminutives is a γ-matrix. If G is regular (semi-regular), so is H with respect to the class of series Σc_k satisfying*

$$(3.6)\qquad g_{n,k}\,c_{k-i}\to 0\quad as\quad k\to\infty\quad for\quad i=-p,\,-p+1,\ldots 0,\,1,\,2,\ldots\,q.$$

Proof. It follows from [1.II] that H is a γ-matrix.

Also since (3.6) holds we can rewrite the sum

$$(3.7)\qquad \sum_{k=1}^{\infty}h_{n,k}\,c_k\equiv\frac{1}{l}\sum_{k=1}^{\infty}\left(\sum_{i=-p}^{q}\lambda_i\,g_{n,\,k+i}\right)c_k\;\overset{\to}{\leftarrow}\;\frac{1}{l}\sum_{k=1}^{\infty}g_{n,k}\left(\sum_{i=-p}^{q}\lambda_i\,c_{k-i}\right)$$

where $c_0,\,c_{-1},\,c_{-2},\ldots$ are a finite number of zero terms. Thus if H sums the series $\Sigma\, c_k$, and G is regular or semi-regular, (3.7) establishes the same property for H.

NOTE: Condition (3.6) is satisfied in particular when

(3.8) G is row-finite,

(3.9) G is regular and sums the series $\Sigma\, c_k$.

4. *The γ-sum of the series $\Sigma\, c_k$.*

In this section we give a few results concerning generalized sums by λ-means; and a theorem on inefficiency.

[4.I] *If the matrices $G^{(i)}$ $(i=1,\,2,\ldots.p)$ sum the series $\Sigma\, c_k$ to $s^{(i)}$*

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[1] *T.S.* 401. [2] *T.S.* 419-420.

respectively, then their λ-mean H sums the series to the λ-mean of the $s^{(i)}$.
This follows from

$$\frac{1}{l}\sum_{i=1}^{p}\lambda_i s^{(i)} = \frac{1}{l}\sum_{i=1}^{p}\lambda_i \lim_{n\to\infty}\sum_{k=1}^{\infty} g_{n,k}^{(i)} c_k \rightrightarrows \lim_{n\to\infty}\sum_{k=1}^{\infty}\left(\frac{1}{l}\sum_{i=1}^{p}\lambda_i g_{n,k}^{(i)}\right) c_k.$$

[4.II] *The λ-mean formed from a semi-regular γ-matrix G, and a finite number of its diminutives, is consistent with G.*

[4.III] *The λ-mean formed from a regular γ-matrix G, and a finite number of its diminutives and proper extensions, is consistent with G.*

Both results follow from [4.I] since $s^{(i)} = s$ for every i.

NOTE: In the last two theorems H was proved to be consistent with G. But G need not be consistent with H. An example will be given in (5.13).

[4.IV] *A semi-regular γ-matrix is inefficient for the series* Σc_k *if* c_k *tends to a finite non-zero limit as* $k \to \infty$.

Proof: We assume that G sums the series, and we have, since G is semi-regular,

$$(4.0) \qquad \lim_{n\to\infty}\sum_{k=1}^{\infty} g_{n,k}\, c_k \rightrightarrows \lim_{n\to\infty}\sum_{k=1}^{\infty} g_{n,k+1}\, c_k \quad = s,$$

or rewriting the right-hand side

$$(4.1) \qquad \lim_{n\to\infty}\sum_{k=2}^{\infty} g_{n,k}\, c_{k-1} = s.$$

Subtraction of (4.1) from (4 0) gives

$$(4.2) \qquad \lim_{n\to\infty}\left\{ g_{n,1}\, c_1 + \sum_{k=2}^{\infty} g_{n,k}\,(c_k - c_{k-1})\right\} = 0.$$

But by hypothesis $c_1 + \sum_{k=2}^{\infty}(c_k - c_{k-1}) = \lim_{k\to\infty} c_k = l \neq 0$, and hence by (2.1) its generalized sum by the γ-matrix G exists and is different from zero. This is contradicted by (4.2), showing that the original assumption, that G is efficient, is not true.

5. *The γ-sum of the binomial series.*

Dienes proved[1] that if the *regular* γ-matrix G sums the series Σz^k, then this sum is the "right" value $(1-z)^{-1}$. We apply his method to the binomial series

$$(5.1) \qquad \sum_{k=0}^{\infty}\binom{p+k-1}{k} z^k \quad (p \text{ any real number}),$$

which is the Taylor expansion about the origin of the function

$$(5.2) \qquad f(z) \equiv (1-z)^{-p}.$$

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[1] *T.S.* 418.

We denote by $S_p(z)$ the γ-sum of the series (5.1) by the *semi-regular* G, *i.e.*

(5.3) $$S_p(z) \equiv \lim_{n\to\infty} \sum_{k=0}^{\infty} g_{n,k} \binom{p+k-1}{k} z^k \qquad (p \text{ real}).$$

[5.I] *If $S_p(z)$ exists for $z=z_0$, then $S_{p-1}(z_0)$ exists and*
$$S_{p-1}(z_0) = (1-z_0)\, S_p(z_0).$$

Proof: The theorem is trivial for $p = 0, -1, -2, \ldots,$ when the series is finite. Assuming $p \neq 0, -1, -2, \ldots,$ by hypothesis (5.3) holds for z_0, and since G is semi-regular, (5.3) implies

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} g_{n,k+1} \binom{p+k-1}{k} z_0^k = S_p(z_0),$$

which multiplied by z_0 can be rewritten as

(5 4) $$\lim_{n\to\infty} \sum_{k=1}^{\infty} g_{n,k} \binom{p+k-2}{k-1} z_0^k = z_0\, S_p(z_0).$$

Thus substituting z_0 into (5.3) and subtracting (5.4) we obtain

(5.5) if $p \neq 1$, $$\lim_{n\to\infty} \sum_{k=0}^{\infty} g_{n,k} \binom{p+k-2}{k} z_0^k = (1-z_0)\, S_p(z_0),$$

(5.6) if $p=1$, $$1 = (1-z_0)\, S_1.$$

Thus the theorem is proved for every real p.

[5.II] *If $S_p(z)$ exists for $z=z_0$ and p is a positive integer, then*
$S_p(z_0) = (1-z_0)^{-p}$, *the "right" value.*

The proof follows from [5.I] by induction.

NOTE: Dienes' theorem is a special case of this theorem for $p=1$, and has now been proved on the weaker supposition that G is semi-regular.

[5.III] *If the γ-matrix G sums the series Σz_0^k to its "right" value $(1-z_0)^{-1}$, then G is regular with respect to this series.*

Proof: By hypothesis

(5.7) $$\lim_{n\to\infty} \sum_{k=0}^{\infty} g_{n,k} z_0^k = \frac{1}{1-z_0}.$$

Hence

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} g_{n,k} z_0^k = \frac{1}{1-z_0} - 1 = \frac{z_0}{1-z_0}$$

since $g_{n,0} \to 1$, so that

(5.8) $$\lim_{n\to\infty} \sum_{k=0}^{\infty} g_{n,k+1} z_0^k = \frac{1}{1-z_0}.$$

Thus (5.7) implies (5.8), and reversing the order of argument we see that (5.8) implies (5.7). This proves the theorem.

[5.IV] *If the γ-matrix G sums the series* $\Sigma \binom{p+k-1}{k} z_0^{\,k}$ *and* $\Sigma \binom{p+k-2}{k} z_0^{\,k}$ *(p real) to their "right" values* $(1-z_0)^{-p}$ *and* $(1-z_0)^{-p+1}$ *respectively, then G is semi-regular with respect to the first series.*

Proof: By hypothesis

$$(5.9)\quad \lim_{n\to\infty} \sum_{k=0}^{\infty} g_{n,k} \binom{p+k-1}{k} z_0^{\,k} = \frac{1}{(1-z_0)^p}$$

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} g_{n,k} \binom{p+k-2}{k} z_0^{\,k} = \frac{1}{(1-z_0)^{p-1}}.$$

Subtraction gives

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} g_{n,k} \binom{p+k-2}{k-1} z_0^{\,k} = \frac{z_0}{(1-z_0)^p}.$$

Dividing by z_0 and rewriting, we obtain

$$(5.10)\qquad \lim_{n\to\infty} \sum_{k=0}^{\infty} g_{n,k+1} \binom{p+k-1}{k} z_0^{\,k} = \frac{1}{(1-z_0)^p}.$$

Thus (5.9) implies (5.10), which proves the theorem.

[5.V] *If the γ-matrix G sums the series* $\Sigma \binom{p+k-1}{k} z^k$ *to $S(z)$ in a domain D, then the matrix*[1]

$$H \equiv (G - z_0\, G^{(1)}) / (1-z_0),$$

where $z_0 \neq 1$ is in D, sums the series $\Sigma \binom{p+k}{k} z_0^{\,k}$ *to the sum* $S(z_0)/(1-z_0)$ *for all real values of p except $p=0$. In particular if G is semi-regular and p a positive integer, the H-sum is the "right" value* $(1-z_0)^{-1-p}$.

Proof: The theorem is trivial if p is a negative integer.

Otherwise we have by hypothesis

$$(5.10)\,a\quad \lim_{n\to\infty} \sum_{k=0}^{\infty} g_{n,k} \binom{p+k-1}{k} z_0^{\,k}$$
$$= \lim_{n\to\infty} \left[g_{n,0} + \sum_{k=1}^{\infty} g_{n,k} \left\{ \binom{p+k}{k} - \binom{p+k-1}{k-1} \right\} \right] z_0^{\,k} = S(z_0),$$

and we can rewrite the right-hand side in the form

$$(5.11)\qquad \lim_{n\to\infty} \sum_{k=0}^{\infty} \binom{p+k}{k} (g_{n,k} - z_0\, g_{n,k+1})\, z_0^{\,k} = S(z_0)$$

(5.12) provided that $g_{n,k} \binom{p+k}{k} z_0^k \to 0$ as $k \to \infty$ for every fixed n.

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[1] $G^{(1)}$ denotes the first diminutive of G, defined in (3.1).

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By hypothesis the power series (5.10)a converges for every fixed n and z in D, and hence it can be differentiated so that

$$g_{n,k}\binom{p+k-1}{k} k z_0^{k-1} \to 0 \text{ as } k \to \infty,$$

$$i.e., \frac{k\, p}{(p+k)\, z_0}\left[g_{n,k}\binom{p+k}{k} z_0^k\right] \to 0,$$

showing that (5.12) holds. We have therefore from (5.11), dividing it by $(1 - z_0)$, the first part of the theorem.

The second part then follows from [5.II].

Corollary 1. By repeated application of the theorem we have *The matrix*

$$H \equiv (1 - z_0)^{-r} \sum_{i=0}^{r} (-1)^i \binom{r}{i} z_0^i G^{(i)} \qquad (z_0 \text{ in } D)$$

sums the series $\Sigma \binom{p+k-1+r}{k} z_0^k$ *to* $S\ (z_0)\ /\ (1 - z_0)^r$, *with the restriction that if* p *is a negative integer*, $p + r \leqq 0$. *If* G *is semi-regular and* p *a positive integer, the* H*-sum is the "right" value* $(1 - z_0)^{-p-r}$.

Corollary 2. *If* G *is row-finite, the restrictions* $p \neq 0$, *and that* G *sums the series in a domain, can be omitted, since* (5.12) *is always satisfied. The theorem then holds for every real* p, *even if* z_0 *is an isolated point of* G*-summability.* (See section 6.)

(5.13) *Example.* The lower-semi-γ-matrix of *arithmetic means*[1]

(5.14) $a_{n,k} = (n - k + 1)\,/\,(n + 1)$ for $k \leqq n$, $a_{n,k} = 0$ for $k > n$, $n, k \geqq 0$, sums the series Σz^k at $z = -1$ to the "right" value $\frac{1}{2}$. It is inefficient for the series $\Sigma\ (k + 1) z^k$ at the same point. But the matrix H given by $h_{n,k} = \frac{1}{2}\ (a_{n,k} + a_{n,k+1})$ sums the second series to its "right" value $\frac{1}{4}$ and the first series to $\frac{1}{2}$. Here H is consistent with A, but not A with H.

Contraction of the binomial series. This method is closely related to the stretching of the matrix (3.4), though not equivalent. Given the series $c_0 + c_1 + c_2 + \ldots$, and writing

(5.15) $$d_k = \sum_{i=kr}^{(k+1)r-1} c_i \qquad (r = 2, 3, 4, \ldots),$$

we call the series $d_0 + d_1 + d_2 + \ldots$ the *r-fold contracted series*, and the r subseries $c_{0+i} + c_{r+i} + c_{2r+i} + \ldots$ $(i = 0, 1, 2, \ldots r - 1)$ *the subseries of r-fold contraction.*

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[1] *T.S.* 399. Theorem VI was used to construct the γ-matrix.

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[5.VI] *The semi-regular γ-matrix G sums all the subseries of the r-fold contracted series* $\Sigma \binom{p+k-1}{k} z^k$ *(p a positive integer) for all values of z for which G sums the series* Σu_k, *where* $u_k = \binom{p+k-1}{k} z^{rk}$. *The sum obtained by contraction is the "right" value* $(1-z)^{-p}$.

We give the proof for $r = 3$. The general case can be proved similarly.

The identity

$$(1 + x + x^2)^p (1 - x^3)^{-p} \equiv (1 - x)^{-p}$$

can be expanded for $|x| < 1$ in the form

$$\left(\sum_{i=0}^{2p} C_i x^i\right) \left\{ \sum_{j=0}^{\infty} \binom{p+j-1}{j} x^{3j} \right\} = \sum_{k=0}^{\infty} \binom{p+k-1}{k} x^k,$$

where $C_i = 0$ for $i > 2p$. Equating coefficients we have

$$(5.16) \quad \sum_{i=0}^{q} C_{3i+m} \binom{p+q-1-i}{q-i} = \binom{p+3q-1+m}{3q+m}, \quad p \geqq 1,\ q \geqq 0,\ m=0,\ 1,\ 2.$$

By hypothesis G sums the series Σu_k; by [5.II] the G-sum is the "right" value, *i.e.*

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} g_{n,k} \binom{p+k-1}{k} z^{3k} = (1 - z^3)^{-p},$$

and since G is semi-regular

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} g_{n,k+i} \binom{p+k-1}{k} z^{3k} = (1 - z^3)^{-p}, \qquad (i = 0, 1, 2, \ldots .),$$

which multiplied by z^{3i} can be rewritten

$$(5.17) \quad \lim_{n \to \infty} \sum_{k=i}^{\infty} g_{n,k} \binom{p+k-1-i}{k-i} z^{3k} = z^{3i} (1 - z^3)^{-p}, \ (i=0, 1, 2, \ldots).$$

Multiplying the series for the different values of i in turn by $C_{3i+m} z^m$ ($m=0$, 1, or 2), and adding them, we obtain by the identity (5.16), for $m = 0, 1, 2$,

$$(5.18) \quad \lim_{n \to \infty} \sum_{k=0}^{\infty} g_{n,k} \binom{p+3k-1+m}{3k+m} z^{3k+m}$$
$$= (C_m z^m + C_{m+3} z^{m+3} + C_{m+6} z^{m+6} + \ldots)(1 - z^3)^{-p},$$

which proves the first statement.

Adding the three equations ($m = 0, 1, 2$) of (5.18) we have

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} g_{n,k} d_k = (1 + z + z^2)^p (1 - z^3)^{-p} = (1 - z)^{-p},$$

where Σd_k is the threefold contracted series.

This concludes the proof.

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(5.19) *Example.* We consider Borel's exponential summation by the γ-matrix (3.5). Write $z = \rho e^{i\theta}$, $(\rho \geqq 0)$. The series Σu_k is summable (B) if the real part of z^r is less than 1, *i.e.* if

(5.20) $$\rho^r \cos r\theta < 1 \qquad (r = 1, 2, 3, \ldots).$$

For $r = 1$, *i.e.* for the original series, the domain of summability is the half-plane $R(z) < 1$. For $r = 2$ the domain lies between the two branches of the hyperbola $x^2 - y^2 = 1$.

Thus the domain of (B) summability for the binomial series varies with the contraction. Given any particular value of z in the star-domain, (*i.e.* excluding $z = 1$ and all points of the real axis to the right of $z = 1$), we can find a suitable contraction for which the series is summable (B). For (5.20) is satisfied if $\cos r\theta \leqq 0$, *i.e.* for $r = 2^q$ if $\pi / 2^{q+1} \leqq |\theta| \leqq \pi / 2^q$, which for $q = 0, 1, 2, \ldots$ covers all the points in question.

6. *γ-matrices efficient at isolated points.*

R. G. Cooke and P. Dienes [1] constructed T-matrices that sum the series Σz^k at an isolated point $z = z_0$ outside the circle of convergence. Similar results are obtained in this section for γ-matrices and extended to the expansion of $(1 - z)^{-p}$, using operations developed in this paper.

We consider the lower semi-γ-matrix G, given by

(6.1) $g_{n,k} = 1$ for $k \leqq n$, $g_{n,k} = 0$ for $k > n$, $(k, n = 0, 1, 2, \ldots)$,

and form from it the γ-matrix $H(p, z_0)$ as in corollary 1 of theorem [5.V]:

(6.2) $$h(p, z_0)_{n,k} = (1 - z_0)^{-p} \sum_{j=0}^{n-k} \binom{p}{j} (-z_0)^j \qquad (n \geqq k \geqq 0),$$
$$= 0 \qquad (n < k).$$

We shall apply this matrix to the series

(6.3) $$\sum_{k=0}^{\infty} \binom{r+k-1}{k} z^k \qquad (r = 1, 2, \ldots).$$

[6.I] *If* $|z_0| > 1$, *then* $H(p, z_0)$ *is a regular* γ*-matrix. If in addition* $1 \leqq r \leqq p$, H *is efficient for the series* (6.3) (*summing it to its "right" value*) *at* $z = z_0$ *and at no other point outside the unit circle: if* $r > p$, H *is inefficient everywhere outside the unit circle.*

Proof: G is obviously regular and so is H by [3.IV] and (3.8).

[1] R. G. Cooke and P. Dienes, "On the effective range of generalized limit processes," Proc. London Math. Soc. (2) 45 (1939), 45-63 (53-55).

G is efficient for the series (6.3) when $r = 0$ at $z = z_0$, *i.e.* for the series $1 + 0.z_0 + 0.z_0^2 + \ldots.$, and thus by theorem [5.V] and both corollaries, H is efficient for the series (6.3) at $z = z_0$ when $r = p$. Hence by [5.I] H is efficient when $r < p$ and by [5.II] the H-sum is the "right" value. To prove that z_0 is an isolated point of efficiency, we consider the transform of Σz^k by the $(n+1)^{th}$ row of H, for $n > p$, *i.e.*

$$\sigma_n(z) = 1 + z + z^2 + \ldots + z^{n-p-1} + (1 - z_0)^{-p} z^{n-p} S,$$

where

$$S = z^p + z^{p-1}\left\{1 - \binom{p}{1} z_0\right\} + z^{p-2}\left\{1 - \binom{p}{1} z_0 + \binom{p}{2} z_0^2\right\} + \ldots + (1 - z_0)^p$$
$$= (1 - z)^{-1}\left\{(1 - z_0)^p - z(z - z_0)^p\right\}.$$

Hence

$$\sigma_n(z) = \frac{1 - z^{n-p}}{1 - z} + \frac{z^{n-p}\left\{(1 - z_0)^p - z(z - z_0)^p\right\}}{(1 - z)(1 - z_0)^p}$$
$$= \frac{1}{1 - z}\left\{1 - \left(\frac{z - z_0}{1 - z_0}\right)^p z^{n-p+1}\right\}.$$

Thus if $|z| \geqq 1$, $\sigma_n(z)$ diverges except for $z = z_0$. Hence H is inefficient for the series (6.3) when $r = 1$, and therefore by [5.I] cannot be efficient for $r > 1$, if $z \neq z_0$. Hence z_0 is an isolated point of efficiency.

Again if H is efficient at z_0 for $r = p + 1$, it follows from the identity (cf. (5.10)a and (5.11))

$$\sum_{k=0}^{\infty} h(p, z_0)_{n,k} \binom{q+k-1}{k} z_0^k \gtrless \sum_{k=0}^{\infty} h(p+1, z_0)_{n,k} \binom{q+k}{k} z_0^k$$

that G is efficient for the series Σz_0^k, which is obviously not the case when $|z_0| > 1$. This proves the last statement.

We may now consider several distinct points $z_1, z_2, \ldots z_m$ outside the circle $|z| = 1$. Replacing the matrix G by $H(p, z_1)$ we obtain the matrix $H(p, z_1, z_2)$ as in corollary 1 of [5.V]. Obviously $H(p, z_1, z_2) = H(p, z_2, z_1)$. Repeating this operation we finally obtain the matrix $H(p, z_1, z_2, \ldots z_m)$, which is given by the formula

$$h(p, z_1, z_2, \ldots z_m)_{n,k} = \sum_{j=0}^{n-k} u_j \Big/ \sum_{j=0}^{mp} u_j, \tag{6.4}$$

where u_j is defined by the identity

$$\left\{(1 - z_1 x)(1 - z_2 x) \ldots (1 - z_m x)\right\}^p \equiv 1 + u_1 x + u_2 x^2 + \ldots + u_{mp} x^{mp}.$$

Thus denoting the matrix briefly by H, we have

$$h_{n,k} = 1 \text{ for } k \leqq n - mp, \quad h_{n,k} = 0 \text{ for } k > n.$$

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[6.II] *If $z_1, z_2, \ldots z_m$ are distinct points outside the unit circle and $1 \leqq r \leqq p$, $H(p, z_1, z_2, \ldots, z_m)$ is regular and efficient for the series* (6.3) *outside the unit circle at these points only. It is inefficient outside the unit circle when $r > p$.*

Proof: $H(p, z_1)$ is efficient for the series $1 + 0.z_2 + 0.z_2^2 + \ldots$ and by [6.I] inefficient for the series (6.3) at $z = z_2$ when $r \geqq 1$. Hence we can replace G by $H(p, z_1)$ for these series and then obtain our result for $m = 2$ in the same way as in [6.I]. Replacing then G by $H(p, z_1, z_2)$ and applying it to the series (6.3) at $z = z_3$ we obtain the result for $m = 3$. Continuing in this way, we obtain the result for the general case.

Examples.

If we take $p = 1$, $m = 2$, we have

$$u_0 = 1,\ u_1 = -(z_1 + z_2), \quad u_2 = z_1 z_2, \quad u_3 = u_4 = \ldots = 0.$$

We then obtain the matrix

$$\begin{aligned} h_{n,k} &= 1/(1-z_1)(1-z_2) && \text{for } k = n \\ &= (1 - z_1 - z_2)/(1 - z_1)(1 - z_2) && \text{for } k = n-1 \\ &= 1 && \text{for } k < n-1 \\ &= 0 && \text{for } k > n. \end{aligned}$$

This matrix transforms the series Σz^k into

$$\sigma_n(z) = \frac{1}{1-z}\left\{1 - \frac{z^{n-1}(z-z_1)(z-z_2)}{(1-z_1)(1-z_2)}\right\},$$

which illustrates the theorem.

If $z_1, z_2, \ldots z_m$ are the m distinct values of $z_0^{1/m}$, we have a simple expression for $h_{n,k}$. For example if $p = 1$, $m = 3$, we obtain

$$\begin{aligned} h_{n,k} &= 1 && \text{for} \quad k < n-2 \\ &= (1 - z_0)^{-1} && \text{for} \quad k = n-2,\ n-1,\ n \\ &= 0 && \text{for} \quad k > n. \end{aligned}$$

This transforms the series Σz^k into

$$\sigma_n(z) = \frac{1}{1-z}\left\{1 - \frac{z^{n-3}(z^3 - z_0)}{1 - z_0}\right\}$$

which is divergent for $|z| > 1$ except for $z^3 = z_0$.

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