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Abstract

The Bogomolov conjecture claims that a closed subvariety containing a dense subset of small points is a special kind of subvariety. In the arithmetic setting over number fields, the Bogomolov conjecture for abelian varieties has already been established as a theorem of Ullmo and Zhang, but in the geometric setting over function fields, it has not yet been solved completely. There are only some partial results known such as the totally degenerate case due to Gubler and our recent work generalizing Gubler's result. The key in establishing the previous results on the Bogomolov conjecture is the equidistribution method due to Szpiro, Ullmo and Zhang with respect to the canonical measures. In this paper we exhibit the limits of this method, making an important contribution to the geometric version of the conjecture. In fact, by the crucial investigation of the support of the canonical measure on a subvariety, we show that the conjecture in full generality holds if the conjecture holds for abelian varieties which have anywhere good reduction. As a consequence, we establish a partial answer that generalizes our previous result.

Introduction

0.1 Background and our main results

The target of this paper is the geometric Bogomolov conjecture, which is the geometric version of the theorem of Ullmo and Zhang. In this subsection we review the conjecture with its background, and then we state our main results.

Let K be a number field, or the function field of a normal projective variety over an algebraically closed base field k. We fix an algebraic closure $\overline{K}$ of K. Let A be an abelian variety over $\overline{K}$ and let L be an ample line bundle on A. Assume that L is even, i.e. $[-1]^*L = L$ for the endomorphism $[-1] : A \to A$ given by $[-1](a) = -a$. Then the canonical height function $\hat{h}_L$, also called the Néron–Tate height, associated with L is a semipositive definite quadratic form on $A(\overline{K})$. It is well known that $\hat{h}_L(x) = 0$ if x is a torsion point. Let X be a closed subvariety of A. We put

$$X(\epsilon; L) := \{x \in X(\overline{K}) \mid \hat{h}_L(x) \leqslant \epsilon\}$$

for a real number $\epsilon > 0$. Then the Bogomolov conjecture for abelian varieties claims that $X(\epsilon; L)$ is not Zariski dense in X for sufficiently small $\epsilon > 0$ unless X is 'special', for example a torsion subvariety. Note that if X is a torsion subvariety, then $X(\epsilon; L)$ is dense in X.

In the arithmetic case, the Bogomolov conjecture has been established by Ullmo for curves inside their Jacobians and by Zhang in general.

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Theorem A ([Zha98, Ull98], arithmetic version of the Bogomolov conjecture). *Let K be a number field. Then $X(\epsilon; L)$ is Zariski dense in X for any $\epsilon > 0$ if and only if X is a torsion subvariety.*[1]

The 'Bogomolov conjecture' over a function field was wide open, while the Bogomolov conjecture over number fields was established. In fact, Moriwaki established in [Mor00] an arithmetic version of the Bogomolov conjecture over a field K *finitely generated over* $\mathbb{Q}$, but it is a different problem because the height in his setting is different from the classical height over a function field, as mentioned in the introduction of [Yam13].

Nine years after Theorem A, Gubler made a breakthrough. In [Gub07b], he established the following theorem, the same statement as Theorem A for abelian varieties totally degenerate at some place over a function field; see § 6.1 for the meaning of 'place'.

Theorem B [Gub07b, Theorem 1.1]. *Let K be a function field. Let A be an abelian variety over $\overline{K}$. Assume that A is totally degenerate at some place. Let $X \subset A$ be a closed subvariety. If $X(\epsilon; L)$ is dense in X for any $\epsilon > 0$, then X is a torsion subvariety.*

Then the next important problem was to generalize Theorem B for any abelian variety, but it was not a trivial task. In fact, Theorem B does not hold for any abelian variety, because in general an abelian variety has subvarieties which are 'constant over k', and subvarieties of this kind have a dense subset of points of height 0.

Therefore we needed to define some notion of subvarieties which should be a counterpart to the torsion subvarieties. In [Yam13] we introduced the notion of special subvarieties: a closed subvariety X of A is said to be *special* if there exist an abelian variety B over k, a closed subvariety $X' \subset B$, a homomorphism $\phi : B_{\overline{K}} \to A$, an abelian subvariety $A' \subset A$, and a torsion point $\tau \in A(\overline{K})$ such that $X = A' + \phi(X'_{\overline{K}}) + \tau$, where $B_{\overline{K}} = B \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{K}$, $X'_{\overline{K}} = X' \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{K}$ (cf. Lemma 7.1(1)). Note that if X is a special subvariety, then $X(\epsilon; L)$ is dense in X for any $\epsilon > 0$ (cf. [Yam13, Corollary 2.8]).

Using the notion of special subvarieties as a counterpart, we formulated in [Yam13] the following conjecture, called the *geometric Bogomolov conjecture*.

Conjecture C (cf. [Yam13, Conjecture 2.9] and Conjecture 7.3 below). Let K be a function field. Let X be a closed subvariety of A. Then if $X(\epsilon; L)$ is Zariski dense in X for any $\epsilon > 0$, then X is a special subvariety.

Conjecture C is still an open problem. The best partial solution known so far is the following, where A_v is the Berkovich analytic space associated to A over v, and $b(A_v)$ is the abelian rank of A_v (cf. § 3.1).

Theorem D [Yam13, Corollary 5.6]. *Let A be an abelian variety over $\overline{K}$. Suppose that there exists a place v such that $b(A_v) \leqslant 1$. Then the geometric Bogomolov conjecture holds for A.*

Theorem D generalizes Theorem B. Indeed, if A_v is totally degenerate, then $b(A_v) = 0 \leqslant 1$ and any special subvariety of A is a torsion subvariety.

In this paper we make a major contribution to the geometric Bogomolov conjecture, exhibiting the limits of the methods of Ullmo, Zhang, Gubler and us on the Bogomolov conjecture. All the results explained so far have been proved by a common method, the equidistribution method due to Szpiro, Ullmo and Zhang (cf. [SUZ97]). We will give a more detailed description of this method in the next subsection.

[1] The essential part is the 'only if' part.

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We are now ready to state our main results. Let K be a function field. For an abelian variety A over $\overline{K}$, it can be seen that there exists a unique maximal nowhere degenerate[2] abelian subvariety $\mathfrak{m}$ of A (cf. § 7.3).

THEOREM E (Corollary 7.22). *Let A be an abelian variety over $\overline{K}$ and let $\mathfrak{m}$ be the maximal nowhere degenerate abelian subvariety of A. Then, the geometric Bogomolov conjecture holds for A if and only if it holds for $\mathfrak{m}$.*

As a consequence of Theorem E, the geometric Bogomolov conjecture for abelian varieties is reduced to the conjecture for those without places of degeneration (cf. Conjecture 7.24). This theorem also shows that the conjecture holds for simple abelian varieties which are degenerate at some place (cf. Remark 7.20).

Theorem E generalizes all the important previous results concerning the geometric Bogomolov conjecture. To see that, we define the nowhere-degeneracy rank of A to be $\operatorname{nd-rk}(A) := \dim \mathfrak{m}$ (cf. Definition 7.10). Since the geometric Bogomolov conjecture holds for all elliptic curves, the following theorem follows from Theorem E.[3]

THEOREM F (cf. Corollary 7.19). *Let A be an abelian variety over $\overline{K}$. Assume that* $\operatorname{nd-rk}(A) \leqslant 1$. *Then the geometric Bogomolov conjecture holds for A.*

We remark that $b(A_v) \geqslant \operatorname{nd-rk}(A)$ for any $v \in M_{\overline{K}}$. Thus Theorem F generalizes Theorem D (cf. Remark 7.20) and hence Theorem B.

This paper also discusses another version of the Bogomolov conjecture, called the Bogomolov conjecture *for curves* over function fields. This conjecture claims that the embedded curve in its Jacobian should have only a finite number of small points unless it is isotrivial (cf. Conjecture 8.1 for the precise statement). It is still an open problem with partial solutions, while the arithmetic version is established by Ullmo in [Ull98]. Under the assumption that K is the function field of a curve over a field of characteristic zero, Cinkir established in [Cin11] an affirmative answer to Conjecture 8.2, an effective version of this conjecture. There is not a satisfactory answer to the conjecture in positive characteristic except for some partial answers. We refer to § 8 for more details. In this paper, applying our arguments on Conjecture C to this problem, we make a non-trivial contribution including the case of positive characteristic.

0.2 Idea of the proof

Our basic strategy for the geometric Bogomolov conjecture is based on the non-archimedean analogue of the proof of Theorem A, as is done in [Gub07b, Yam13]. Zhang's proof of Theorem A relies on the equidistribution theorem of Szpiro, Ullmo and Zhang with respect to the canonical measure over an archimedean place. Although we do not have an archimedean place over function fields, we can define the canonical measure on the analytic space over a non-archimedean place, and it plays a crucial role as a counterpart to the canonical measure over an archimedean place.

Let K be a function field. Let $\mathbb{K} = \overline{K}_v$ be the completion of $\overline{K}$ with respect to a place v of $\overline{K}$ (cf. § 6.1). Let A be an abelian variety over $\mathbb{K}$, $X \subset A$ a closed subvariety of dimension d, and L an even ample line bundle on A. By the theorem of the cube, we have $[n]^*L = L^{n^2}$ for any $n \in \mathbb{Z}$. Suppose that all of them can be defined over $\overline{K}$. Then it is known that there exists a canonical metric on L defined by the condition $[n]^*\|\cdot\| = \|\cdot\|^{n^2}$ via the above identification. Let $\overline{L}$ be

[2] An abelian variety B over $\overline{K}$ is said to be *nowhere degenerate* if B has good reduction at any place.

[3] In this paper we give a direct proof of Theorem F, rather than via Theorem E.

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the line bundle L with a canonical metric. Then we have a measure $c_1(\overline{L}|_X)^{\wedge d}$ on the associated Berkovich analytic space X^{an}, which was originally introduced by Chambert-Loir in [Cha06]. It is a semipositive Borel measure, and thus the probability measure

$$\mu_{X^{\mathrm{an}},\overline{L}} := \frac{1}{\deg_L X} c_1(\overline{L}|_X)^{\wedge d}$$

on X^{an} is defined. This is also called a canonical measure.

With respect to the canonical measures over non-archimedean analytic spaces, an argument analogous to Zhang's proof works to some extent. We refer to the introduction to [Yam13] for details of the argument. However, if we wish to establish some results concerning the geometric Bogomolov conjecture in the same way, we need certain information on canonical measures. In the setting of Theorem D, actually, we focused on the minimal dimension of the components of the support of the tropicalized canonical measure (see [Yam13] for details). In the setting of Theorems F and E, we will introduce strict supports of a canonical measure. Their study will be crucial in this paper and leads to the required information.

Our main results Theorems F and E are consequences of Theorem 6.2. We say that a closed subvariety of an abelian variety is tropically trivial if the canonical tropicalization map of the abelian variety contracts the subvariety to a point for every place of the function field. Theorem 6.2 says that, if the closed subvariety X of A has dense small points,[4] then X/G_X is tropically trivial, where G_X is the stabilizer of X, i.e. $G_X = \{a \in A \mid a + X \subset X\}$.

The proof of Theorem 6.2 follows the same strategy as Zhang's proof. In fact, the strategy works well by virtue of Proposition 5.12, which gives us the crucial information about the canonical measure. In the rest of this subsection, we describe what Proposition 5.12 says and how it is used to prove Theorem 6.2. Let A be an abelian variety over $\mathbb{K}$ and let $X \subset A$ be a closed subvariety. Let $\mathscr{X}'$ be a strictly semistable proper formal scheme with Raynaud generic fiber X' viewed as a Berkovich analytic space (cf. § 1), and let $f : X' \to A^{\mathrm{an}}$ be a generically finite morphism such that $f(X') = X^{\mathrm{an}}$ with some technical assumptions. Let $\overline{L}$ be an even ample line bundle on A with a canonical metric. Then we have the probability measure $\mu_{X',f^*\overline{L}}$ on X', which has the property that $f_*\mu_{X',f^*\overline{L}} = \mu_{X^{\mathrm{an}},L}$.

Let $S(\mathscr{X}')$ be the skeleton of $\mathscr{X}'$ (cf. § 2.1). It is a simplicial set and a subspace of X'; we have a canonical simplex Δ_S for each stratum S of the special fiber of $\mathscr{X}'$ and $S(\mathscr{X}') = \bigcup_S \Delta_S$. In [Gub10], Gubler defined the notion of non-degenerate canonical simplices with respect to f, and showed that $\mu_{X',f^*\overline{L}}$ is a finite sum of the Lebesgue measures on the non-degenerate canonical simplices. It follows that the support $S_{X^{\mathrm{an}}}$ of $\mu_{X^{\mathrm{an}},L}$ coincides with the image of the union of the non-degenerate canonical simplices by f. Further, he showed that $S_{X^{\mathrm{an}}}$ has a unique piecewise linear structure such that the restriction of f to each non-degenerate canonical simplex is a piecewise linear map.

We fix a sufficiently refined polytopal decomposition of $S_{X^{\mathrm{an}}}$.[5] A polytope σ in $S_{X^{\mathrm{an}}}$ is called a *strict support* of $\mu_{X^{\mathrm{an}}}$ if $\mu_{X^{\mathrm{an}}} - \epsilon\delta_\sigma$ is semipositive for small $\epsilon > 0$ (cf. Definition 5.9), where δ_σ is the push-out to $S_{X^{\mathrm{an}}}$ of the Lebesgue measure on σ. Suppose that σ is a strict support of $\mu_{X^{\mathrm{an}},L}$. Then we see that there is a non-degenerate canonical simplex Δ_S with $\sigma \subset f(\Delta_S)$ and $\dim \sigma = \dim f(\Delta_S)$ (cf. Lemma 5.10). Furthermore, Proposition 5.12, together with Lemma 5.13,

[4] We can show that the property '$X(\epsilon; L)$ is dense in X for any $\epsilon > 0$' does not depend on an even ample line bundle L. We say X has dense small points if, for some (and hence any) L, $X(\epsilon; L)$ is dense for any $\epsilon > 0$ (cf. [Yam13, Definition 2.2]).

[5] It should be a subdivisional one with the terminology in § 5.4.

shows that, for *any* stratum Δ_S of $S(\mathscr{X}')$ with $\sigma \subset f(\Delta_S)$ and $\dim \sigma = \dim f(\Delta_S)$, we have $\dim f(\Delta_S) = \dim \Delta_S$.[6]

Let us now give an outline of the proof of Theorem 6.2. To argue by contradiction, we suppose that X/G_X is not tropically trivial but $X \subset A$ has dense small points. Replacing X with X/G_X, we may further assume that X has trivial stabilizer. Then it follows from the tropical non-triviality of X that S_{X_v} has positive dimension for some place v, where X_v is the associated Berkovich space at v. Thus μ_{X_v} has a strict support of positive dimension.

Consider the homomorphism $A^N \to A^{N-1}$ defined by $(x_1, \dots, x_N) \mapsto (x_2 - x_1, \dots, x_N - x_{N-1})$. We set $Z := X^N$ and let Y be the image of Z by this homomorphism. Let $\alpha : Z \to Y$ be the restriction of this homomorphism. By the triviality of G_X, there exists $N \in \mathbb{N}$ such that α is generically finite. Note that α contracts the diagonal of $Z = X^N$ to a point. Since μ_{X_v} has a strict support of positive dimension and since μ_{Z_v} is the product of N copies of μ_{X_v}, it follows that there is a strict support σ of μ_{Z_v} with $\dim \alpha(\sigma) < \dim \sigma$.

Since Z has dense small points, we obtain $\alpha_*(\mu_{Z_v}) = \mu_{Y_v}$ by the equidistribution theorem. Since σ is a strict support of μ_{Z_v}, it follows that $\alpha(\sigma)$ is a strict support of μ_{Y_v}.

By de Jong's alteration theorem, we have a proper strictly semistable formal scheme $\mathscr{X}'$ with a generically finite surjective morphism $g : (\mathscr{X}')^{\mathrm{an}} \to Z_v$. Since α is a generically finite surjective morphism, the morphism $h := \alpha \circ g$ is also a generically finite surjective morphism. Since σ is a strict support, there exists a non-degenerate canonical simplex Δ_S of $S(\mathscr{X}')$ with respect to g such that $\sigma \subset g(\Delta_S)$ and $\dim \sigma = \dim g(\Delta_S)$. Then we have $\alpha(\sigma) \subset h(\Delta_S)$ and $\dim \alpha(\sigma) = \dim h(\Delta_S)$. Since $\alpha(\sigma)$ is a strict support of μ_{Y_v}, it follows form Proposition 5.12 that $\dim h(\Delta_S) = \dim \Delta_S$. On the other hand, the inequality $\dim \alpha(\sigma) < \dim \sigma$ tells us that $\dim h(\Delta_S) = \dim \alpha(g(\Delta_S)) < \dim \Delta_S$. This is a contradiction. Thus Theorem 6.2 follows.

Finally in this subsection, we notice a limit of this strategy. As mentioned in § 0.2, Conjecture C can be reduced to Conjecture 7.24, namely, the geometric Bogomolov conjecture *for nowhere degenerate abelian varieties*. However, the strategy used here is of no use for such abelian varieties because the support of the canonical measure of a closed subvariety is the Dirac measure of a finite set. This fact suggests that our Theorems F and E are as far as we can go with the strategy based on equidistribution theorems.

0.3 Organization

This rest of this paper consists of eight sections. In § 1 we recall some basic facts on non-archimedean geometry. In § 2, we introduce an affinoid torus action on a formal fiber of the Berkovich analytic space associated to a strictly semistable formal scheme, and we show that this action is compatible with the reduction (cf. Lemma 2.7). In § 3 we recall the Raynaud extension of an abelian variety and its tropicalization. We also recall Mumford models of the Raynaud extension. In § 4 we establish that a morphism from a strictly semistable formal scheme to a Mumford model of an abelian variety induces a torus-equivariant morphism between their strata (cf. Proposition 4.4) with the use of Lemma 2.7. Furthermore, we show a key lemma, Lemma 4.5, which will be crucially used to show that a stratum over a strict support does not collapse. We introduce the notion of strict support in § 5, and prove Proposition 5.12 which was explained in § 0.2. In § 6, using Proposition 5.12, we show that if the closed subvariety X of A has dense small points, then X/G_X is tropically trivial (cf. Theorem 6.2). In § 7, using Theorem 6.2, we establish results concerning the geometric Bogomolov conjecture, including the

[6] To be precise, we should write $\overline{f}_{\mathrm{aff}}(\Delta_S)$ instead of $f(\Delta_S)$, etc., but we use this notation in this subsection despite its imprecision; we are just sketching the idea of the proof.

main theorems mentioned in § 0.1. Some non-trivial remarks on the conjecture for curves are made in § 8.

1. Preliminary

1.1 Convention and terminology

When we write $\mathbb{K}$, it is an algebraically closed field which is complete with respect to a non-trivial non-archimedean absolute value $|\cdot| : \mathbb{K} \to \mathbb{R}_{\geqslant 0}$. We put $\mathbb{K}^{\circ} := \{a \in \mathbb{K} \mid |a| \leqslant 1\}$, the ring of integers of $\mathbb{K}$, $\mathbb{K}^{\circ\circ} := \{a \in \mathbb{K} \mid |a| < 1\}$, the maximal ideal of the valuation ring $\mathbb{K}^{\circ}$, and $\tilde{\mathbb{K}} := \mathbb{K}^{\circ}/\mathbb{K}^{\circ\circ}$, the residue field. We put $\Gamma := \{-\log |a| \mid a \in \mathbb{K}^{\times}\}$, the value group of $\mathbb{K}$.

We also fix the notation used in convex geometry; see [Gub07a, 6.1 and Appendix A] for more details. A polytope Δ of $\mathbb{R}^n$ is said to be Γ-*rational* if it can be given as an intersection of subsets of the form $\{\mathbf{u} \mid \mathbf{m} \cdot \mathbf{u} \geqslant c\}$ (closed half-space) for some $\mathbf{m} \in \mathbb{Z}^n$ and $c \in \Gamma$. When $\Gamma = \mathbb{Q}$, a Γ-rational polytope is called a *rational* polytope. A *closed face* of Δ is a polytope which is Δ itself or is of the form $H \cap \Delta$ where H is the boundary of a closed half-space of $\mathbb{R}^n$ containing Δ. An *open face* of Δ means the relative interior of any closed face of Δ.

Let Ω be a subset of $\mathbb{R}^n$. A Γ-*rational polytopal decomposition* $\mathscr{C}$ *of* Ω is a locally finite family of Γ-rational polytopes of $\mathbb{R}^n$ such that:

- $\bigcup_{\Delta \in \mathscr{C}} \Delta = \Omega$;
- for any $\Delta, \Delta' \in \mathscr{C}$, $\Delta \cap \Delta'$ is an empty set or a face of Δ and Δ'; and
- for any $\Delta \in \mathscr{C}$, any closed face of Δ is in $\mathscr{C}$.

Let $\mathscr{C}$ be a Γ-rational polytopal decomposition of $\mathbb{R}^n$. Let Λ be a lattice of $\mathbb{R}^n$. The polytopal decomposition $\mathscr{C}$ is said to be Λ-*periodic* if, for any $\Delta \in \mathscr{C}$, we have $\lambda + \Delta \in \mathscr{C}$ for any $\lambda \in \Lambda$, and the restriction of the quotient map $\mathbb{R}^n \to \mathbb{R}^n/\Lambda$ to Δ is a homeomorphism to its image. For a Λ-periodic Γ-rational polytopal decomposition $\mathscr{C}$ of $\mathbb{R}^n$, we set $\overline{\mathscr{C}} := \{\overline{\Delta} \mid \Delta \in \mathscr{C}\}$, where $\overline{\Delta}$ is the image of Δ by the quotient map $\mathbb{R}^n \to \mathbb{R}^n/\Lambda$. Such a $\overline{\mathscr{C}}$ is called a Γ-*rational polytopal decomposition of* $\mathbb{R}^n/\Lambda$.

By an algebraic variety over a field we mean an irreducible, reduced and separated scheme of finite type over the field unless otherwise specified.

1.2 Berkovich analytic spaces

We recall some notions and properties on analytic spaces associated to admissible formal schemes and those associated to algebraic varieties, for later use. For details, we refer to Berkovich's original papers [Ber90, Ber93, Ber94, Ber99] or Gubler's expositions in [Gub07a, Gub10].

Let $\mathbb{K}\langle x_1, \ldots, x_n\rangle$ be the Tate algebra over $\mathbb{K}$, i.e. the completion of the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ with respect to the Gauss norm. A $\mathbb{K}$-algebra $\mathfrak{A}$ isomorphic to $\mathbb{K}\langle x_1, \ldots, x_n\rangle/I$ for some ideal I of $\mathbb{K}\langle x_1, \ldots, x_n\rangle$ is called a $\mathbb{K}$-*affinoid algebra*. Let $\mathrm{Max}(\mathfrak{A})$ be the maximal spectrum of $\mathfrak{A}$. Let $|\cdot|_{\sup} : \mathfrak{A} \to \mathbb{R}$ denote the supremum seminorm over $\mathrm{Max}(\mathfrak{A})$. The Berkovich spectrum of $\mathfrak{A}$ is the set of multiplicative seminorms $\mathfrak{A}$ bounded with $|\cdot|_{\sup}$. A (Berkovich) analytic space over $\mathbb{K}$ is given by an atlas of Berkovich spectrums of $\mathbb{K}^{\circ}$-affinoid algebras.

A $\mathbb{K}^{\circ}$-algebra is called an *admissible* $\mathbb{K}^{\circ}$-*algebra* if it does not have any $\mathbb{K}^{\circ}$-torsions and it is isomorphic to $\mathbb{K}^{\circ}\langle x_1, \ldots, x_n\rangle/I$ for some $n \in \mathbb{N}$ and for some ideal I of $\mathbb{K}^{\circ}\langle x_1, \ldots, x_n\rangle$. Note that an admissible $\mathbb{K}^{\circ}$-algebra is flat over $\mathbb{K}^{\circ}$. The formal spectrum of an admissible $\mathbb{K}^{\circ}$-algebra is called an *affine admissible formal scheme*. A formal scheme over $\mathbb{K}^{\circ}$ is called an *admissible formal scheme* [7] if it has a locally finite open atlas of affine admissible formal schemes. Note that an admissible formal scheme is flat over $\mathbb{K}^{\circ}$. For an admissible formal scheme $\mathscr{X}$, we write

[7] In this paper, when we say a formal scheme, we always mean an admissible formal scheme.

$\tilde{\mathscr{X}} := \mathscr{X} \times_{\operatorname{Spf} \mathbb{K}^\circ} \operatorname{Spec} \tilde{\mathbb{K}}$ for the special fiber. For a morphism $\varphi : \mathscr{X} \to \mathscr{Y}$ of admissible formal schemes, we write $\tilde{\varphi} : \tilde{\mathscr{X}} \to \tilde{\mathscr{Y}}$ for the induced morphism between their special fibers.

Let $\mathscr{X}$ be an admissible formal scheme over $\mathbb{K}^\circ$. Then we can associate a Berkovich analytic space $\mathscr{X}^{\mathrm{an}}$, called the *(Raynaud) generic fiber* of $\mathscr{X}$. Further, we have a map $\operatorname{red}_{\mathscr{X}} : \mathscr{X}^{\mathrm{an}} \to \tilde{\mathscr{X}}$, called the *reduction map*. It is known that the reduction map is surjective. Let Z be a dense open subset of an irreducible component of $\tilde{\mathscr{X}}$ with the generic point $\eta_Z \in Z$. Then there is a point $\xi_Z \in \mathscr{X}^{\mathrm{an}}$ with $\operatorname{red}_{\mathscr{X}}(\xi_Z) = \eta_Z$. If the special fiber $\tilde{\mathscr{X}}$ is reduced, then such a point ξ_Z is unique, and we refer to it as *the point corresponding to (the generic point of) Z*.

We can associate an analytic space to an algebraic variety X over $\mathbb{K}$ as well, and we write X^{an} for the analytic space associated to X. There is a natural inclusion $X(\mathbb{K}) \subset X^{\mathrm{an}}$. We recall the relationship between the analytic space associated to an algebraic variety and that associated to an admissible formal scheme. Let $\mathbf{X}$ be a scheme flat and of finite type over $\mathbb{K}^\circ$ with the generic fiber X. Let $\widehat{\mathbf{X}}$ be the formal completion with respect to a principal open ideal of $\mathbb{K}^\circ$ contained in $\mathbb{K}^{\circ\circ}$. Then $\widehat{\mathbf{X}}$ is an admissible formal scheme and $\widehat{\mathbf{X}}^{\mathrm{an}}$ is an analytic subdomain of X^{an}. If $\mathbf{X}$ is proper over $\mathbb{K}^\circ$, we have $\widehat{\mathbf{X}}^{\mathrm{an}} = X^{\mathrm{an}}$.

For a given analytic space X, an admissible formal scheme having X as the generic fiber is called a *formal model* of X. Note that a formal model is flat over $\mathbb{K}^\circ$ by definition. Let Y be a closed analytic subvariety of X and let $\mathscr{X}$ be a formal model of X. Then there exists a unique admissible formal subscheme $\mathscr{Y} \subset \mathscr{X}$ with $\mathscr{Y}^{\mathrm{an}} = Y$. We call $\mathscr{Y}$ the *closure* of Y in $\mathscr{X}$.

1.3 Tori

Let $\mathbb{G}_m^n$ denote the split torus of rank n over $\mathbb{K}$. Let $x_1, \ldots, x_n$ denote the standard coordinates of $\mathbb{G}_m^n$ unless otherwise noted, so that $\mathbb{G}_m^n = \operatorname{Spec} \mathbb{K}[x_1^{\pm}, \ldots, x_n^{\pm}]$. Let $(\mathbb{G}_m^n)^{\mathrm{an}}$ be the analytic space associated to $\mathbb{G}_m^n$. We set

$$(\mathbb{G}_m^n)_1^{\text{f-sch}} := \operatorname{Spf} \mathbb{K}^\circ[x_1^{\pm}, \ldots, x_n^{\pm}],$$

the formal torus over $\mathbb{K}^\circ$, writing $(\mathbb{G}_m^n)_1^{\mathrm{an}}$ for the generic fiber of $(\mathbb{G}_m^n)_1^{\text{f-sch}}$. We call $(\mathbb{G}_m^n)_1^{\text{f-sch}}$ the *canonical model* of $(\mathbb{G}_m^n)_1^{\mathrm{an}}$. The analytic space $(\mathbb{G}_m^n)_1^{\mathrm{an}}$ is an analytic subgroup of $(\mathbb{G}_m^n)^{\mathrm{an}}$ as well as an affinoid subdomain of $(\mathbb{G}_m^n)^{\mathrm{an}}$. The special fiber of $(\mathbb{G}_m^n)_1^{\text{f-sch}}$ is an algebraic torus $(\mathbb{G}_m^n)_{\tilde{\mathbb{K}}} = \operatorname{Spec} \tilde{\mathbb{K}}[x_1^{\pm}, \ldots, x_n^{\pm}]$, which we call the *canonical reduction* of $(\mathbb{G}_m^n)_1^{\mathrm{an}}$.

Each element $p \in (\mathbb{G}_m^n)^{\mathrm{an}}$ is regarded as a seminorm on $\mathbb{K}[x_1^{\pm}, \ldots, x_n^{\pm}]$. We define a map $\operatorname{val} : (\mathbb{G}_m^n)^{\mathrm{an}} \to \mathbb{R}^n$, called the *valuation map*, by

$$p \mapsto (-\log p(x_1), \ldots, -\log p(x_n)).$$

Let Δ be a Γ-rational polytope of $\mathbb{R}^n$. Then $U_\Delta := \operatorname{val}^{-1}(\Delta)$ is an analytic subdomain of $(\mathbb{G}_m^n)^{\mathrm{an}}$. In fact, it is the Berkovich spectrum of the affinoid algebra

$$\mathbb{K}\langle U_\Delta \rangle := \left\{ \sum_{\mathbf{m} \in \mathbb{Z}^n} a_{\mathbf{m}} x_1^{m_1} \cdots x_n^{m_n} \,\middle|\, \lim_{|\mathbf{m}| \to \infty} v(a_{\mathbf{m}}) + \mathbf{m} \cdot \mathbf{u} = \infty \text{ for any } \mathbf{u} \in \Delta \right\}.$$

Note that $\operatorname{val}^{-1}(\mathbf{0}) = (\mathbb{G}_m^n)_1^{\mathrm{an}}$. We refer to [Gub07b, 4.3] for more details.

2. Torus action on the formal fiber

An admissible formal scheme $\mathscr{X}'$ is called a *strictly semistable* formal scheme if any point of $\mathscr{X}'$ has an open neighborhood $\mathscr{U}'$ and an étale morphism

$$\psi : \mathscr{U}' \to \mathscr{S} := \operatorname{Spf} \mathbb{K}^\circ \langle x_0', \ldots, x_d' \rangle / (x_0' \cdots x_r' - \pi), \tag{2.1}$$

where $\pi \in \mathbb{K}^{\circ\circ} \setminus \{0\}$.

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Throughout this section, let $\mathscr{X}'$ be a strictly semistable formal scheme over $\mathbb{K}^\circ$ with generic fiber $X' = (\mathscr{X}')^{\mathrm{an}}$. Let $\mathrm{red}_{\mathscr{X}'} : X' \to \tilde{\mathscr{X}}'$ be the reduction map. For a closed point $\tilde{p} \in \tilde{\mathscr{X}}'$, we put $X'_+(\tilde{p}) := (\mathrm{red}_{\mathscr{X}'})^{-1}(\tilde{p})$ and call it the *formal fiber* over $\tilde{p}$. It is an open subdomain of X'. See [Gub07a, 2.8] for more details.

The purpose of this section is to define a torus action on $X'_+(\tilde{p})$ and show that it is compatible with the reduction as Lemma 2.7 claims. This lemma will be used in the proof of Proposition 4.4.

2.1 Strictly semistable formal schemes, their skeletons, and subdivision

We begin by recalling the notion of stratification of a reduced separated scheme Y of finite type over a field. We start with $Y^{(0)} := Y$. For each $r \in \mathbb{N}$, let $Y^{(r)} \subset Y^{(r-1)}$ be the complement of the set of normal points in $Y^{(r-1)}$. Since the set of normal points is open and dense, we obtain a chain of closed subsets:

$$Y = Y^{(0)} \supsetneq Y^{(1)} \supsetneq \cdots \supsetneq Y^{(s)} \supsetneq Y^{(s+1)} = \emptyset.$$

The irreducible components of $Y^{(r)} \backslash Y^{(r+1)}$ $(0 \leqslant r \leqslant s)$ are called the *strata of* Y, and the set of strata is denoted by $\mathrm{str}(Y)$.

We take an étale morphism as in (2.1). Putting $\mathscr{S}_1 := \mathrm{Spf}\, \mathbb{K}^\circ\langle x'_0, \ldots, x'_r\rangle/(x'_0 \cdots x'_r - \pi)$ and $\mathscr{S}_2 := \mathrm{Spf}\, \mathbb{K}^\circ\langle x'_{r+1}, \ldots, x'_d\rangle$, we have $\mathscr{S} = \mathscr{S}_1 \times \mathscr{S}_2$. Let $\tilde{o}$ denote the point of $\tilde{\mathscr{S}}_1 = \mathrm{Spec}\, \tilde{\mathbb{K}}[x'_0, \ldots, x'_r]/(x'_0 \cdots x'_r)$ defined by $x'_0 = \cdots = x'_r = 0$.

The following proposition is due to Gubler.

Proposition 2.1 [Gub10, Proposition 5.2]. *Any formal open covering of $\mathscr{X}'$ admits a refinement $\{\mathscr{U}'\}$ by formal open subsets $\mathscr{U}'$ with étale morphisms as in (2.1) which have the following properties.*

(a) *Any $\mathscr{U}'$ is a formal affine open subscheme of $\mathscr{X}'$.*
(b) *There exists a distinguished stratum S of $\tilde{\mathscr{X}}'$ associated to $\mathscr{U}'$ such that for any stratum T of $\tilde{\mathscr{X}}'$, we have $S \subset \overline{T}$ if and only if $\tilde{\mathscr{U}}' \cap \overline{T} \neq \emptyset$, where $\overline{T}$ is the closure of T in $\tilde{\mathscr{X}}'$.*
(c) *The subset $\tilde{\psi}^{-1}(\{\tilde{o}\} \times \tilde{\mathscr{S}}_2)$ is the stratum of $\tilde{\mathscr{U}}'$ which is equal to $\tilde{\mathscr{U}}' \cap S$ for the distinguished stratum S from (b).*
(d) *Any stratum of $\tilde{\mathscr{X}}'$ is the distinguished stratum of a suitable $\mathscr{U}'$.*

We can define a subspace $S(\mathscr{X}')$ of $X' = (\mathscr{X}')^{\mathrm{an}}$ called the *skeleton*. It has a canonical structure of an abstract simplicial set which reflects the incidence relations between the strata of $\tilde{\mathscr{X}}'$. We briefly recall some properties of skeletons here, and refer to [Ber99] or [Gub10, 5.3] for more details.

First, we recall the skeletons $S(\mathscr{S}_1)$ and $S(\mathscr{S})$. We set $\mathbb{G}_m^r := \mathrm{Spec}\, \mathbb{K}[(x'_1)^{\pm}, \ldots, (x'_r)^{\pm}]$, the algebraic torus with the standard coordinates $x'_1, \ldots, x'_r$, and consider the associated analytic group $(\mathbb{G}_m^r)^{\mathrm{an}}$. Let $\mathrm{val}' : (\mathbb{G}_m^r)^{\mathrm{an}} \to \mathbb{R}^r$ denote the valuation map with respect to the coordinates $x'_1, \ldots, x'_r$ (cf. § 1.3).[8] We regard $\mathscr{S}_1^{\mathrm{an}}$ as a rational subdomain of $(\mathbb{G}_m^r)^{\mathrm{an}}$ by omitting x'_0. We put $\Delta' := \{(u'_1, \ldots, u'_r) \in \mathbb{R}^r_{\geqslant 0} \mid u'_1 + \cdots + u'_r \leqslant v(\pi)\}$. Then val' induces a map $\mathscr{S}_1^{\mathrm{an}} \to \Delta'$ by restriction. It is known that this map restricts to an isomorphism $S(\mathscr{S}_1) \cong \Delta'$, and that the first projection $\mathscr{S} \to \mathscr{S}_1$ restricts to an isomorphism $S(\mathscr{S}) \cong S(\mathscr{S}_1)$. Thus we have identifications $S(\mathscr{S}) = S(\mathscr{S}_1) = \Delta'$.

[8] We write here val' instead of val to emphasize that it is the valuation map with respect to the coordinates $x'_1, \ldots, x'_r$.

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Recall that the vertices of Δ' correspond to the irreducible components of $\tilde{\mathscr{S}}$. In fact, let v_0 denote the origin in Δ' and let v_i denote the vertex of Δ' whose ith coordinate is $v(\pi)$ ($i = 0, \ldots, r$). The points $v_0, v_1, \ldots, v_r$ are the vertices of Δ'. Let $\xi_i \in S(\mathscr{S})$ be the point corresponding to the irreducible component of $\operatorname{Spec} \tilde{\mathbb{K}}[x'_0, \ldots, x'_d]/(x'_0 \cdots x'_r)$ defined by $x_i = 0$ for each $i = 0, \ldots, r$ (cf. § 1.2). Then we have $\xi_i = v_i$ via the identification $S(\mathscr{S}) = \Delta'$.

Let S be a stratum of $\tilde{\mathscr{X}}'$ of codimension r. We take a formal affine open subset $\mathscr{U}'$ of $\mathscr{X}'$ such that S is the distinguished stratum of $\mathscr{U}'$ in Proposition 2.1, and an étale morphism as in (2.1). The skeleton $S(\mathscr{U}')$ of $(\mathscr{U}')^{\mathrm{an}}$ is a subset of $S(\mathscr{X}')$, and it is known that ψ in (2.1) induces an isomorphism $S(\mathscr{U}) \cong S(\mathscr{S})$ between skeletons. Thus

$$S(\mathscr{U}') \cong S(\mathscr{S}) = S(\mathscr{S}_1) = \Delta' \tag{2.2}$$

and, in particular, the subset $S(\mathscr{U}')$ of $S(\mathscr{X}')$ is a simplex. It is also known that $S(\mathscr{U}')$ depends only on S and not on the choice of $\mathscr{U}'$, so that we write $\Delta_S = S(\mathscr{U}')$ and call it the *canonical simplex* corresponding to S. The canonical simplices $\{\Delta_S\}_{S \in \mathrm{str}(\tilde{\mathscr{X}}')}$ cover $S(\mathscr{X}')$, which gives a canonical structure of an abstract simplicial set to the skeleton $S(\mathscr{X}')$.

We have a continuous map $\mathrm{Val} : X' \to S(\mathscr{X}')$ which restricts to the identity on $S(\mathscr{X}')$. If S is a distinguished stratum of $\tilde{\mathscr{X}}'$ associated to $\mathscr{U}'$ in the sense of Proposition 2.1, then the restriction $(\mathscr{U}')^{\mathrm{an}} \to \Delta_S$ of Val to $(\mathscr{U}')^{\mathrm{an}}$ is described as follows: regarding $\mathscr{S}_1^{\mathrm{an}}$ as a rational subdomain of $(\mathbb{G}_m^r)^{\mathrm{an}}$ and using the identification

$$\Delta_S = \Delta' \tag{2.3}$$

given by (2.2), we can describe Val as $\mathrm{Val}(p) = \mathrm{val}'(\psi^{\mathrm{an}}(p))$ for $p \in (\mathscr{U}')^{\mathrm{an}}$.

Let $\mathscr{D}$ be a Γ-rational subdivision of the skeleton $S(\mathscr{X}')$. This means that $\mathscr{D}$ is a family of polytopes, each contained in a canonical simplex, such that $\{\Delta \in \mathscr{D} \mid \Delta \subset \Delta_S\}$ is a Γ-rational polytopal decomposition of Δ_S for any stratum S of $\tilde{\mathscr{X}}'$ (cf. [Gub10, 5.4]).

Remark 2.2. By [Gub10, Proposition 5.5], we have a unique formal model $\mathscr{X}''$ of $(\mathscr{X}')^{\mathrm{an}}$ associated to $\mathscr{D}$. Note that there is a morphism $\iota' : \mathscr{X}'' \to \mathscr{X}'$ extending the identity on the generic fiber $(\mathscr{X}')^{\mathrm{an}}$.

Although we omit the precise construction of $\mathscr{X}''$, we recall two important propositions concerning $\mathscr{X}''$.

PROPOSITION 2.3 [Gub10, Proposition 5.7]. *Let $\mathscr{X}''$ be the formal scheme associated to $\mathscr{D}$ as in Remark 2.2. Let $\mathrm{red}_{\mathscr{X}''} : (\mathscr{X}'')^{\mathrm{an}} \to \tilde{\mathscr{X}}''$ be the reduction map. Then there exists a bijective correspondence between open faces τ of $\mathscr{D}$ and strata R of $\tilde{\mathscr{X}}''$ given by*

$$R = \mathrm{red}_{\mathscr{X}''}(\mathrm{Val}^{-1}(\tau)), \quad \tau = \mathrm{Val}(\mathrm{red}_{\mathscr{X}''}^{-1}(Y)),$$

where Y is any non-empty subset of R.

For a canonical simplex Δ_S, let $\operatorname{relin} \Delta_S$ denote the relative interior of Δ_S.[9]

PROPOSITION 2.4 (cf. [Gub10, Corollary 5.9]). *Let $\mathscr{X}''$ be the formal scheme associated to $\mathscr{D}$ and let $\iota' : \mathscr{X}'' \to \mathscr{X}'$ be the morphism extending the identity on $(\mathscr{X}')^{\mathrm{an}}$ (cf. Remark 2.2). Let $u \in \mathscr{D}$ be a vertex and let R be the stratum of $\tilde{\mathscr{X}}''$ corresponding to u in Proposition 2.3. Then $S := \tilde{\iota}'(R)$ is a unique stratum of $\tilde{\mathscr{X}}'$ with $u \in \operatorname{relin} \Delta_S$. Furthermore, if we put $r := \dim R - \dim S = \dim \Delta_S$, then $\tilde{\iota}'|_R : R \to S$ has a $(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$-torsor structure.*

[9] For a polytope P, let $\operatorname{relin}(P)$ denote the relative interior of P in this paper.

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Finally in this subsection, we show a lemma.

Lemma 2.5. *Let $\mathscr{D}$ and $\mathscr{X}''$ be as above. Let u be a vertex of $\mathscr{D}$ of $S(\mathscr{X}')$ and let R be the stratum of $\tilde{\mathscr{X}}''$ corresponding to u. Let $\xi_R \in (\mathscr{X}'')^{\mathrm{an}}$ be the point corresponding to the generic point of R (cf. § 1.2). Then $u = \xi_R$.*

Proof. It follows from [Gub10, Proposition 5.7 and Corollary 5.9(a)] that $u = \mathrm{Val}(\xi_R)$. Since Val restricts to the identity on $S(\mathscr{X}')$, it only remains to show $\xi_R \in S(\mathscr{X}'')$, but that is done in the proof of [Gub10, Corollary 5.9(g)]. □

2.2 Construction of the action and the compatibility lemma

Let S be a stratum of $\tilde{\mathscr{X}}'$ of codimension r. We take a formal affine open subset $\mathscr{U}'$ of $\mathscr{X}'$ such that S is the distinguished stratum of $\mathscr{U}'$ in Proposition 2.1, and an étale morphism as in (2.1). Let $\tilde{p} \in S \cap \tilde{\mathscr{U}}'$ be a closed point. In this subsection, we define an affinoid torus action of the formal fiber $X'_+(\tilde{p})$ over $\tilde{p}$. Since our interest is local at $\tilde{p}$, we may and do assume that $\mathscr{X}' = \mathscr{U}'$.[10]

Recall from § 2.1 that $\mathscr{S} = \mathscr{S}_1 \times \mathscr{S}_2$. Since $(\mathscr{S}_1)^{\mathrm{an}} \subset (\mathbb{G}_m^r)^{\mathrm{an}}$ and $(\mathscr{S}_1)^{\mathrm{an}}$ is stable under the $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-action, we have a $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-action on $(\mathscr{S}_1)^{\mathrm{an}}$. We put the trivial $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-action on $(\mathscr{S}_2)^{\mathrm{an}}$. Then, via the isomorphism $\mathscr{S}^{\mathrm{an}} = (\mathscr{S}_1)^{\mathrm{an}} \times (\mathscr{S}_2)^{\mathrm{an}}$, we have a $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-action on $\mathscr{S}^{\mathrm{an}}$. Let $(\mathscr{S}^{\mathrm{an}})_+(\tilde{\psi}(\tilde{p})) := \mathrm{red}_{\mathscr{S}}^{-1}(\tilde{\psi}(\tilde{p}))$ be the formal fiber of $\mathscr{S}^{\mathrm{an}}$ over $\tilde{\psi}(\tilde{p})$. Since $\tilde{\psi}(\tilde{p}) = (\tilde{o}, \tilde{c})$ for some $\tilde{c} \in \tilde{\mathscr{S}}_2$ via the identification $\tilde{\mathscr{S}} = \tilde{\mathscr{S}}_1 \times \tilde{\mathscr{S}}_2$, we see that $(\mathscr{S}^{\mathrm{an}})_+(\tilde{\psi}(\tilde{p}))$ is stable under the $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-action on $\mathscr{S}^{\mathrm{an}}$, and thus $(\mathbb{G}_m^r)_1^{\mathrm{an}}$ acts on $(\mathscr{S}^{\mathrm{an}})_+(\tilde{\psi}(\tilde{p}))$.

We define the $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-action on $X'_+(\tilde{p})$. By [Ber99, Lemma 4.4] or [Gub10, Proposition 2.9], the restricted morphism

$$\psi'^{\mathrm{an}}|_{X'_+(\tilde{p})} : X'_+(\tilde{p}) \to (\mathscr{S}^{\mathrm{an}})_+(\tilde{\psi}(\tilde{p})) \tag{2.4}$$

is an isomorphism, and then we define the $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-action on $X'_+(\tilde{p})$ by pulling back the $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-action on $(\mathscr{S}^{\mathrm{an}})_+(\tilde{\psi}(\tilde{p}))$ via this isomorphism.

Let Δ_S be the canonical simplex corresponding to S. Let $\mathscr{D}$ be a Γ-rational polytopal subdivision of the skeleton $S(\mathscr{X}')$, $\mathscr{X}''$ the formal model of $(\mathscr{X}')^{\mathrm{an}}$ associated to $\mathscr{D}$, $\iota' : \mathscr{X}'' \to \mathscr{X}'$ the morphism extending the identity on X' (cf. Remark 2.2), and $\mathrm{red}_{\mathscr{X}''} : X' \to \tilde{\mathscr{X}}''$ the reduction map. Suppose that $u \in \mathscr{D}$ is a vertex with $u \in \mathrm{relin}\, \Delta_S$ and let R be the stratum of $\tilde{\mathscr{X}}''$ corresponding to u in Proposition 2.3. Recall that we have a $(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$-torsor $\tilde{\iota}' : R \to S$ (cf. Proposition 2.4). If $\tilde{p} \in S$ is a closed point, then $(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$ acts on the fiber $\{\tilde{p}\} \times_S R$.

Remark 2.6. We have $\mathrm{red}_{\mathscr{X}'} = \tilde{\iota}' \circ \mathrm{red}_{\mathscr{X}''}$ and $\tilde{\iota}'(\{\tilde{p}\} \times_S R) = \{\tilde{p}\}$. It follows that $(\mathrm{red}_{\mathscr{X}'})^{-1}(\tilde{p}) \supset (\mathrm{red}_{\mathscr{X}''})^{-1}(\{\tilde{p}\} \times_S R)$, and thus

$$\mathrm{red}_{\mathscr{X}''}(X'_+(\tilde{p})) = \mathrm{red}_{\mathscr{X}''}((\mathrm{red}_{\mathscr{X}'})^{-1}(\tilde{p})) \supset \{\tilde{p}\} \times_S R.$$

Next we show that the $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-action on $X'_+(\tilde{p})$ defined above is compatible with the action of $(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$ on $\{\tilde{p}\} \times_S R$ with respect to the reduction map $\mathrm{red}_{\mathscr{X}''}$. To be precise, we show the following lemma.

[10] We have $X'_+(\tilde{p}) = (\mathscr{U}')^{\mathrm{an}}_+(\tilde{p})$ in fact.

LEMMA 2.7. *With the notation above, we have a commutative diagram*

$$\begin{tikzcd}
(\mathbb{G}_m^r)_1^{\mathrm{an}} \times (\mathrm{red}_{\mathscr{X}''})^{-1}(\{\tilde{p}\} \times_S R) \arrow[r] \arrow[d, "\mathrm{red}_{(\mathbb{G}_m^r)_1^{\text{f-sch}} \times \mathscr{X}''}"'] & (\mathrm{red}_{\mathscr{X}''})^{-1}(\{\tilde{p}\} \times_S R) \arrow[d, "\mathrm{red}_{\mathscr{X}''}"] \\
(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}} \times (\{\tilde{p}\} \times_S R) \arrow[r] & \{\tilde{p}\} \times_S R
\end{tikzcd}$$

where the first row is the restriction of the $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-action on $X'_+(\tilde{p})$, and the second row is the $(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$-action on $\{\tilde{p}\} \times_S R$ (cf. Remark 2.6).

Proof. Recall that we have an isomorphism $p_1^{\mathrm{an}} \circ \psi^{\mathrm{an}} : S(\mathscr{X}') \cong S(\mathscr{S}_1)$ (cf. (2.2)), where $p_1 : \mathscr{S} \to \mathscr{S}_1$ is the canonical projection. Let $\mathscr{D}_1$ be the subdivision of $S(\mathscr{S}_1)$ induced from $\mathscr{D}$ via the isomorphism $p_1^{\mathrm{an}} \circ \psi^{\mathrm{an}}$ between the skeletons, and let $\mathscr{S}'_1$ be the formal scheme corresponding to the subdivision $\mathscr{D}_1$ of $S(\mathscr{S}_1)$. As in [Gub10, Remark 5.6], we have a cartesian diagram,

$$\begin{tikzcd}
\mathscr{X}'' \arrow[r, "\psi'"] \arrow[d, "\iota'"'] & \mathscr{S}' \arrow[d, "\iota"] \\
\mathscr{X}' \arrow[r, "\psi"] & \mathscr{S}
\end{tikzcd}$$

where ι is the morphism $\mathscr{S}'_1 \times \mathscr{S}_2 \to \mathscr{S}_1 \times \mathscr{S}_2$ given by the base-change of the morphism $\mathscr{S}'_1 \to \mathscr{S}_1$ arising from the subdivision $\mathscr{D}_1$.

We set $u_1 := p_1^{\mathrm{an}}(\psi^{\mathrm{an}}(u))$, which is a vertex of $\mathscr{D}_1$ and is in the interior of $\Delta' = S(\mathscr{S}_1)$. Let T'_1 be the stratum of $\tilde{\mathscr{S}}'_1$ corresponding to u_1. Recall that $\tilde{\psi}(\tilde{p}) = (\tilde{o}, \tilde{c})$. From Remark 2.6 and the proof of [Gub10, Proposition 5.7], we obtain a commutative diagram,

$$\begin{tikzcd}
X'_+(\tilde{p}) \arrow[d, "\psi^{\mathrm{an}}|_{X'_+(\tilde{p})}"'] & (\mathrm{red}_{\mathscr{X}''})^{-1}(\{\tilde{p}\} \times_S R) \arrow[l, "\supset"'] \arrow[r, "\mathrm{red}_{\mathscr{X}''}"] \arrow[d] & \{\tilde{p}\} \times_S R \arrow[d, "\tilde{\psi}'|_{\{\tilde{p}\} \times_S R}"] \\
\mathscr{S}'_+(\tilde{\psi}(\tilde{p})) & (\mathrm{red}_{\mathscr{S}'})^{-1}(T'_1 \times \{\tilde{c}\}) \arrow[l, "\supset"'] \arrow[r, "\mathrm{red}_{\mathscr{S}'}"] & T'_1 \times \{\tilde{c}\}
\end{tikzcd} \tag{2.5}$$

where the first column is the isomorphism (2.4), and the left rows are inclusions.

By (17) in the proof of [Gub10, Proposition 5.7], we have $\tilde{\psi}'^{-1}(T'_1 \times \tilde{\mathscr{S}}_2) = R$. It follows that the last column $\{\tilde{p}\} \times_S R \to T'_1 \times \{\tilde{c}\}$ in (2.5) is an isomorphism. Furthermore, it follows that the middle column is also an isomorphism.

Noting (14) in the proof of [Gub10, Proposition 5.7], we find that $(\mathrm{red}_{\mathscr{S}'})^{-1}(T'_1 \times \{\tilde{c}\})$ is $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-invariant. It follows that $(\mathrm{red}_{\mathscr{X}''})^{-1}(\{\tilde{p}\} \times_S R)$ is $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-invariant. Since the isomorphism in the middle (respectively, right) column is $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-equivariant (respectively, $(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$-equivariant) and since $\mathrm{red}_{\mathscr{S}'}$ is equivariant with respect to the reduction map $(\mathbb{G}_m^r)_1^{\mathrm{an}} \to (\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$, the right upper row is also equivariant with respect to the reduction map $(\mathbb{G}_m^r)_1^{\mathrm{an}} \to (\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$. We conclude that Lemma 2.7 holds. □

Remark 2.8. The $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-action on $X'_+(\tilde{p})$ preserves $(\mathrm{red}_{\mathscr{X}''}|_{X'_+(\tilde{p})})^{-1}(\{\tilde{p}\} \times_S R)$, as is shown above.

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3. Raynaud extension and Mumford models

In this section, we put together some properties of the Raynaud extensions of abelian varieties and their Mumford models, which will be used later. We recall, in § 3.1 and § 3.3, basic facts discussed in [Gub10, § 4]. We show some properties of the torus rank and the abelian rank of an abelian variety in § 3.2.

3.1 Raynaud extension and the valuation map

We recall some notions on Raynaud extensions; see [BL91, § 1] and [Gub10, § 4] for details.

Let A be an abelian variety over $\mathbb{K}$. By [BL91, Theorem 1.1], there exists a unique analytic subgroup $A^\circ \subset A^{\mathrm{an}}$ with a formal model $\mathscr{A}^\circ$ having the following properties.

- $\mathscr{A}^\circ$ is a formal group scheme and $(\mathscr{A}^\circ)^{\mathrm{an}} \cong A^\circ$ as analytic groups.
- There is a short exact sequence

$$1 \longrightarrow \mathscr{T}^\circ \longrightarrow \mathscr{A}^\circ \longrightarrow \mathscr{B} \longrightarrow 0 \tag{3.1}$$

where $\mathscr{T}^\circ \cong (\mathbb{G}_m^n)_1^{\text{f-sch}}$ for some $n \geqslant 0$, and $\mathscr{B}$ is a formal abelian variety over $\mathbb{K}^\circ$.

By [Bos76, Satz 1.1], such an $\mathscr{A}^\circ$ is unique, and $\mathscr{T}^\circ$ and $\mathscr{B}$ are also uniquely determined. Taking the generic fiber of (3.1), we obtain an exact sequence

$$1 \longrightarrow T^\circ \longrightarrow A^\circ \xrightarrow{(q^\circ)^{\mathrm{an}}} B \longrightarrow 0 \tag{3.2}$$

of analytic groups. We call $\mathscr{T}^\circ$, $\mathscr{A}^\circ$ and $\mathscr{B}$ the *canonical formal models* of T°, A° and B, respectively.

Since $T^\circ \cong (\mathbb{G}_m^n)_1^{\mathrm{an}}$, we regard T° as an analytic subgroup of the analytic torus $T = (\mathbb{G}_m^n)^{\mathrm{an}}$. Pushing (3.2) out by $T^\circ \hookrightarrow T$, we obtain an exact sequence

$$1 \longrightarrow T \longrightarrow E \xrightarrow{q^{\mathrm{an}}} B \longrightarrow 0 \tag{3.3}$$

which we call the *Raynaud extension* of A. The natural morphism $A^\circ \to E$ is an immersion of analytic groups. The assertion [BL91, Theorem 1.2] says that the homomorphism $T^\circ \hookrightarrow A^{\mathrm{an}}$ extends uniquely to a homomorphism $T \to A^{\mathrm{an}}$ and hence to a homomorphism $p^{\mathrm{an}} : E \to A^{\mathrm{an}}$. It is known that p^{an} is a surjective homomorphism and $M := \operatorname{Ker} p^{\mathrm{an}}$ is a lattice in $E(\mathbb{K})$. Thus A^{an} can be described as a quotient of E by a lattice. This p^{an} is called the *uniformization* of A.

The dimension of T is called the *torus rank* of A, and the dimension of B is called the *abelian rank* of A. We denote the abelian rank of A by $b(A)$. Note that the torus rank of A equals $\dim A - b(A)$. The abelian variety A is said to be *degenerate* if $b(A) < \dim A$, or equivalently if the torus rank of A is positive. Note that 'being non-degenerate' means 'having good reduction'.

We can take transition functions of the T-torsor (3.3) valued in T°, and thus we can define a continuous map

$$\mathrm{val} : E \to \mathbb{R}^n \tag{3.4}$$

as in [BL91], where n is the torus rank of A. We recall here how it is constructed. Fix an isomorphism $T^\circ \cong (\mathbb{G}_m^n)_1^{\mathrm{an}}$, with the standard coordinates $x_1, \ldots, x_n$. We can take a covering $\{V\}$ of B consisting of rational subdomains with trivializations

$$((q^\circ)^{\mathrm{an}})^{-1}(V) \cong V \times (\mathbb{G}_m^n)_1^{\mathrm{an}} \tag{3.5}$$

as $(\mathbb{G}_m^n)_1^{\mathrm{an}}$-torsors for all V. Since the Raynaud extension is the push-out of (3.2) by the canonical inclusion $(\mathbb{G}_m^n)_1^{\mathrm{an}} \hookrightarrow (\mathbb{G}_m^n)^{\mathrm{an}}$, the isomorphisms (3.5) extend to isomorphisms

$$(q^{\mathrm{an}})^{-1}(V) \cong V \times (\mathbb{G}_m^n)^{\mathrm{an}} \tag{3.6}$$

of $(\mathbb{G}_m^n)^{\mathrm{an}}$-torsors. Thus we obtain morphisms

$$r_V : (q^{\mathrm{an}})^{-1}(V) \cong V \times (\mathbb{G}_m^n)^{\mathrm{an}} \to (\mathbb{G}_m^n)^{\mathrm{an}} \tag{3.7}$$

for all V by composing with the second projection. A different choice of (3.5) gives a different isomorphism in (3.6) and hence a different morphism in (3.7) for each V, but the difference is only the multiplication of an element of $(\mathbb{G}_m^n)_1^{\mathrm{an}}$. Therefore, the morphisms $(q^{\mathrm{an}})^{-1}(V) \to \mathbb{R}^n$ given by

$$e \mapsto (-\log r_V(e)(x_1), \ldots, -\log r_V(e)(x_n))$$

patch together to form a morphism from E to $\mathbb{R}^n$. This is our valuation map $\mathrm{val} : E \to \mathbb{R}^n$.

We set $\Lambda := \mathrm{val}(M) \subset \mathbb{R}^n$, where we recall that $M = \operatorname{Ker} p^{\mathrm{an}}$ is a lattice of E. Then Λ is a complete lattice in $\mathbb{R}^n$ and is contained in Γ^n. There is a diagram

$$\begin{tikzcd} E \arrow[r, "\mathrm{val}"] \arrow[d] & \mathbb{R}^n \arrow[d] \\ A^{\mathrm{an}} \arrow[r, "\overline{\mathrm{val}}"] & \mathbb{R}^n/\Lambda \end{tikzcd}$$

that commutes. The homomorphism $\overline{\mathrm{val}}$ is also called the *valuation map*. It follows from the constructions of val and $\overline{\mathrm{val}}$ that $A^\circ = \overline{\mathrm{val}}^{-1}(\overline{\mathbf{0}}) \cong \mathrm{val}^{-1}(\mathbf{0})$.

3.2 Homomorphism, products and abelian ranks

Let A_1 and A_2 be abelian varieties over $\mathbb{K}$ and let

$$1 \longrightarrow T_i \longrightarrow E_i \xrightarrow{q_i^{\mathrm{an}}} B_i \longrightarrow 0$$

be the Raynaud extension of A_i for $i = 1, 2$. We consider a homomorphism $\phi : A_1 \to A_2$. Then [BL86, Proposition 2.2] tells us that ϕ induces a homomorphism $A_1^\circ \to A_2^\circ$ and hence a homomorphism $T_1^\circ \to T_2^\circ$. Thus we have an induced homomorphism $\phi_{ab} : B_1 \to B_2$ between the abelian parts. Furthermore, it follows from [BL91, Theorem 1.2] that ϕ gives rise to a homomorphism $\Phi : E_1 \to E_2$.

It follows from the construction of the valuation map that Φ descends to a linear map $\phi_{\mathrm{aff}} : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ via the valuation maps, where n_1 and n_2 are the torus ranks of A_1 and A_2, respectively. Further, $\phi_{\mathrm{aff}}(\Lambda_1) \subset \Lambda_2$ holds. Thus we obtain a homomorphism $\overline{\phi}_{\mathrm{aff}} : \mathbb{R}^{n_1}/\Lambda_1 \to \mathbb{R}^{n_2}/\Lambda_2$ which makes the diagram

$$\begin{tikzcd} A_1^{\mathrm{an}} \arrow[r, "\phi"] \arrow[d] & A_2^{\mathrm{an}} \arrow[d] \\ \mathbb{R}^{n_1}/\Lambda_1 \arrow[r, "\overline{\phi}_{\mathrm{aff}}"] & \mathbb{R}^{n_2}/\Lambda_2 \end{tikzcd} \tag{3.8}$$

commutative.

Next we consider the direct product. We set $A := A_1 \times A_2$. The Raynaud extensions of A_1 and A_2 gives rise to an exact sequence

$$1 \to T_1^\circ \times T_2^\circ \to A_1^\circ \times A_2^\circ \to B_1 \times B_2 \to 0.$$

It follows from their definitions that $A^\circ = A_1^\circ \times A_2^\circ$ and $T^\circ = T_1^\circ \times T_2^\circ$ hold, and that

$$1 \to T_1 \times T_2 \to E_1 \times E_2 \to B_1 \times B_2 \to 0 \tag{3.9}$$

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is the Raynaud extension of A. Further,

$$\overline{\mathrm{val}} : (A_1 \times A_2)^{\mathrm{an}} \to \mathbb{R}^{n_1+n_2}/(\Lambda_1 \oplus \Lambda_2)$$

coincides with the map

$$A_1^{\mathrm{an}} \times A_2^{\mathrm{an}} \to \mathbb{R}^{n_1}/\Lambda_1 \times \mathbb{R}^{n_2}/\Lambda_2$$

given by the product of $\overline{\mathrm{val}}_1 : A_1^{\mathrm{an}} \to \mathbb{R}^{n_1}/\Lambda_1$ and $\overline{\mathrm{val}}_2 : A_2^{\mathrm{an}} \to \mathbb{R}^{n_2}/\Lambda_2$.

We show some properties of abelian ranks which will be used later.

Lemma 3.1. *Let A_1 and A_2 be abelian varieties over $\mathbb{K}$.*

(1) *We have $b(A_1 \times A_2) = b(A_1) + b(A_2)$.*

(2) *Suppose that $\phi : A_1 \to A_2$ is an isogeny. Then $b(A_1) = b(A_2)$.*

Proof. The equality in (1) follows from the fact that (3.9) is the Raynaud extension of $A_1 \times A_2$.

To show (2), note that there is an isogeny $A_2 \to A_1$ as well as an isogeny $A_1 \to A_2$ (cf. [Mum08, p. 157, Remark]). Thus it is enough to show that $b(A_1) \geqslant b(A_2)$, so that it suffices to show that the homomorphism $\phi_{ab} : B_1 \to B_2$ induced from ϕ is surjective.

Let $y \in B_2$ be a point. Since $q_2^{\mathrm{an}}|_{A_2^\circ} : A_2^\circ \to B_2$, $\phi^{\mathrm{an}} : A_1^{\mathrm{an}} \to A_2^{\mathrm{an}}$ and $p_1 : E_1 \to A_1^{\mathrm{an}}$ are surjective, there exists a point $x \in E_1$ with $q_2^{\mathrm{an}}(\phi^{\mathrm{an}}(p_1(x))) = y$. Then we have $\phi_{ab}(q_1^{\mathrm{an}}(x)) = y$, which implies that ϕ_{ab} is surjective. □

Remark 3.2. If $\phi : A_1 \to A_2$ is an isogeny, then ϕ_{aff} is an isomorphism of vector spaces, and thus $\overline{\phi}_{\mathrm{aff}}$ is a finite surjective homomorphism.

Proposition 3.3. *Let*

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

be an exact sequence of abelian varieties over $\mathbb{K}$. Then we have $b(A_2) = b(A_1) + b(A_3)$ and $n_2 = n_1 + n_3$, where n_i denote the torus rank of A_i.

Proof. Recall that Poincaré's complete reducibility theorem (cf. [Mil86, Proposition 12.1] or [Mum08, § 19, Theorem 1]) gives us an abelian subvariety A' of A_2 such that the natural homomorphism $A_1 \times A' \to A_2$ given by $(a_1, a') \mapsto a_1 + a'$ is an isogeny. Then $b(A_2) = b(A_1 \times A') = b(A_1) + b(A')$ by Lemma 3.1. Since the composite $A' \hookrightarrow A_2 \to A_3$ is an isogeny, we have $b(A') = b(A_3)$ by Lemma 3.1(2). It follows that $b(A_2) = b(A_1) + b(A_3)$, which shows one equality of this proposition. The equality for torus ranks follows from that for abelian ranks. □

3.3 Mumford models, Torus torsors, and initial degenerations

In this subsection we recall some properties of Mumford models associated to a Λ-periodic Γ-rational polytopal decomposition. The basic reference is [Gub10, § 4]. We also define the initial degeneration of a closed subvariety of an abelian variety.

Let A be an abelian variety over $\mathbb{K}$, and let

$$1 \longrightarrow T \longrightarrow E \xrightarrow{q^{\mathrm{an}}} B \longrightarrow 0$$

be the Raynaud extension of A. We use the notation in § 3.1: the affinoid torus T° is the maximal affinoid subtorus of T; the morphism $p^{\mathrm{an}} : E \to A^{\mathrm{an}}$ is the uniformization; the map $\mathrm{val} : E \to \mathbb{R}^n$ is the valuation map, where n is the torus rank of A; further, $\Lambda := \mathrm{val}(\mathrm{Ker}\, p^{\mathrm{an}})$ is the lattice in $\mathbb{R}^n$, and $\overline{\mathrm{val}} : A \to \mathbb{R}^n/\Lambda$ is the valuation map induced from val by quotient.

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Let $\mathscr{C}$ be a Λ-periodic Γ-rational polytopal decomposition of $\mathbb{R}^n$ (cf. § 1.1). For a polytope $\Delta \in \mathscr{C}$, the subset $\mathrm{val}^{-1}(\Delta) \subset E$ is an analytic subdomain, and there exists a natural surjective morphism

$$q_\Delta^{\mathrm{an}} := q^{\mathrm{an}}|_{\mathrm{val}^{-1}(\Delta)} : \mathrm{val}^{-1}(\Delta) \to B.$$

Since val is invariant under the action of T°, we have a natural T°-action on $\mathrm{val}^{-1}(\Delta)$, which is an action over B with respect to q_Δ^{an}.

Let $\overline{\mathscr{C}}$ denote the polytopal decomposition of $\mathbb{R}^n/\Lambda$ induced from $\mathscr{C}$ by the quotient and let $\overline{\Delta} \in \overline{\mathscr{C}}$ be a polytope. Then $\overline{\mathrm{val}}^{-1}(\overline{\Delta})$ is an analytic subdomain of A^{an} with a T°-action. Fix a representative $\Delta \in \mathscr{C}$ of $\overline{\Delta}$. The uniformization map $p^{\mathrm{an}} : E \to A^{\mathrm{an}}$ restricts to an isomorphism $p^{\mathrm{an}}|_{\mathrm{val}^{-1}(\Delta)} : \mathrm{val}^{-1}(\Delta) \to \overline{\mathrm{val}}^{-1}(\overline{\Delta})$, via which we define

$$\overline{q_\Delta^{\mathrm{an}}} := q_\Delta^{\mathrm{an}} \circ (p^{\mathrm{an}}|_{\mathrm{val}^{-1}(\Delta)})^{-1} : \overline{\mathrm{val}}^{-1}(\overline{\Delta}) \to B.$$

The T°-action on $\overline{\mathrm{val}}^{-1}(\overline{\Delta})$ is a T°-action over B.

There are open subsets $\mathrm{val}^{-1}(\mathrm{relin}(\Delta)) \subset \mathrm{val}^{-1}(\Delta)$ and $\overline{\mathrm{val}}^{-1}(\mathrm{relin}(\overline{\Delta})) \subset \overline{\mathrm{val}}^{-1}(\overline{\Delta})$ with T°-actions, where $\mathrm{relin}(\Delta)$ and $\mathrm{relin}(\overline{\Delta})$ are the relative interior of Δ and $\overline{\Delta}$, respectively. We have morphisms

$$q_\Delta^{\mathrm{an}}|_{\mathrm{val}^{-1}(\mathrm{relin}(\Delta))} : \mathrm{val}^{-1}(\mathrm{relin}(\Delta)) \to B, \quad \overline{q_\Delta^{\mathrm{an}}}|_{\overline{\mathrm{val}}^{-1}(\mathrm{relin}(\overline{\Delta}))} : \overline{\mathrm{val}}^{-1}(\mathrm{relin}(\overline{\Delta})) \to B \tag{3.10}$$

with T°-actions over B.

The *Mumford model* $p : \mathscr{E} \to \mathscr{A}$ associated to $\mathscr{C}$ is constructed in [Gub10]. It is a formal model of $p^{\mathrm{an}} : E \to A^{\mathrm{an}}$, and there exists a unique morphism $q : \mathscr{E} \to \mathscr{B}$ extending the morphism $q^{\mathrm{an}} : E \to B$ in the Raynaud extension. Although we do not repeat the precise definition of it here, we recall some properties which will be used later. See [Gub10, § 4 (especially 4.7)] for more details.

For a formal affine open subset $\mathscr{V}$ of $\mathscr{B}$, its generic fiber $V = \mathscr{V}^{\mathrm{an}}$ is an analytic subdomain of B. Such an analytic subdomain is called a *formal affinoid subdomain* of B. Using this notion, we can write $\mathrm{val}^{-1}(\Delta) = \bigcup_V (\mathrm{val}^{-1}(\Delta) \cap (q^{\mathrm{an}})^{-1}(V))$, where V runs through the formal affinoid subdomains of B. Further, we have

$$\mathrm{val}^{-1}(\Delta) \cap (q^{\mathrm{an}})^{-1}(V) \cong U_\Delta \times V,$$

where U_Δ is the rational subdomain of $(\mathbb{G}_m^n)^{\mathrm{an}}$ as in § 1.3. We set $U_{\Delta,V} := U_\Delta \times V$. Then $\mathscr{E}$ has a suitable formal affine open covering by the sets $\mathscr{U}_{V,\Delta}$, where $\mathscr{U}_{V,\Delta}$ satisfies $(\mathscr{U}_{V,\Delta})^{\mathrm{an}} = U_{V,\Delta}$. We set $\mathscr{E}_\Delta := \bigcup_V \mathscr{U}_{V,\Delta}$, where V runs through the formal affinoid subdomains of B. We have a natural morphism $q|_{\mathscr{E}_\Delta} : \mathscr{E}_\Delta \to \mathscr{B}$ by restriction. The restriction of $p : \mathscr{E} \to \mathscr{A}$ to $\mathscr{E}_\Delta$ is an isomorphism onto its image $\mathscr{A}_{\overline{\Delta}} := p(\mathscr{E}_\Delta)$. Using the isomorphism $p|_{\mathscr{E}_\Delta}$, we define

$$\overline{q_\Delta} := q|_{\mathscr{E}_\Delta} \circ (p|_{\mathscr{E}_\Delta})^{-1} : \mathscr{A}_{\overline{\Delta}} \to \mathscr{B}.$$

We notice that $\overline{q_\Delta}$ depends not only on $\overline{\Delta}$ but also on the choice of a representative Δ of $\overline{\Delta}$ over $\mathscr{B}$.

The T°-action on E over B extends to the $\mathscr{T}^\circ$-action on $\mathscr{E}$ over $\mathscr{B}$, where $\mathscr{T}^\circ$ is the canonical model of T° (cf. § 3.1). The $\mathscr{T}^\circ$-action on $\mathscr{E}$ restricts to a $\mathscr{T}^\circ$-action on $\mathscr{E}_\Delta$ over $\mathscr{B}$. Via the isomorphism $p|_{\mathscr{E}_\Delta} : \mathscr{E}_\Delta \to \mathscr{A}_{\overline{\Delta}}$, we have a $\mathscr{T}^\circ$-action on $\mathscr{A}_{\overline{\Delta}}$ over $\mathscr{B}$. Note that $\overline{\mathrm{val}}^{-1}(\overline{\Delta}) = (\mathscr{A}_{\overline{\Delta}})^{\mathrm{an}}$ as an analytic subspace of A^{an} and that this $\mathscr{T}^\circ$-action on $\mathscr{A}_{\overline{\Delta}}$ induces the canonical T°-action on $\overline{\mathrm{val}}^{-1}(\overline{\Delta})$ on the Raynaud generic fiber.

We recall that [Gub10, Proposition 4.8] gives us a bijective correspondence between the strata of $\tilde{\mathscr{E}}$ and the set of relative interiors of the polytopes in $\mathscr{C}$, and a bijective correspondence between the strata of $\tilde{\mathscr{A}}$ and the set of relative interiors of the polytopes in $\overline{\mathscr{C}}$. In fact, if we set $Z_{\mathrm{relin}(\Delta)} := \mathrm{red}_{\mathscr{E}}(\mathrm{val}^{-1}(\mathrm{relin}(\Delta)))$ and $Z_{\mathrm{relin}(\overline{\Delta})} := \mathrm{red}_{\mathscr{A}}(\overline{\mathrm{val}}^{-1}(\mathrm{relin}(\overline{\Delta})))$, then $Z_{\mathrm{relin}(\Delta)}$ and $Z_{\mathrm{relin}(\overline{\Delta})}$ are the strata of $\tilde{\mathscr{E}}$ and $\tilde{\mathscr{A}}$ corresponding to $\mathrm{relin}(\Delta)$ and $\mathrm{relin}(\overline{\Delta})$ respectively via these bijective correspondences.

Taking the reductions of the morphisms in (3.10), we obtain surjective morphisms

$$\tilde{q_\Delta}|_{Z_{\mathrm{relin}(\Delta)}} : Z_{\mathrm{relin}(\Delta)} \to \tilde{\mathscr{B}}, \quad \tilde{\overline{q_\Delta}}|_{Z_{\mathrm{relin}(\overline{\Delta})}} : Z_{\mathrm{relin}(\overline{\Delta})} \to \tilde{\mathscr{B}}. \tag{3.11}$$

The torus $\tilde{\mathscr{T}}^\circ$ over $\tilde{\mathbb{K}}$ acts on $Z_{\mathrm{relin}(\Delta)}$ and $Z_{\mathrm{relin}(\overline{\Delta})}$, and these actions are over $\tilde{\mathscr{B}}$ with respect to $\tilde{q_\Delta}$ and $\tilde{\overline{q_\Delta}}|_{Z_{\mathrm{relin}(\overline{\Delta})}}$, respectively. If Δ consists of a single point w, then we write Z_w and $Z_{\overline{w}}$, etc. instead of $Z_{\{w\}}$ and $Z_{\{\overline{w}\}}$, etc. for simplicity.

Remark 3.4. It follows from [Gub10, Remark 4.9] that $\tilde{q_w} : Z_w \to \tilde{\mathscr{B}}$ and $\tilde{\overline{q_w}} : Z_{\overline{w}} \to \tilde{\mathscr{B}}$ are $\tilde{\mathscr{T}}^\circ$-torsors.

We have seen that the affinoid torus T° acts on $\overline{\mathrm{val}}^{-1}(\mathrm{relin}(\overline{\Delta}))$ and that the algebraic torus $\tilde{\mathscr{T}}^\circ$ acts on the reduction $Z_{\mathrm{relin}(\overline{\Delta})}$. These actions are compatible with respect to the reduction map. To be precise, we have the following lemma.

LEMMA 3.5. *With the notation above, the diagram,*

$$\begin{tikzcd}
T^\circ \times \overline{\mathrm{val}}^{-1}(\mathrm{relin}(\overline{\Delta})) \arrow[r] \arrow[d, "\mathrm{red}_{\mathscr{T}^\circ \times \mathscr{A}}"'] & \overline{\mathrm{val}}^{-1}(\mathrm{relin}(\overline{\Delta})) \arrow[d, "\mathrm{red}_{\mathscr{A}}"] \\
\tilde{\mathscr{T}}^\circ \times Z_{\mathrm{relin}(\overline{\Delta})} \arrow[r] & Z_{\mathrm{relin}(\overline{\Delta})}
\end{tikzcd}$$

where the first row is the T°-action and the second row is the $\tilde{\mathscr{T}}^\circ$-action, is commutative.

Proof. Let $\mu : \mathscr{T}^\circ \times \mathscr{A}_{\overline{\Delta}} \to \mathscr{A}_{\overline{\Delta}}$ denote the $\mathscr{T}^\circ$-action recalled above. Then the T°-action on $\overline{\mathrm{val}}^{-1}(\mathrm{relin}(\overline{\Delta}))$ is induced from μ by taking the Raynaud generic fiber, and the $\tilde{\mathscr{T}}^\circ$-action on $Z_{\mathrm{relin}(\overline{\Delta})}$ is induced from μ by taking the special fiber. It follows from the definition of reduction maps that the diagram is commutative. □

Let A_1 and A_2 be abelian varieties over $\mathbb{K}$ and let $\phi : A_1 \to A_2$ be a homomorphism. Let n_i be the torus rank of A_i, let $\overline{\mathrm{val}_i} : A_i^{\mathrm{an}} \to \mathbb{R}^{n_i}/\Lambda_i$ be the valuation map for $i = 1, 2$, and let $\overline{\phi}_{\mathrm{aff}} : \mathbb{R}^{n_1}/\Lambda_1 \to \mathbb{R}^{n_2}/\Lambda_2$ be the homomorphism as in (3.8). Let $\overline{\mathscr{C}_i}$ be a Γ-rational polytopal decomposition of $\mathbb{R}^{n_i}/\Lambda_i$ and let $\mathscr{A}_i$ be the Mumford model associated to $\overline{\mathscr{C}_i}$. Then ϕ extends to a morphism $\mathscr{A}_1 \to \mathscr{A}_2$ if, for any $\overline{\Delta_1} \in \overline{\mathscr{C}_1}$, there exists a polytope $\overline{\Delta_2} \in \overline{\mathscr{C}_2}$ such that $\overline{\phi}_{\mathrm{aff}}(\overline{\Delta_1}) \subset \overline{\Delta_2}$. In fact, for any $\overline{\Delta_1} \in \overline{\mathscr{C}_1}$ and $\overline{\Delta_2} \in \overline{\mathscr{C}_2}$ with $\overline{\phi}_{\mathrm{aff}}(\overline{\Delta_1}) \subset \overline{\Delta_2}$, we can construct a unique morphism $(\mathscr{A}_1)_{\overline{\Delta_1}} \to (\mathscr{A}_2)_{\overline{\Delta_2}}$ which extends the morphism $\phi|_{\overline{\mathrm{val}_1}^{-1}(\overline{\Delta_1})} : \overline{\mathrm{val}_1}^{-1}(\overline{\Delta_1}) \to \overline{\mathrm{val}_2}^{-1}(\overline{\Delta_2})$, and these morphisms patch together to form a morphism $\mathscr{A}_1 \to \mathscr{A}_2$ extending ϕ.

Finally in this subsection, we define the notion of initial degenerations of a closed subvariety X of A. For any Γ-rational polytope $\overline{\Delta}$ of $\mathbb{R}^n/\Lambda$, let $\mathscr{X}_{\overline{\Delta}}$ be the closure of $X^{\mathrm{an}} \cap \overline{\mathrm{val}}^{-1}(\overline{\Delta})$ in $\mathscr{A}_{\overline{\Delta}}$. We define the *initial degeneration* $\mathrm{in}_{\overline{\Delta}}(X)$ *of* X *over* $\overline{\Delta}$ by

$$\mathrm{in}_{\overline{\Delta}}(X) := \tilde{\mathscr{X}}_{\overline{\Delta}}, \tag{3.12}$$

the special fiber of $\mathscr{X}_\Delta$. It is determined from X and $\overline{\Delta}$ and is independent of the polytopal decomposition $\mathscr{C}$ that was used before. If $\overline{\Delta} = \{\overline{w}\}$, we write $\mathrm{in}_{\overline{w}}(X)$ for $\mathrm{in}_{\overline{\Delta}}(X)$. Note that $\mathrm{in}_{\overline{w}}(X)$ is a closed subscheme of $Z_{\overline{w}}$. Moreover, since $\mathscr{X}_{\overline{\Delta}} = \mathscr{X} \cap \mathscr{A}_{\overline{\Delta}}$, we have $\mathrm{in}_{\overline{\Delta}}(X) = \tilde{\mathscr{X}} \cap \tilde{\mathscr{A}}_{\overline{\Delta}}$ and $\mathrm{in}_{\overline{w}}(X) = \tilde{\mathscr{X}} \cap Z_{\overline{w}}$.

Remark 3.6. Set $d := \dim X$. Let W be an irreducible component of $\mathrm{in}_{\overline{\Delta}}(X)$. Since a formal model of X is flat over $\mathbb{K}^\circ$, it follows from the definition of $\mathrm{in}_{\overline{\Delta}}(X)$ that $\dim W = d$.

Remark 3.7. Let W be an irreducible component of $\mathrm{in}_{\overline{\Delta}}(X)$. By the definition of $\mathrm{in}_{\overline{\Delta}}(X)$, we have the reduction map $X^{\mathrm{an}} \cap \overline{\mathrm{val}}^{-1}(\overline{\Delta}) \to \mathrm{in}_{\overline{\Delta}}(X)$, which is surjective. Thus there is a point $\xi_W \in X^{\mathrm{an}} \cap \overline{\mathrm{val}}^{-1}(\overline{\Delta})$ which maps to the generic point of W by this reduction map. Then we say that ξ_W *reduces to the generic point of* W.

4. Torus equivalence between strata and canonical simplices

We begin by fixing the notation throughout this section. Let A be an abelian variety over $\mathbb{K}$ with Raynaud extension

$$1 \longrightarrow (\mathbb{G}_m^n)^{\mathrm{an}} \longrightarrow E \xrightarrow{q^{\mathrm{an}}} B \longrightarrow 0.$$

Let $p^{\mathrm{an}} : E \to A^{\mathrm{an}}$ be the uniformization. Let $\mathrm{val} : E \to \mathbb{R}^n$ and $\overline{\mathrm{val}} : A^{\mathrm{an}} \to \mathbb{R}^n/\Lambda$ be the valuation maps with respect to the standard coordinates $x_1, \ldots, x_n$ (cf. (3.4)), where $\Lambda = \mathrm{val}(\mathrm{Ker}\, p^{\mathrm{an}})$. Fix a Λ-periodic Γ-rational polytopal decomposition $\mathscr{C}_0$ of $\mathbb{R}^n$, and let $\overline{\mathscr{C}_0}$ denote the polytopal decomposition of $\mathbb{R}^n/\Lambda$ induced from $\mathscr{C}_0$ by the quotient. Let $p_0 : \mathscr{E}_0 \to \mathscr{A}_0$ be the Mumford model associated to $\mathscr{C}_0$ (cf. § 3.3). Let $\mathscr{X}'$ be a connected strictly semistable formal scheme over $\mathbb{K}^\circ$ (cf. § 2.1) and let $\varphi_0 : \mathscr{X}' \to \mathscr{A}_0$ be a generically finite morphism. We put $X' := (\mathscr{X}')^{\mathrm{an}}$ and $d := \dim X'$. Let $f : X' \to A^{\mathrm{an}}$ be the restriction of φ_0 to the Raynaud generic fiber.

In this section we show that f induces a torus-equivariant morphism over the formal fiber over a point in a stratum of $\tilde{\mathscr{X}}'$ (cf. Lemma 4.3). Using this fact together with Lemma 2.7, we show that the morphism between some models of X' and A^{an} induces a torus-equivariant morphism between their strata (cf. Proposition 4.4). Then we give in Lemma 4.5 a sufficient condition for a canonical simplex Δ_S of $S(\mathscr{X}')$ not to be collapsed by f. This lemma will play a key role in the sequel.

4.1 Associated affine map

Recall that $\mathrm{Val} : X' \to S(\mathscr{X}')$ is the retraction map to the skeleton (cf. § 2.1). The following assertion is due to Gubler.

Proposition 4.1 [Gub10, Proposition 5.11]. *Under the setting above, there is a unique map* $\overline{f}_{\mathrm{aff}} : S(\mathscr{X}') \to \mathbb{R}^n/\Lambda$ *with* $\overline{f}_{\mathrm{aff}} \circ \mathrm{Val} = \overline{\mathrm{val}} \circ f$ *on* X'. *The map* $\overline{f}_{\mathrm{aff}}$ *is continuous. For any* $S \in \mathrm{str}(\tilde{\mathscr{X}}')$, *the restriction of* $\overline{f}_{\mathrm{aff}}$ *to the canonical simplex* Δ_S *(cf. § 2.1) is an affine map and there exists a unique* $\overline{\Delta} \in \mathscr{C}_0$ *with* $\overline{f}_{\mathrm{aff}}(\mathrm{relin}(\Delta_S)) \subset \mathrm{relin}(\overline{\Delta})$.

We recall how $\overline{f}_{\mathrm{aff}}$ is described over Δ_S. Set $r := d - \dim S$, the codimension of S in $\tilde{\mathscr{X}}'$. We take an affine open subset $\mathscr{U}' \subset \mathscr{X}'$ and an étale morphism $\psi : \mathscr{U}' \to \mathscr{S} = \mathscr{S}_1 \times \mathscr{S}_2$, where $\mathscr{S}_1 = \mathbb{K}^\circ\langle x_0', \ldots, x_r'\rangle/(x_0' \cdots x_r' - \pi)$ with $\pi \in \mathbb{K}^{\circ\circ}\setminus\{0\}$, such that S is a distinguished stratum associated to $\mathscr{U}'$ as in Proposition 2.1. Since $p^{\mathrm{an}} : E \to A^{\mathrm{an}}$ is a local isomorphism, replacing $\mathscr{U}'$ by a non-empty open subset of it, we can take a local lift $F : (\mathscr{U}')^{\mathrm{an}} \to q^{-1}(V)$ of $f : X' \to A^{\mathrm{an}}$,

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where V is a formal affinoid subdomain of B. We fix an identification $q^{-1}(V) \cong (\mathbb{G}_m^n)^{\mathrm{an}} \times V$. By [Gub07a, Proposition 2.11], we have an expression

$$F^*(x_i) = \lambda_i v_i \psi^*(x'_1)^{m_{i1}} \cdots \psi^*(x'_r)^{m_{ir}} \tag{4.1}$$

with some $\lambda_i \in \mathbb{K}^\times$, $v_i \in \mathcal{O}(\mathscr{U}')^\times$ and $\mathbf{m}_i = (m_{i1}, \ldots, m_{ir}) \in \mathbb{Z}^r$. We define $f_{\mathrm{aff}} : S(\mathscr{U}') \to \mathbb{R}^n$, via the identification

$$S(\mathscr{U}') = \Delta_S = \{(u'_1, \ldots, u'_r) \in \mathbb{R}^r_{\geqslant 0} \mid u'_1 + \cdots + u'_r \leqslant v(\pi)\}$$

in (2.3), by

$$f_{\mathrm{aff}}(\mathbf{u}') = (\mathbf{m}_1 \cdot \mathbf{u}' + v(\lambda_1), \ldots, \mathbf{m}_n \cdot \mathbf{u}' + v(\lambda_n)), \quad \mathbf{u}' \in \Delta_S = S(\mathscr{U}'). \tag{4.2}$$

Then $\overline{f}_{\mathrm{aff}}$ is the composite of $f_{\mathrm{aff}} : S(\mathscr{U}') \to \mathbb{R}^n$ with the quotient $\mathbb{R}^n \to \mathbb{R}^n/\Lambda$.

Let $\operatorname{rk} \overline{f}_{\mathrm{aff}}$ denote the rank of the linear part of f_{aff}, i.e. the rank of the matrix (m_{ij}). Note that $\operatorname{rk} \overline{f}_{\mathrm{aff}}$ does not depend on the choice of a lift F or the identification $q^{-1}(V) \cong (\mathbb{G}_m^n)^{\mathrm{an}} \times V$, and thus is well defined from f and Δ_S.

4.2 Torus equivariance between strata

Let S be a stratum of $\tilde{\mathscr{X}}'$ of codimension r and let Δ_S be the canonical simplex corresponding to S. Let m_{ij}, for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant r$, be integers which give the linear part of f_{aff} (cf. (4.2)). We define a homomorphism $h_{f,\Delta_S} : (\mathbb{G}_m^r)_1^{\mathrm{an}} \to (\mathbb{G}_m^n)_1^{\mathrm{an}}$ by

$$h^*_{f,\Delta_S}(x_i) = (x'_1)^{m_{i1}} \cdots (x'_r)^{m_{ir}}, \quad i = 1, \ldots, n.$$

Since $(\mathbb{G}_m^n)^{\mathrm{an}}$ is the torus part of the Raynaud extension of A, the affinoid torus $(\mathbb{G}_m^n)_1^{\mathrm{an}}$ acts on A^{an}. Then we make $(\mathbb{G}_m^r)_1^{\mathrm{an}}$ act on A^{an} via h_{f,Δ_S}.

Remark 4.2. Let $\tilde{h}_{f,\Delta_S} : (\mathbb{G}_m^r)_{\tilde{\mathbb{K}}} \to (\mathbb{G}_m^n)_{\tilde{\mathbb{K}}}$ be the homomorphism given by the reduction of h_{f,Δ_S}. Then the image $\tilde{h}_{f,\Delta_S}((\mathbb{G}_m^r)_{\tilde{\mathbb{K}}})$ is a subtorus of $(\mathbb{G}_m^n)_{\tilde{\mathbb{K}}}$ of $\dim \overline{f}_{\mathrm{aff}}(\Delta_S)$, which is the rank of (the linear part of) f_{aff}.

Lemma 4.3. *With the notation above, let $\tilde{p} \in S$ be a closed point and let $X'_+(\tilde{p}) := \mathrm{red}^{-1}_{\mathscr{X}'}(\tilde{p})$ be the formal fiber over $\tilde{p}$. Then the morphism*

$$f|_{X'_+(\tilde{p})} : X'_+(\tilde{p}) \to A^{\mathrm{an}}$$

is $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-equivariant with respect to the $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-action on $X'_+(\tilde{p})$ given in § 2.2 and that on A^{an} induced by h_{f,Δ_S}.

Proof. We take a formal affine open subscheme $\mathscr{U}' \subset \mathscr{X}'$ such that $\tilde{p} \in \tilde{\mathscr{U}}'$. We then note that $X'_+(\tilde{p}) \subset (\mathscr{U}')^{\mathrm{an}}$. Replacing $\mathscr{U}'$ by a formal open subscheme containing $\tilde{p}$, we have a local lift F of f as in § 4.1, so that we have a commutative diagram,

$$\begin{tikzcd} (\mathscr{U}')^{\mathrm{an}} \arrow[r, "F"] & (\mathbb{G}_m^n)^{\mathrm{an}} \times V \arrow[d, "p^{\mathrm{an}}"] \\ X'_+(\tilde{p}) \arrow[u, "\cup"] \arrow[r, "f"] & A^{\mathrm{an}} \end{tikzcd}$$

where V is a formal affinoid subdomain of B, and $(\mathbb{G}_m^n)^{\mathrm{an}} \times V$ is regarded as a subdomain of E.

Since the right column in the above diagram is $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-equivariant, it suffices to show that F is a $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-equivariant morphism. Recall that we have an isomorphism $\psi'^{\mathrm{an}}|_{X'_+(\tilde{p})} : X'_+(\tilde{p}) \to (\mathscr{S}^{\mathrm{an}})_+(\tilde{\psi}(\tilde{p}))$ in (2.4). Let G be the composite

$$(\mathscr{S}^{\mathrm{an}})_+(\tilde{\psi}(\tilde{p})) \xrightarrow{(\psi'^{\mathrm{an}}|_{X'_+(\tilde{p})})^{-1}} X'_+(\tilde{p}) \xrightarrow{F} (\mathbb{G}_m^n)^{\mathrm{an}} \times V \xrightarrow{pr_1} (\mathbb{G}_m^n)^{\mathrm{an}},$$

where pr_1 is the first projection. We note that if G is $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-equivariant, then F is also $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-equivariant. Indeed, suppose that G is $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-equivariant. Since the $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-action on $X'_+(\tilde{p})$ in Lemma 2.7 is defined through $\psi'^{\mathrm{an}}|_{X'_+(\tilde{p})}$, this morphism $\psi'^{\mathrm{an}}|_{X'_+(\tilde{p})}$ is $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-equivariant by definition. Thus $G \circ \psi'^{\mathrm{an}}|_{X'_+(\tilde{p})}$ is $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-equivariant, and hence $pr_1 \circ F$, which equals $G \circ \psi'^{\mathrm{an}}|_{X'_+(\tilde{p})}$, is $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-equivariant. Since the $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-action on $(\mathbb{G}_m^n)^{\mathrm{an}} \times V$ is the pull-back of the $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-action on $(\mathbb{G}_m^n)^{\mathrm{an}}$ by pr_1, it follows that F is $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-equivariant.

Thus we only have to show that G is $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-equivariant. By the description (4.1), we have $G^*(x_i) = \lambda_i v'_i (x'_1)^{m_{i1}} \cdots (x'_r)^{m_{ir}}$, where $\lambda_i \in \mathbb{K}^\times$ and $v'_i \in \mathcal{O}(\mathscr{S})^\times$. It follows from this description and the definition of h_{f,Δ_S} that G is $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-equivariant. Thus we obtain the lemma. $\square$

Let $\mathscr{C}$ be a Λ-periodic Γ-rational polytopal subdivision of $\mathscr{C}_0$, with the induced polytopal subdivision $\overline{\mathscr{C}}$ of $\overline{\mathscr{C}_0}$, and let $p : \mathscr{E} \to \mathscr{A}$ be the Mumford model of the uniformization $p^{\mathrm{an}} : E \to A^{\mathrm{an}}$ associated to $\mathscr{C}$. Let $\mathscr{B}$ be the formal abelian scheme with $\mathscr{B}^{\mathrm{an}} = B$. Recall that there is a morphism $q : \mathscr{E} \to \mathscr{B}$ which restricts to the morphism $q^{\mathrm{an}} : E \to A^{\mathrm{an}}$ on the Raynaud extension. Also recall that, for any $\overline{\Delta} \in \overline{\mathscr{C}}$, we obtain a natural $(\mathbb{G}_m^n)_1^{\mathrm{an}}$-action on $\overline{\mathrm{val}}^{-1}(\mathrm{relin}(\overline{\Delta}))$ (cf. § 3.3). Via the homomorphism h_{f,Δ_S}, we have a $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-action on $\overline{\mathrm{val}}^{-1}(\mathrm{relin}(\overline{\Delta}))$.

Let $\mathscr{D}$ be the subdivision of $S(\mathscr{X}')$ given by

$$\mathscr{D} := \{\Delta_{S'} \cap \overline{f}_{\mathrm{aff}}^{-1}(\overline{\Delta}) \mid S' \in \mathrm{str}(\tilde{\mathscr{X}}'), \overline{\Delta} \in \overline{\mathscr{C}}\}.$$

Let $\mathscr{X}''$ be the formal model of X' associated to $\mathscr{D}$ and let $\iota' : \mathscr{X}'' \to \mathscr{X}'$ be the morphism extending the identity on X' as in Remark 2.2. Then we have a morphism $\varphi' : \mathscr{X}'' \to \mathscr{A}$ extending $f : X' \to A^{\mathrm{an}}$ by [Gub10, Proposition 5.14].

Let $\tilde{h}_{f,\Delta_S} : (\mathbb{G}_m^r)_{\tilde{\mathbb{K}}} \to (\mathbb{G}_m^n)_{\tilde{\mathbb{K}}}$ be the homomorphism given by the reduction of h_{f,Δ_S}. Since $Z_{\mathrm{relin}\,\overline{\Delta}} = \mathrm{red}_{\mathscr{A}}(\overline{\mathrm{val}}^{-1}(\mathrm{relin}(\overline{\Delta})))$ has a $(\mathbb{G}_m^n)_{\tilde{\mathbb{K}}}$-action (cf. § 3.3), we obtain a $(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$-action on $Z_{\mathrm{relin}\,\overline{\Delta}}$ via $\tilde{h}_{f,\Delta_S}$.

Proposition 4.4. *Let $u \in \mathscr{D}$ be a vertex, Δ_S the canonical simplex of $S(\mathscr{X}')$ with $u \in \mathrm{relin}\,\Delta_S$, and R the stratum of $\tilde{\mathscr{X}}''$ corresponding to u (cf. Proposition 2.3). Let $\overline{\Delta} \in \overline{\mathscr{C}}$ be the polytope with $\overline{f}_{\mathrm{aff}}(u) \in \mathrm{relin}(\overline{\Delta})$. Take a representative $\Delta \in \mathscr{C}$ of $\overline{\Delta}$ and let $\overline{\tilde{q}_\Delta}|_{Z_{\mathrm{relin}\,\overline{\Delta}}} : Z_{\mathrm{relin}\,\overline{\Delta}} \to \tilde{\mathscr{B}}$ be the morphism in (3.11). Then there exists a unique morphism $\beta_\Delta : S \to \tilde{\mathscr{B}}$ such that the diagram*

$$\begin{tikzcd} R \arrow[r, "\tilde{\varphi}'"] \arrow[d, "\tilde{\iota}'"'] & Z_{\mathrm{relin}\,\overline{\Delta}} \arrow[d, "\overline{\tilde{q}_\Delta}|_{Z_{\mathrm{relin}\,\overline{\Delta}}}"] \\ S \arrow[r, "\beta_\Delta"] & \tilde{\mathscr{B}} \end{tikzcd}$$

commutes. Moreover, this diagram is $(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$-equivariant with respect to the $(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$-action on R in Proposition 2.4, that on $Z_{\mathrm{relin}\,\overline{\Delta}}$ induced by $\tilde{h}_{f,\Delta_S}$ and the trivial actions on S and $\tilde{\mathscr{B}}$.

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Proof. We first define the morphism $S \to \tilde{\mathscr{B}}$. Since there is a local section of $\tilde{\iota}'$, we define locally on S a morphism from S to $\tilde{\mathscr{B}}$ which is compatible with $\tilde{\overline{q_\Delta}}|_{Z_{\mathrm{relin}\,\overline{\Delta}}} \circ \tilde{\varphi}'$. Since the fiber of $\tilde{\iota}'$ is an algebraic torus and $\tilde{\mathscr{B}}$ is an abelian variety, the morphism $\tilde{\overline{q_\Delta}}|_{Z_{\mathrm{relin}\,\overline{\Delta}}} \circ \tilde{\varphi}'$ contracts any fiber of $\tilde{\iota}'$ to a point. This implies that the local morphism from S to $\tilde{\mathscr{B}}$ defined in this way does not depend on the choice of local sections of $\tilde{\iota}'$. It follows that the local morphisms patch together to form a global morphism $\beta_\Delta : S \to \tilde{\mathscr{B}}$, which satisfies the commutativity of the diagram.

The uniqueness of β_Δ follows from the construction. It only remains to show that $\tilde{\varphi}' : R \to Z_{\mathrm{relin}\,\overline{\Delta}}$ is $(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$-equivariant. Take an arbitrary closed point $\tilde{p} \in S$. We have relations $R = \mathrm{red}_{\mathscr{X}''}(\mathrm{Val}^{-1}(u))$ and $\{u\} = \mathrm{Val}((\mathrm{red}_{\mathscr{X}''})^{-1}(\{\tilde{p}\} \times_S R))$ by Proposition 2.3. We also have $\overline{\mathrm{val}} \circ f = \overline{f}_{\mathrm{aff}} \circ \mathrm{Val}$ by Proposition 4.1. It follows that

$$\overline{\mathrm{val}}(f((\mathrm{red}_{\mathscr{X}''})^{-1}(\{\tilde{p}\} \times_S R))) = \overline{f}_{\mathrm{aff}}(\mathrm{Val}((\mathrm{red}_{\mathscr{X}''})^{-1}(\{\tilde{p}\} \times_S R))) = \{\overline{f}_{\mathrm{aff}}(u)\} \subset \mathrm{relin}\,\overline{\Delta},$$

and thus f restricts to a morphism

$$(\mathrm{red}_{\mathscr{X}''})^{-1}(\{\tilde{p}\} \times_S R) \to \overline{\mathrm{val}}^{-1}(\mathrm{relin}(\overline{\Delta})). \tag{4.3}$$

Recall that we have a $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-action on $(\mathrm{red}_{\mathscr{X}''})^{-1}(\{\tilde{p}\} \times_S R) \subset X'_+(\tilde{p})$ (cf. Remark 2.8). Since $\overline{\mathrm{val}}^{-1}(\mathrm{relin}(\overline{\Delta}))$ is stable under the $(\mathbb{G}_m^n)_1^{\mathrm{an}}$-action (cf. § 3.3), it is stable under the $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-action induced by h_{f,Δ_S}. It follows from Lemma 4.3 that the morphism (4.3) is $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-equivariant.

We deduce the $(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$-equivariance of $\tilde{\varphi}' : R \to Z_{\mathrm{relin}\,\overline{\Delta}}$ from the $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-equivariance of (4.3). Indeed, it follows from Lemma 3.5 that the diagram

$$\begin{tikzcd}
(\mathbb{G}_m^r)_1^{\mathrm{an}} \times \overline{\mathrm{val}}^{-1}(\mathrm{relin}(\overline{\Delta})) \arrow[r] \arrow[d, "\mathrm{red}_{(\mathbb{G}_m^r)_1^{\text{f-sch}} \times \mathscr{A}}"'] & \overline{\mathrm{val}}^{-1}(\mathrm{relin}(\overline{\Delta})) \arrow[d, "\mathrm{red}_{\mathscr{A}}"] \\
(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}} \times Z_{\mathrm{relin}(\overline{\Delta})} \arrow[r] & Z_{\mathrm{relin}(\overline{\Delta})}
\end{tikzcd}$$

where the rows are the torus actions given by h_{f,Δ_S} and $\tilde{h}_{f,\Delta_S}$, is commutative, that is, the $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-action on $\overline{\mathrm{val}}^{-1}(\mathrm{relin}(\overline{\Delta}))$ and the $(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$-action on $Z_{\mathrm{relin}(\overline{\Delta})}$ are compatible with respect to reduction. On the other hand, by Lemma 2.7, the $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-action on $(\mathrm{red}_{\mathscr{X}''})^{-1}(\{\tilde{p}\} \times_S R)$ and the $(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$-action on $\{\tilde{p}\} \times_S R$ are compatible with respect to the reduction map $\mathrm{red}_{\mathscr{X}''}$. Since (4.3) is $(\mathbb{G}_m^r)_1^{\mathrm{an}}$-equivariant, it follows that $\{\tilde{p}\} \times_S R \to Z_{\mathrm{relin}\,\overline{\Delta}}$ is $(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$-equivariant. Since $\tilde{p} \in S$ is any closed point, this implies that $R \to Z_{\mathrm{relin}\,\overline{\Delta}}$ is $(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$-equivariant. Thus the proof is complete. □

4.3 Key lemma

The following is the key lemma in this paper. It is crucially used in the proof of Proposition 5.12.

Lemma 4.5. *Let X be a d-dimensional closed subvariety of A and let $\mathscr{X}_0$ be the closure of X in $\mathscr{A}_0$. Assume that the morphism $\varphi_0 : \mathscr{X}' \to \mathscr{A}_0$ factors through $\mathscr{X}_0$ to be a generically finite surjective morphism $\mathscr{X}' \to \mathscr{X}_0$.*[11] *Let Δ_S be a canonical simplex of $S(\mathscr{X}')$. Suppose that there exist a Γ-rational point $\overline{w} \in \overline{\mathrm{val}}(X^{\mathrm{an}})$, an irreducible component W of $\mathrm{in}_{\overline{w}}(X)$,*[12] *and a Γ-rational point $u \in \mathrm{relin}(\Delta_S)$ such that $f(u)$ reduces to the generic point of W (cf. Remark 3.7). Then we have $\dim \overline{f}_{\mathrm{aff}}(\Delta_S) = \dim \Delta_S$.*

[11] Moreover, if this morphism $\mathscr{X}' \to \mathscr{X}_0$ is proper, it is called a semistable alteration (cf. § 5.2).

[12] Recall that $\mathrm{in}_{\overline{w}}(X)$ denotes the initial degeneration of X at $\overline{w}$ (cf. (3.12)).

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Proof. Let $\overline{\mathscr{C}}$ be a Γ-rational subdivision of $\overline{\mathscr{C}_0}$ such that $\overline{w}$ itself is a vertex of $\overline{\mathscr{C}}$. Let $\mathscr{X}''$ be the formal model of X' associated to the polytopal subdivision

$$\mathscr{D} := \{\Delta_{S'} \cap \overline{f}_{\mathrm{aff}}^{-1}(\overline{\Delta}) \mid S' \in \mathrm{str}(\tilde{\mathscr{X}}'), \overline{\Delta} \in \overline{\mathscr{C}}\}$$

of $S(\mathscr{X}')$, and let $\iota' : \mathscr{X}'' \to \mathscr{X}'$ denote the natural morphism extending the identity on the generic fiber. Then u is a vertex of $\mathscr{D}$. Let $R \in \mathrm{str}(\tilde{\mathscr{X}}'')$ be the stratum corresponding to u. Then $(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$ acts on R, which makes $\tilde{\iota}' : R \to S$ a torus bundle of relative dimension $r = \dim \Delta_S$ (cf. Proposition 2.4).

Let $\mathscr{E}$ and $\mathscr{A}$ be the Mumford models of E and A associated with $\overline{\mathscr{C}}$. Let $q : \mathscr{E} \to \mathscr{B}$ be the surjective morphism obtained from the Raynaud extension (cf. § 3.3). By [Gub10, Proposition 5.14], there is a unique morphism $\varphi' : \mathscr{X}'' \to \mathscr{A}$ extending f. Since $f(u)$ reduces to the generic point of W, we have

$$\overline{f}_{\mathrm{aff}}(u) = \overline{\mathrm{val}}(f(u)) = \overline{w} \in \mathrm{relin}\{\overline{w}\}.$$

Thus we have a morphism $\tilde{\varphi}' : R \to Z_{\overline{w}}$, which is $(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$-equivariant by Proposition 4.4.

Since $f(u)$ reduces to the generic point of W, Lemma 2.5 tells us that the generic point of R maps to that of W. Since W is a closed subset of $Z_{\overline{w}}$, it follows that W coincides with the closure of $\tilde{\varphi}'(R)$ in $Z_{\overline{w}}$. Since the morphism $\tilde{\varphi}' : R \to Z_{\overline{w}}$ is $(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$-equivariant, we see that W is stable under the $(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$-action induced by $\tilde{h}_{f,\Delta_S}$. It follows that W is stable under the action of $\mathbb{T}'' := \mathrm{Image}\, \tilde{h}_{f,\Delta_S}$, the image of $\tilde{h}_{f,\Delta_S}$. Thus, we obtain a $\mathbb{T}''$-action on W. Since the action of $\mathbb{T}''$ on $Z_{\overline{w}}$ is free by Remark 3.4, we see that the $\mathbb{T}''$-action on W is free.

We set $\Xi := W/\mathbb{T}''$. Since $\mathbb{T}''$-action on W is free, we have $\dim \Xi = d - \dim \overline{f}_{\mathrm{aff}}(\Delta_S)$ (cf. Remarks 3.6 and 4.2). Thus we have

$$\dim S = d - \dim \Delta_S \leqslant d - \dim \overline{f}_{\mathrm{aff}}(\Delta_S) = \dim \Xi. \tag{4.4}$$

Since Ξ is the quotient of W by the action of $(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$ induced by $\tilde{h}_{\Delta_S,f} : (\mathbb{G}_m^r)_{\tilde{\mathbb{K}}} \to \mathbb{T}''$ and since $R/(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}} = S$, the composite $R \to W \to \Xi$ factors through $R \to S$. Since $\tilde{\varphi}' : R \to W$ is dominant and since $W \to \Xi$ is surjective, the morphism $S \to \Xi$ is dominant. Thus we have $\dim S \geqslant \dim \Xi$. This inequality, together with (4.4), shows that $\dim S = \dim \Xi$. Further, this equality, together with (4.4), gives us $\dim \Delta_S = \dim \overline{f}_{\mathrm{aff}}(\Delta_S)$. This completes the proof of the key lemma. □

Remark 4.6. In the proof above, we have shown that $\mathrm{Image}\, \tilde{h}_{f,\Delta_S}$ acts freely on W.

4.4 Non-degenerate canonical simplices

In this subsection we recall the notion of non-degeneracy with respect to f for canonical simplices introduced by Gubler in [Gub10, 6.3]. We also show some properties of it.

We begin by recalling what the non-degenerate canonical simplices are. The morphism $\varphi_0 : \mathscr{X}' \to \mathscr{A}_0$ gives us a morphism $\tilde{\varphi}_0 : S \to \tilde{\mathscr{A}}_0$. Then [Gub10, Lemma 5.15] gives us a morphism $\tilde{\Phi}_0 : S \to \tilde{\mathscr{E}}_0$ with $\tilde{p}_0 \circ \tilde{\Phi}_0 = \tilde{\varphi}_0$, called a lift of $\tilde{\varphi}_0$. Let $q_0 : \mathscr{E}_0 \to \mathscr{B}$ be the extension of $q^{\mathrm{an}} : E \to A^{\mathrm{an}}$ between the models. A canonical simplex Δ_S is said to be *non-degenerate with respect to* f if $\dim \overline{f}_{\mathrm{aff}}(\Delta_S) = \dim \Delta_S$ and $\dim \tilde{q}_0(\tilde{\Phi}_0(S)) = \dim S$. This notion does not depend on the choice of $\tilde{\Phi}_0$.

Let $\mathscr{C}$, $\mathscr{A}$, $\mathscr{D}$, and $\iota' : \mathscr{X}'' \to \mathscr{X}'$ be as in § 4.2. Recall that φ_0 lifts to $\varphi' : \mathscr{X}'' \to \mathscr{A}$ by [Gub10, Proposition 5.14].

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Lemma 4.7. *With the notation above, let u be a vertex of $\mathscr{D}$ and let Δ_S be the canonical simplex of $S(\mathscr{X}')$ with $u \in \operatorname{relin} \Delta_S$. Let R be the stratum of $\tilde{\mathscr{X}}''$ corresponding to u. Assume that $\overline{w} := \overline{f}_{\mathrm{aff}}(u)$ is a vertex of $\overline{\mathscr{C}}$. Let $w \in \mathbb{R}^n$ be a representative of $\overline{w}$ and let $\beta_w : S \to \tilde{\mathscr{B}}$ be the morphism in Proposition 4.4 for $\{w\} \in \mathscr{C}$. Then Δ_S is non-degenerate with respect to f if and only if $\dim \overline{f}_{\mathrm{aff}}(\Delta_S) = \dim \Delta_S$ and $\dim \beta_w(S) = \dim S$.*

Proof. It suffices to show that $\beta_w = \tilde{q_0} \circ \tilde{\Phi}_0$ for some lift $\tilde{\Phi}_0$ of $\tilde{\varphi_0}$. Since $\mathscr{C}$ is a Γ-rational subdivision of $\mathscr{C}_0$, we have an extension $\overline{\iota_0} : \mathscr{A} \to \mathscr{A}_0$ of the identity morphism of A between the Mumford models (cf. [Gub10, § 4]). Let $\Delta_0 \in \mathscr{C}_0$ be the polytope with $w \in \operatorname{relin} \Delta_0$. Then $\tilde{\overline{\iota_0}}$ restricts to $Z_{\overline{w}} \to Z_{\operatorname{relin} \overline{\Delta_0}}$, and there is a commutative diagram

$$\begin{tikzcd}
R \arrow[r, "\tilde{\varphi}'"] \arrow[d, "\tilde{\iota}'"] & Z_{\overline{w}} \arrow[d, "\tilde{\overline{\iota_0}}"] & Z_w \arrow[l, "\tilde{p}|_{Z_w}"', "\cong"] \arrow[r, "\tilde{q}|_{Z_w}"] \arrow[d, "\tilde{\iota_0}"] & \tilde{\mathscr{B}} \arrow[d, "\mathrm{id}"] \\
S \arrow[r, "\tilde{\varphi_0}"] & Z_{\operatorname{relin} \overline{\Delta_0}} & Z_{\operatorname{relin} \Delta_0} \arrow[l, "\tilde{p_0}|_{Z_{\operatorname{relin} \Delta_0}}"', "\cong"] \arrow[r, "\tilde{q_0}|_{Z_{\operatorname{relin} \Delta_0}}"] & \tilde{\mathscr{B}}
\end{tikzcd}$$

We take the lift $\tilde{\Phi}_0$ of $\tilde{\varphi_0}$ such that $\tilde{\Phi}_0(S) \subset Z_{\operatorname{relin} \Delta_0}$. Then

$$\tilde{q_0} \circ \tilde{\Phi}_0 \circ \tilde{\iota}' = \tilde{q_0} \circ (\tilde{p_0}|_{Z_{\operatorname{relin} \Delta_0}})^{-1} \circ \tilde{\varphi_0} \circ \tilde{\iota}' = \tilde{q} \circ (\tilde{p}|_{Z_w})^{-1} \circ \tilde{\varphi}'. \tag{4.5}$$

Since $\tilde{q} \circ (\tilde{p}|_{Z_w})^{-1} = \tilde{\overline{q_w}}$ by definition and since $\tilde{\overline{q_w}} \circ \tilde{\varphi}' = \beta_w \circ \tilde{\iota}'$ by Proposition 4.4, it follows from (4.5) that $\tilde{q_0} \circ \tilde{\Phi}_0 \circ \tilde{\iota}' = \beta_w \circ \tilde{\iota}'$. Since $\tilde{\iota}'$ is surjective, we conclude that $\tilde{q_0} \circ \tilde{\Phi}_0 = \beta_w$, as required. □

Lemma 4.8. *Let u, Δ_S and R be as in Lemma 4.7. Assume that $\overline{w} := \overline{f}_{\mathrm{aff}}(u)$ is a vertex of $\overline{\mathscr{C}}$. If Δ_S is non-degenerate with respect to f, then $\dim \tilde{\varphi}'(R) = d$.*

Proof. Recall that $(\mathbb{G}_m^n)_{\tilde{\mathbb{K}}}$ acts on $Z_{\overline{w}}$ and that the $(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$-action on $Z_{\overline{w}}$ is given by the homomorphism $\tilde{h}_{f,\Delta_S}$. We set $\mathbb{T}'' := \operatorname{Image} \tilde{h}_{f,\Delta_S}$. Then there is a natural $\mathbb{T}''$-action on $Z_{\overline{w}}$. By Proposition 4.4, the morphism $\tilde{\varphi}' : R \to Z_{\overline{w}}$ is $(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$-equivariant. It follows that $\tilde{\varphi}'(R) \subset Z_{\overline{w}}$ is stable under the $\mathbb{T}''$-action. Thus we have a $\mathbb{T}''$-action on $\tilde{\varphi}'(R)$. Since the $(\mathbb{G}_m^n)_{\tilde{\mathbb{K}}}$-action on $Z_{\overline{w}}$ is free (cf. Remark 3.4), the $\mathbb{T}''$-action on $\tilde{\varphi}'(R)$ is free.

We take a representative $w \in \mathscr{C}$ of $\overline{w}$. The morphism $\tilde{\overline{q_w}} : Z_{\overline{w}} \to \tilde{\mathscr{B}}$ in (3.11) is the quotient by $(\mathbb{G}_m^n)_{\tilde{\mathbb{K}}}$, so that any $\mathbb{T}''$-orbit in $\tilde{\varphi}'(R)$ contracts to a point by $\tilde{\overline{q_w}}$. It follows that $\dim \tilde{\varphi}'(R) \geqslant \dim \tilde{\overline{q_w}}(\tilde{\varphi}'(R)) + \dim \mathbb{T}''$. Since $\dim \mathbb{T}'' = \dim \overline{f}_{\mathrm{aff}}(\Delta_S)$ by Remark 4.2, we thus obtain $\dim \tilde{\varphi}'(R) \geqslant \dim \tilde{\overline{q_w}}(\tilde{\varphi}'(R)) + \dim \overline{f}_{\mathrm{aff}}(\Delta_S)$.

Suppose that Δ_S is non-degenerate with respect to f. Then $\dim \overline{f}_{\mathrm{aff}}(\Delta_S) = \dim \Delta_S$ and $\dim S = \dim \beta_w(S)$ by Lemma 4.7. Since $\beta_w(S) = \tilde{\overline{q_w}}(\tilde{\varphi}'(R))$ (cf. Proposition 4.4), we obtain $\dim S = \dim \tilde{\overline{q_w}}(\tilde{\varphi}'(R))$. Thus we have

$$d \geqslant \dim \tilde{\varphi}'(R) \geqslant \dim \tilde{\overline{q_w}}(\tilde{\varphi}'(R)) + \dim \overline{f}_{\mathrm{aff}}(\Delta_S) = \dim S + \dim \Delta_S = d,$$

and we conclude that $\dim \tilde{\varphi}'(R) = d$. □

5. Strict supports of canonical measures

In this section, let $\mathbf{K}$ be a subfield of $\mathbb{K}$ such that the absolute value of $\mathbb{K}$ restricts to a discrete absolute value on $\mathbf{K}$, and assume that $\mathbf{K}$ is complete. The value group $\Gamma_{\mathbf{K}}$ of $\mathbf{K}$ is a discrete subgroup of $\mathbb{Q}$. Let $\mathbf{K}^\circ$ denote the ring of integers of $\mathbf{K}$, which is a discrete valuation ring. For a variety X over $\mathbf{K}$ and a flat formal scheme $\mathcal{X}$ of finite type over $\mathbf{K}^\circ$, we let X^{an} and $\mathcal{X}^{\mathrm{an}}$ stand for $(X \times_{\operatorname{Spec}\mathbf{K}} \operatorname{Spec}\mathbb{K})^{\mathrm{an}}$ and $(\mathcal{X} \times_{\operatorname{Spf}\mathbf{K}^\circ} \operatorname{Spf}\mathbb{K}^\circ)^{\mathrm{an}}$, respectively. They are analytic spaces over $\mathbb{K}$. We deal only with analytic spaces which arise from varieties defined over $\mathbf{K}$ and formal schemes defined over $\mathbf{K}^\circ$ in this section because it is enough for our later applications.

5.1 Mumford models over a discrete valuation ring

Let A be an abelian variety over $\mathbf{K}$. We recall that, replacing $\mathbf{K}$ by a finite extension in $\mathbb{K}$ if necessary, we have the Raynaud extension over $\mathbf{K}$. Indeed, by [BL91, Theorem 1.1], replacing $\mathbf{K}$ by a finite extension in $\mathbb{K}$ if necessary, we have an exact sequence

$$1 \longrightarrow (\mathbb{G}_m^n)^{\text{f-sch}}_{\mathbf{K}^\circ} \longrightarrow \mathcal{A}^\circ \longrightarrow \mathcal{B} \longrightarrow 0$$

of formal group schemes over $\mathbf{K}^\circ$, where n is the torus rank of A, $(\mathbb{G}_m^n)^{\text{f-sch}}_{\mathbf{K}^\circ}$ is a split formal torus over $\mathbf{K}^\circ$, $\mathcal{A}^\circ$ is the formal completion of a semiabelian scheme over $\mathbf{K}^\circ$, and $\mathcal{B}$ is the formal completion of an abelian scheme over $\mathbf{K}^\circ$. Further, the base-change to $\mathbb{K}^\circ$ is nothing but the exact sequence (3.1) for $A \times_{\operatorname{Spec}\mathbf{K}} \operatorname{Spec}\mathbb{K}$. We can also construct, in the same way as we did in § 3.1, a short exact sequence of analytic groups over $\mathbf{K}$ whose base-change to $\mathbb{K}$ coincides with (3.3).

Let $p^{\mathrm{an}} : E \to A^{\mathrm{an}}$ be the uniformization and consider the lattice $\Lambda = \operatorname{val}(\operatorname{Ker} p^{\mathrm{an}})$ in $\mathbb{R}^n$. We see from [BL91, Theorem 1.2] and its proof that $\Lambda = \operatorname{val}(\operatorname{Ker} p^{\mathrm{an}}) \subset \Gamma_{\mathbf{K}}^n$, and hence we have $\Lambda \subset \mathbb{Q}^n$. Thus there exists a Λ-periodic rational polytopal decomposition of $\mathbb{R}^n$ in this setting.

LEMMA 5.1. *Let A be an abelian variety over $\mathbf{K}$ of torus rank n. Let $\mathscr{C}$ be a Λ-periodic rational polytopal decomposition of $\mathbb{R}^n$ and let $\overline{\mathscr{C}}$ be the induced rational polytopal decomposition of $\mathbb{R}^n/\Lambda$. Then, replacing $\mathbf{K}$ with a finite extension in $\mathbb{K}$ if necessary, we have a proper flat formal scheme $\mathcal{A}$ over $\mathbf{K}^\circ$ such that $\mathcal{A} \times_{\operatorname{Spf}\mathbf{K}^\circ} \operatorname{Spf}\mathbb{K}^\circ$ is the Mumford model of A associated to $\overline{\mathscr{C}}$.*

Proof. Replacing $\mathbf{K}$ by a finite extension in $\mathbb{K}$, we may assume that any vertex in $\mathscr{C}$ is in $\Gamma_{\mathbf{K}}^n$. Then the construction of the Mumford models in [Gub10, 4.7] works over $\mathbf{K}^\circ$ without any change, so that the Mumford model $\mathscr{A}$ associated to $\overline{\mathscr{C}}$ is defined over $\mathbf{K}^\circ$, that is, there exists a formal model $\mathcal{A}$ of A over $\mathbf{K}^\circ$ such that $\mathscr{A} = \mathcal{A} \times_{\operatorname{Spf}\mathbf{K}^\circ} \operatorname{Spf}\mathbb{K}^\circ$. □

Remark 5.2. In the sequel, when we talk about a Mumford model of A, this means that it is the Mumford model associated to a Λ-periodic rational polytopal decomposition of $\mathbb{R}^n$, so that this Mumford model can be defined over a finite extension of $\mathbf{K}^\circ$ by Lemma 5.1.

5.2 Semistable alterations

Let $\mathscr{X}_0$ be a connected admissible formal scheme over $\mathbb{K}^\circ$. A morphism $\mathscr{X}' \to \mathscr{X}_0$ of formal schemes over $\mathbb{K}^\circ$ is called a *semistable alteration* for $\mathscr{X}_0$ if $\mathscr{X}'$ is a connected strictly semistable formal scheme and $\mathscr{X}' \to \mathscr{X}_0$ is a proper surjective generically finite morphism. We say that a proper surjective generically finite morphism $\mathcal{X}' \to \mathcal{X}_0$ of flat formal schemes of finite type over $\mathbf{K}^\circ$ is a semistable alteration if the base-change of this morphism to $\mathbb{K}^\circ$ is a semistable alteration in the above sense.

In this subsection we discuss the existence of semistable alteration for a model of a closed subvariety of an abelian variety defined over $\mathbf{K}$.

Lemma 5.3. *Let A be an abelian variety over $\mathbf{K}$ and let X be a closed subvariety of A. Let $\mathcal{A}_0$ be a proper flat formal scheme over $\mathbf{K}$ with $\mathcal{A}_0^{\mathrm{an}} = A^{\mathrm{an}}$, and let $\mathcal{X}_0$ be the closure of X in $\mathcal{A}_0$. Then there exist a projective formal scheme $\mathcal{X}'$ flat over $\operatorname{Spf} \mathbf{K}^\circ$ and a semistable alteration $\varphi_0 : \mathcal{X}' \to \mathcal{X}_0$.*

Proof. Note that [Gub03, Proposition 10.5] gives us a projective scheme $\mathbf{X}_1 \to \operatorname{Spec} \mathbf{K}^\circ$ with generic fiber X, and a dominating morphism $\widehat{\mathbf{X}}_1 \to \mathcal{X}_0$ extending the identity on the generic fiber, where $\widehat{\mathbf{X}}_1$ is the formal completion of $\mathbf{X}_1$ along its special fiber. Indeed, the statement of [Gub03, Proposition 10.5] is given under the assumption that the base field $\mathbf{K}$ is algebraically closed, but the proof given there also works well in our situation without any change.

Since $\widehat{\mathbf{X}}_1$ is algebraizable, we apply [dJ96, Theorem 6.5] to $\widehat{\mathbf{X}}_1$ to obtain a semistable alteration $\mathcal{X}' \to \widehat{\mathbf{X}}_1$ over $\mathbf{K}^\circ$ such that $\mathcal{X}'$ is a projective formal scheme flat over $\operatorname{Spf} \mathbf{K}^\circ$. Let $\varphi_0 : \hat{\mathcal{X}}' \to \hat{\mathcal{X}}_0$ be the composite. Then it is a semistable alteration, as required. □

Remark 5.4. Let $\mathscr{A}_0$ be such a Mumford model as in Remark 5.2. Let X be a closed subvariety of A, and let $\mathscr{X}_0$ the closure of X. Since $\mathscr{A}_0$ is a formal scheme which can be defined over a finite extension of $\mathbf{K}^\circ$, $\mathscr{X}_0$ is a formal scheme which can be defined over the finite extension as well. By Lemma 5.3, we have a semistable alteration $\varphi_0 : \mathscr{X}' \to \mathscr{X}_0$ which can be defined over a finite extension of $\mathbf{K}^\circ$. Further, the restriction of φ_0 to the Raynaud generic fibers is regarded as a morphism of projective varieties over a finite extension of $\mathbf{K}$. In the sequel, we consider such semistable alterations only.

5.3 Canonical measures and the canonical subset

Let X be a proper variety over $\mathbf{K}$ and let L be a line bundle on X. In [Gub10], Gubler defined the notion of *admissible metrics on* L (cf. [Gub10, 3.5]). If $\overline{L}$ is a line bundle on X endowed with an admissible metric, then one can define a regular Borel measure $c_1(\overline{L})^{\wedge d}$ on X^{an} with suitable properties [Gub10, Proposition 3.8]. It was originally introduced by Chambert-Loir in [Cha06]. These measures satisfy the projection formula: if $f : X' \to X$ is a morphism of d-dimensional geometrically integral proper varieties over $\mathbf{K}$, then $f^*\overline{L}$ is an admissibly metrized line bundle on X', and

$$f_*(c_1(f^*\overline{L})^{\wedge d}) = \deg(f) c_1(\overline{L})^{\wedge d}.$$

Let A be an abelian variety over $\mathbf{K}$ and let L be a line bundle on A. We say that L is *even* if $[-1]L^* \cong L$, where, for any $m \in \mathbb{Z}$, $[m] : A \to A$ denotes the homomorphism given by $[m](a) = ma$. Suppose that L is even and ample. As mentioned in [Gub10, Example 3.7], there is an important metric, called a canonical metric, on L. This metric is described as follows. Let m be a positive integer. Since L is even, we have an identification $[m]^*L = L^{m^2}$. Then a metric $\| \cdot \|$ on L is called a *canonical metric* if $[m]^*\| \cdot \| = \| \cdot \|^{m^2}$ via that identification. Note that a canonical metric depends on the choice of the identification $[m]^*L = L^{m^2}$, but it is unique up to positive rational multiples.

In the sequel, let $\overline{L}$ always denote a line bundle endowed with a canonical metric for a line bundle L on an abelian variety. For a closed subvariety X of A of dimension d, the restriction $\overline{L}|_X$ is a line bundle on X with an admissible metric (cf. [Gub10, Proposition 3.6]). We define a *canonical measure* on X^{an} to be

$$\mu_{X^{\mathrm{an}},L} := \frac{1}{\deg_L X} c_1(\overline{L}|_X)^{\wedge d},$$

which is a probability measure. Although the canonical metric is unique only up to a positive rational multiple, the canonical measure is uniquely determined from L.

We fix the notation which is used in the rest of this section. Let $p^{\mathrm{an}} : E \to A^{\mathrm{an}}$ be the uniformization and let $\mathrm{val} : E \to \mathbb{R}^n$ be the valuation map, where n is the torus rank of A. We set $\Lambda := \mathrm{val}(\operatorname{Ker} p^{\mathrm{an}})$ and let $\overline{\mathrm{val}} : A^{\mathrm{an}} \to \mathbb{R}^n/\Lambda$ be the valuation map. Let $\mathscr{C}_0$ be a Λ-periodic rational polytopal decomposition of $\mathbb{R}^n$ and let $\overline{\mathscr{C}_0}$ be the rational polytopal decomposition of $\mathbb{R}^n/\Lambda$ induced by the quotient. We take the Mumford model $\mathscr{A}_0$ of A associated to $\overline{\mathscr{C}_0}$, which can be defined over a finite extension of $\mathbf{K}^\circ$ (cf. Remark 5.2). Let $\mathscr{X}_0$ be the closure of $X \subset A$ in $\mathscr{A}_0$. We take a semistable alteration $\varphi_0 : \mathscr{X}' \to \mathscr{X}_0$ as in Remark 5.4. Put $X' := (\mathscr{X}')^{\mathrm{an}}$ and let $f : X' \to A^{\mathrm{an}}$ be the composite $(\mathscr{X}')^{\mathrm{an}} \to X^{\mathrm{an}} \to A^{\mathrm{an}}$. Then we have a measure

$$\mu_{X', f^*\overline{L}} := \frac{1}{\deg_{f^*L} X'} c_1(f^*\overline{L})^{\wedge d}$$

on X'.

Since the morphism $X' \to X^{\mathrm{an}}$ is the analytification of a morphism between projective varieties over a finite extension of $\mathbf{K}$, we have

$$f_*\mu_{X', f^*\overline{L}} = \mu_{X^{\mathrm{an}}, L} \tag{5.1}$$

by the projection formula noted above.[13] Recall that we have an expression

$$\mu_{X', f^*\overline{L}} = \sum_{\Delta_S} r_S \delta_{\Delta_S} \quad (r_S > 0) \tag{5.2}$$

by [Gub10, Corollary 6.9], where Δ_S runs through the set of non-degenerate canonical simplices with respect to f, and δ_{Δ_S} is the push-out of the Lebesgue measure on Δ_S. Note that all the coefficients r_S are positive. Let $S(\mathscr{X}')_{\text{nd-}f}$ be the union of the non-degenerate canonical simplices of $S(\mathscr{X}')$ with respect to f. Then (5.2) shows that $\mu_{X', f^*\overline{L}}$ is supported by $S(\mathscr{X}')_{\text{nd-}f}$. Since the notion of non-degeneracy of Δ_S is independent of L, the support of $\mu_{X', f^*\overline{L}}$ is independent of L. Thus the support $S_{X^{\mathrm{an}}}$ of $\mu_{X^{\mathrm{an}}, L}$ is exactly the image of $S(\mathscr{X}')_{\text{nd-}f}$, and in particular, it does not depend of L. Thus $S_{X^{\mathrm{an}}}$ depends only on X. It is called the *canonical subset* of X^{an} in [Gub10, Remark 6.11].

By [Gub10, Theorem 6.12], $S_{X^{\mathrm{an}}}$ has a canonical rational piecewise linear structure. This is characterized by the property that, for any model $\mathscr{X}$ of X in a Mumford model and for any semistable alteration $\psi : \mathscr{Z} \to \mathscr{X}$, the induced map $f^{\mathrm{an}}|_{S(\mathscr{Z})_{\text{nd-}f}} : S(\mathscr{Z})_{\text{nd-}f} \to S_{X^{\mathrm{an}}}$ is a finite rational piecewise linear map (cf. [Gub10, Theorem 6.12]).

If Δ_S is non-degenerate with respect to f, then $f|_{\Delta_S} : \Delta_S \to f(\Delta_S)$ is bijective. Indeed, suppose that Δ_S is non-degenerate. Since $\overline{f}_{\mathrm{aff}}|_{\Delta_S} : \Delta_S \to \mathbb{R}^n/\Lambda$ is an affine map by Proposition 4.1 and since $\dim \overline{f}_{\mathrm{aff}}(\Delta_S) = \dim \Delta_S$, the map $\overline{f}_{\mathrm{aff}}|_{\Delta_S}$ is a finite affine map, which means that $\overline{f}_{\mathrm{aff}}|_{\Delta_S}$ is an injective affine map. Since $\overline{\mathrm{val}} \circ f|_{\Delta_S} = \overline{f}_{\mathrm{aff}}|_{\Delta_S}$, it follows that $f|_{\Delta_S} : \Delta_S \to f(\Delta_S)$ is injective, and hence bijective.

Remark 5.5. The valuation map $\overline{\mathrm{val}}$ restricts to a finite piecewise linear map $\overline{\mathrm{val}} : S_{X^{\mathrm{an}}} \to \mathbb{R}^n/\Lambda$. Further, for any non-degenerate stratum Δ_S with respect to f, we have

$$\dim \Delta_S = \dim \overline{f}_{\mathrm{aff}}(\Delta) = \dim f(\Delta_S).$$

[13] We often identify $\mu_{X^{\mathrm{an}}, L}$ with its push-forward by the canonical closed immersion $X^{\mathrm{an}} \hookrightarrow A^{\mathrm{an}}$.

Since $S_{X^{\mathrm{an}}}$ has a canonical rational piecewise linear structure, we can consider a rational polytopal decomposition of $S_{X^{\mathrm{an}}}$. In describing the canonical measure, the following notion will be convenient.

Definition 5.6. A rational polytopal decomposition Σ of $S_{X^{\mathrm{an}}}$ is said to be φ_0-*subdivisional* if, for any non-degenerate simplex Δ_S of $S(\mathscr{X}')$, the image $f(\Delta_S)$ is a finite union of polytopes in Σ.

Remark 5.7. Let Σ be a φ_0-subdivisional rational polytopal decomposition of $S_{X^{\mathrm{an}}}$ and let $\sigma \in \Sigma$ be a polytope. Then there exists a non-degenerate canonical simplex Δ_S such that $f(\Delta_S) \supset \sigma$. Since $\overline{f}_{\mathrm{aff}}|_{\Delta_S} = \overline{\mathrm{val}} \circ f|_{\Delta_S}$ is injective, it follows that $\overline{\mathrm{val}}|_\sigma$ is injective.

The following lemma gives us a φ_0-subdivisional rational polytopal decomposition of $S_{X^{\mathrm{an}}}$.

Lemma 5.8. *For any rational polytopal decomposition Σ_0 of $S_{X^{\mathrm{an}}}$, there exists a rational polytopal subdivision of Σ_0 which is a φ_0-subdivisional rational polytopal decomposition of $S_{X^{\mathrm{an}}}$.*

Proof. Let Δ_S be a stratum non-degenerate with respect to f. Since $f|_{\Delta_S} : \Delta_S \to f(\Delta_S)$ is a bijective piecewise linear map, there exists a rational subdivision Σ'_S of Δ_S such that, for each $\sigma' \in \Sigma'_S$, the restriction $f|_{\sigma'}$ is an affine map. Then $f(\Sigma'_S)$ is a rational polytopal decomposition of $f(\Delta_S)$. Since $f : S(\mathscr{X}')_{\text{nd-}f} \to S_{X^{\mathrm{an}}}$ is a finite rational piecewise linear map, we can take a rational polytopal subdivision Σ of Σ_0 such that, for any non-degenerate Δ_S, $f(\Delta_S) \cap \Sigma := \{\sigma \in \Sigma \mid \sigma \subset f(\Delta_S)\}$ is a subdivision of $f(\Sigma'_S)$. Then we see that Σ is a φ_0-subdivisional rational polytopal decomposition. $\square$

Let Σ be a φ_0-subdivisional rational polytopal decomposition of $S_{X^{\mathrm{an}}}$. Then we have an expression

$$\mu_{X^{\mathrm{an}},L} = \sum_{\sigma \in \Sigma} r'_\sigma \delta_\sigma, \tag{5.3}$$

where $r'_\sigma \geqslant 0$ and δ_σ is the push-out of the Lebesgue measure on σ. Indeed, since Σ is φ_0-subdivisional, we have $\delta_{\Delta_S} = \sum_\sigma \delta_{(f|_{\Delta_S})^{-1}(\sigma)}$, where σ runs through the polytopes in Σ such that $\sigma \subset f(\Delta_S)$ and $\dim \sigma = \dim \Delta_S$. Thus we write $f_* \delta_{\Delta_S} = \sum_\sigma \alpha_\sigma \delta_\sigma$ for some $\alpha_\sigma \geqslant 0$, where σ runs through polytopes in Σ such that $\sigma \subset f(\Delta_S)$. It follows from (5.2) and (5.1) that we can write $\mu_{X^{\mathrm{an}},L} = f_* \mu_{X',f^*\overline{L}} = \sum_{\sigma \in \Sigma} r'_\sigma \delta_\sigma$ for some $r'_\sigma \geqslant 0$.

5.4 Strict supports of canonical measures

In this subsection we define the notion of strict supports and investigate the strict supports of a canonical measure on the canonical subset. We follow the notation in § 5.3 for A, L, X, $\varphi_0 : \mathscr{X}' \to \mathscr{A}_0$, $f : X' \to A^{\mathrm{an}}$, $\mu_{X^{\mathrm{an}},L}$ and so on.

Definition 5.9. Let $\mathscr{P}$ be a polytopal set with a finite polytopal decomposition Σ. Let μ be a semipositive Borel measure on $\mathscr{P}$. We say that $\sigma \in \Sigma$ is a *strict support* of μ if there exists an $\epsilon > 0$ such that $\mu - \epsilon \delta_\sigma$ is semipositive, where δ_σ is the push-out of the Lebesgue measure on σ.

For example, let Σ be a φ_0-subdivisional rational polytopal decomposition of $S_{X^{\mathrm{an}}}$. Then $\sigma \in \Sigma$ is a strict support of the canonical measure $\mu_{X^{\mathrm{an}},L}$ if and only if $r'_\sigma > 0$ in the expression (5.3).

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Lemma 5.10. *Let Σ be a φ_0-subdivisional rational polytopal decomposition of $S_{X^{\mathrm{an}}}$ and take any $\sigma \in \Sigma$. Then σ is a strict support of $\mu_{X^{\mathrm{an}},L}$ if and only if there exists a canonical simplex Δ_S non-degenerate with respect to f such that $\dim \Delta_S = \dim \sigma$ and $\sigma \subset f(\Delta_S)$.*

Proof. Recall that a canonical simplex Δ_S of $S(\mathscr{X}')$ is a strict support of $\mu_{X',f^*\overline{L}}$ if and only if Δ_S is non-degenerate with respect to f (cf. (5.2)). Since Σ is φ_0-subdivisional and $\mu_{X^{\mathrm{an}},L} = f_*\mu_{X',f^*\overline{L}}$, it follows that σ is a strict support of $\mu_{X^{\mathrm{an}},L}$ if and only if there exists a non-degenerate Δ_S such that $\dim f(\Delta_S) = \dim \sigma$ and that $\sigma \subset f(\Delta_S)$. Since $\dim f(\Delta_S) = \dim \Delta_S$ for non-degenerate Δ_S (cf. Remark 5.5), that shows our lemma. □

Recall that, for a rational point $\overline{w} \in \overline{\mathrm{val}}(X^{\mathrm{an}})$, $\mathrm{in}_{\overline{w}}(X)$ denotes the initial degeneration of X over $\overline{w}$.

Lemma 5.11. *Let Σ be a φ_0-subdivisional rational polytopal decomposition of $S_{X^{\mathrm{an}}}$. Suppose that $\sigma \in \Sigma$ is a strict support of $\mu_{X^{\mathrm{an}},L}$. Then for any rational point $t \in \mathrm{relin}\,\sigma$, there exists an irreducible component W of $\mathrm{in}_{\overline{\mathrm{val}}(t)}(X)$ such that t reduces to the generic point of W (cf. Remark 3.7).*

Proof. Since Σ is φ_0-subdivisional and σ is a strict support of $\mu_{X^{\mathrm{an}},L}$, Lemma 5.10 gives us a non-degenerate stratum Δ_S of $S(\mathscr{X}')$ with respect to f such that $\dim \Delta_S = \dim \sigma$ and that $f(\Delta_S) \supset \sigma$. Note that $\mathrm{relin}\,\sigma \subset f(\mathrm{relin}\,\Delta_S)$, and we take a point $u \in \mathrm{relin}\,\Delta_S$ with $f(u) = t$. We put $\overline{w} := \overline{\mathrm{val}}(t) = \overline{f}_{\mathrm{aff}}(u)$.

Let $\overline{\mathscr{C}}$ be a rational subdivision of $\overline{\mathscr{C}_0}$ such that $\overline{w}$ itself is a vertex of $\overline{\mathscr{C}}$. Let $\mathscr{A}$ be the Mumford model of A associated to $\overline{\mathscr{C}}$. Let $\mathscr{D}$ be the subdivision of $S(\mathscr{X}')$ given by

$$\mathscr{D} = \{\Delta_{S'} \cap (\overline{f}_{\mathrm{aff}})^{-1}(\overline{\Delta}) \mid S' \in \mathrm{str}(\tilde{\mathscr{X}}'), \overline{\Delta} \in \overline{\mathscr{C}}\}.$$

Since Δ_S is non-degenerate, the map $\overline{f}_{\mathrm{aff}}|_{\Delta_S}$ is injective, which shows that u is a vertex of $\mathscr{D}$.

Let $\mathscr{X}''$ be the formal model associated to the subdivision $\mathscr{D}$ (cf. Remark 2.2). Then we have an extension $\varphi' : \mathscr{X}'' \to \mathscr{A}$ of $f : X' \to A^{\mathrm{an}}$ by [Gub10, Proposition 5.13]. Let R be the stratum of $\tilde{\mathscr{X}}''$ corresponding to u in Proposition 2.3. Since $\{\overline{w}\}$ is an open face of $\overline{\mathscr{C}}$ with $\overline{f}_{\mathrm{aff}}(u) = \overline{w}$, we have $\tilde{\varphi}'(R) \subset Z_{\overline{w}}$ by [Gub10, Proposition 5.14].

Let $\mathscr{X}$ be the closure of X in $\mathscr{A}$. Since $\tilde{\varphi}'(R)$ is also contained in $\tilde{\mathscr{X}}$ and since $\mathrm{in}_{\overline{w}}(X) = \tilde{\mathscr{X}} \cap Z_{\overline{w}}$, we have $\tilde{\varphi}'(R) \subset \mathrm{in}_{\overline{w}}(X)$. Let W be the closure of $\tilde{\varphi}'(R)$ in $\mathrm{in}_{\overline{w}}(X)$. It is an irreducible closed subset of $\mathrm{in}_{\overline{w}}(X)$. Since Δ_S is non-degenerate, it follows from Lemma 4.8 that $\dim W = d := \dim X$. Since any irreducible component of $\mathrm{in}_{\overline{w}}(X)$ has dimension d (cf. Remark 3.6), W is an irreducible component of $\mathrm{in}_{\overline{w}}(X)$.

It only remains to show that t reduces to the generic point of W. By Lemma 2.5, the point $u \in (\mathscr{X}')^{\mathrm{an}}$ reduces to the generic point of R. Since $\tilde{\varphi}'$ maps the generic point of R to that of W, it follows that $t = f(u)$ reduces to the generic point of W. □

Now we can show the following statement.

Proposition 5.12. *Let Σ be a φ_0-subdivisional rational polytopal decomposition of $S_{X^{\mathrm{an}}}$ and let $\sigma \in \Sigma$ be a polytope. Let Δ_S be a canonical simplex of $S(\mathscr{X}')$. Assume that there exists a rational point $u \in \mathrm{relin}(\Delta_S)$ with $f(u) \in \mathrm{relin}(\sigma)$. Then if σ is a strict support of $\mu_{X^{\mathrm{an}},L}$, then $\dim \overline{f}_{\mathrm{aff}}(\Delta_S) = \dim \Delta_S$ holds.*

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Proof. Let $\sigma \in \Sigma$ be a strict support of $\mu_{X^{\mathrm{an}},L}$. We set $t := f(u)$ and $\overline{w} := \overline{f}_{\mathrm{aff}}(u) = \overline{\mathrm{val}}(f(u))$. Note that they are rational points in $\mathrm{relin}\, \sigma$ and $\overline{\mathrm{val}}(X^{\mathrm{an}})$, respectively. Since σ is a strict support of $\mu_{X^{\mathrm{an}},L}$, Lemma 5.11 gives us an irreducible component W of $\mathrm{in}_{\overline{w}}(X)$ such that t reduces to the generic point of W. Then Lemma 4.5 concludes that $\dim \overline{f}_{\mathrm{aff}}(\Delta_S) = \dim \Delta_S$. □

The following lemma gives us a sufficient condition for the assumption in Proposition 5.12.

Lemma 5.13. *Let Σ be a φ_0-subdivisional rational polytopal decomposition of S_{an} and let $\sigma \in \Sigma$ be a polytope. Let Δ_S be a canonical simplex in $S(\mathscr{X}')$ such that $f(\Delta_S) \supset \sigma$ and $\dim \overline{f}_{\mathrm{aff}}(\Delta_S) = \dim \overline{\mathrm{val}}(\sigma)$. Then there exists a rational point $u \in \mathrm{relin}\, \Delta_S$ such that $f(u) \in \mathrm{relin}(\sigma)$.*

Proof. Since $\overline{\mathrm{val}}|_\sigma : \sigma \to \mathbb{R}^n/\Lambda$ is a piecewise linear map, there exists a polytope $\sigma' \subset \sigma$ with $\dim \sigma' = \dim \sigma$ such that $\overline{\mathrm{val}}$ is affine over σ'. Then $\overline{\mathrm{val}}(\sigma')$ is a polytope in $\mathbb{R}^n/\Lambda$ such that $\mathrm{relin}(\overline{\mathrm{val}}(\sigma')) \subset \mathrm{relin}(\overline{f}_{\mathrm{aff}}(\Delta_S))$, so that there exists a rational point $u \in \mathrm{relin}(\Delta_S)$ such that $\overline{f}_{\mathrm{aff}}(u) \in \mathrm{relin}(\overline{\mathrm{val}}(\sigma'))$. Since $\overline{\mathrm{val}}|_{\sigma'}$ is an injective affine map (cf. Remark 5.7), we have $f(u) \in \mathrm{relin}(\sigma') \subset \mathrm{relin}(\sigma)$, as required. □

6. Tropical triviality and density of small points

6.1 Notation, convention and remarks

In the sequel, we fix the following notation and convention. Let k be an algebraically closed field, $\mathfrak{B}$ a normal projective variety over k, and $\mathcal{H}$ an ample line bundle on $\mathfrak{B}$.[14] Let K be the function field of $\mathfrak{B}$, and fix an algebraic closure $\overline{K}$ of K.

For a finite extension K' in $\overline{K}$ of K, let $\mathfrak{B}_{K'}$ denote the normalization of $\mathfrak{B}$ in K'. Let $M_{K'}$ denote the set of points in $\mathfrak{B}_{K'}$ of codimension one. For any $w \in M_{K'}$, the local ring $\mathcal{O}_{\mathfrak{B}_{K'},w}$ is a discrete valuation ring having K' as the fraction field, and the order function $\mathrm{ord}_w : (K')^\times \to \mathbb{Z}$ gives an additive discrete valuation. If K'' is a finite extension of K', then we have a canonical finite surjective morphism $\mathfrak{B}_{K''} \to \mathfrak{B}_{K'}$, which induces a surjective map $M_{K''} \to M_{K'}$. Thus we have an inverse system $(M_{K'})_{K'}$, where K' runs through the finite extensions of K in $\overline{K}$. We set $M_{\overline{K}} := \varprojlim_{K'} M_{K'}$, and call an element of $M_{\overline{K}}$ a *place* of $\overline{K}$.

Each place $v = (v_{K'})_{K'} \in M_{\overline{K}}$ determines a unique non-archimedean multiplicative absolute value $|\cdot|_v$ on $\overline{K}$ in such a way that the following conditions are satisfied.

- The restriction of $|\cdot|_v$ to K' is equivalent to the absolute value associated with the order function $\mathrm{ord}_{v_{K'}}$.
- For any $x \in K^\times$, $|x|_v = e^{-\mathrm{ord}_{v_K} x}$.

Through this correspondence, we regard a place of $\overline{K}$ as an absolute value of $\overline{K}$. For a $v \in M_{\overline{K}}$, let $\overline{K}_v$ denote the completion of $\overline{K}$ with respect to v. It is an algebraically closed field complete with respect to the non-archimedean absolute value $|\cdot|_v$.

For each $v_K \in M_K$, let $|\cdot|_{v_K,\mathcal{H}}$ be the absolute value normalized in such a way that

$$|x|_{v_K,\mathcal{H}} := e^{-(\mathrm{ord}_{v_K} x)(\deg_{\mathcal{H}} v_K)},$$

where $\deg_{\mathcal{H}} v_K$ stands for the degree of the closure of v_K in $\mathfrak{B}$ with respect to $\mathcal{H}$. It is well known that the set $\mathfrak{V} := \{|\cdot|_{v_K,\mathcal{H}}\}_{v \in M_K}$ of absolute values satisfies the product formula, and hence we define the notion of heights with respect to this set of absolute values, namely, an

[14] We assume $\mathfrak{B}$ to be a curve in § 8.

absolute logarithmic height with respect to $\mathfrak{V}$ (cf. [Lan83b, ch. 3, § 3]). By 'height' in this paper we always mean this height.

Let F/k be any field extension. For a scheme X over k, we write $X_F := X \times_{\operatorname{Spec} k} \operatorname{Spec} F$. If $\phi : X \to Y$ is a morphism of schemes over k, we write $\phi_F : X_F \to Y_F$ for the base extension to F.

Let X be an algebraic scheme over $\overline{K}$. For each place v of $\overline{K}$, we have a Berkovich analytic space associated to $X \times_{\operatorname{Spec} \overline{K}} \operatorname{Spec} \overline{K}_v$. We write X_v for this analytic space.

Let A be an abelian variety over $\overline{K}$ and suppose that X is a subvariety of A. Then A and X can be defined over a finite extension of K in $\overline{K}$, so that A_v and X_v can be defined over a subfield of $\overline{K}_v$ over which the valuation is a discrete valuation. Thus the assumptions in § 5 are fulfilled for them, and we can apply the arguments in § 5 in this setting.

6.2 Tropically trivial subvarieties and density of small points

In this subsection we introduce the notion of tropically trivial subvarieties and investigate in Theorem 6.2 the relationship between tropical triviality and density of small points.

Let A be an abelian variety over $\overline{K}$. Let L be an even ample line bundle on A. Then we can define the canonical height function $\hat{h}_L : A(\overline{K}) \to \mathbb{R}$ associated to L. Refer to [Lan83b] for the definition and properties. Note that it is a non-negative function.

Let X be a closed subvariety of A. We say X *has dense small points* if $X(\epsilon; L)$ is Zariski dense in X for any $\epsilon > 0$ (cf. [Yam13, Definition 2.2]), where $X(\epsilon; L) := \{x \in X(\overline{K}) \mid \hat{h}_L(x) \leqslant \epsilon\}$. Note that this notion does not depend on the choice of an even ample line bundle L.

From the viewpoint of the geometric Bogomolov conjecture (cf. Conjecture C), it is interesting to ask what properties a closed subvariety with dense small points has. Theorem 6.2 gives an answer to this. For the statement, we make the following definition.

Definition 6.1. Let A be an abelian variety over $\overline{K}$ and let X be a closed subvariety of A. We say that X is *tropically trivial* if $\overline{\operatorname{val}}(X_v)$ consists of a single point for any $v \in M_{\overline{K}}$, where $\overline{\operatorname{val}}$ is the valuation map for A_v (cf. § 3.1).

For a closed subvariety X of A, let G_X be the stabilizer of X, i.e. $G_X := \{a \in A \mid a + X \subset X\}$. We regard it as a reduced closed subgroup scheme of A.

Theorem 6.2. *Let A be an abelian variety over $\overline{K}$ and let X be a closed subvariety of A. If X has dense small points, then the closed subvariety X/G_X of A/G_X is tropically trivial.*

Proof. To argue by contradiction, suppose that we have a counterexample, that is, there exists a closed subvariety X satisfying the following: X has dense small points, and there exists a place $v \in M_{\overline{K}}$ such that $\overline{\operatorname{val}}((X/G_X)_v)$ is not a single point, where $\overline{\operatorname{val}} : A_v \to \mathbb{R}^n/\Lambda$ is the valuation map for A_v (and n is the torus rank of A_v); cf. § 3.1. Then X/G_X is also a counterexample by [Yam13, Lemma 2.1], so we may and do assume that the stabilizer G_X is trivial. We consider a homomorphism $\alpha : A^N \to A^{N-1}$ defined by $\alpha : (x_1, \ldots, x_N) \mapsto (x_2 - x_1, \ldots, x_N - x_{N-1})$. Put $Z := X^N \subset A^N$ and $Y := \alpha(Z)$. Since $G_X = 0$, the restriction $Z \to Y$ of α is a generically finite surjective morphism for a large N (cf. [Zha98, Lemma 3.1]). We fix such an N. Let the same symbol α denote the morphism $Z_v \to Y_v$ between the associated analytic spaces over $\overline{K}_v$.

Let $\hat{h}_{A^N}$ and $\hat{h}_{A^{N-1}}$ be the canonical height functions associated to any even ample line bundles on A^N and A^{N-1}, respectively. Since X has dense small points, so does Z (cf. [Yam13, Lemma 2.4]), and hence Z has a generic net $(P_m)_{m \in I}$, where I is a directed set, such that

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$\lim_m \hat{h}_Z(P_m) = 0$ (cf. [Gub07b, Proof of Theorem 1.1]). The image $(\alpha(P_m))_{m\in I}$ is also a generic net of Y with $\lim_m \hat{h}_Y(\alpha(P_m)) = 0$.

Let K' be a finite extension of K in $\overline{K}$ over which A and X, and hence Z and Y, are defined. For a point P in $Z(\overline{K})$ or in $Y(\overline{K})$, let $O(P)$ denote the $\mathrm{Gal}(\overline{K}/K')$-orbit of P. Then, by the equidistribution theorem [Gub08, Theorem 1.1], we find that

$$\nu_{Z_v,m} := \frac{1}{|O(P_m)|}\sum_{z\in O(P_m)}\delta_z \quad\text{and}\quad \nu_{Y_v,m} := \frac{1}{|O(\alpha(P_m))|}\sum_{y\in O(\alpha(P_m))}\delta_y$$

weakly converge, as $m\to\infty$, to the canonical measures μ_{Z_v} on Z_v and to μ_{Y_v} on Y_v associated to the even ample line bundles, respectively. Since $\alpha_*(\nu_{Z_v,m}) = \nu_{Y_v,m}$, we obtain $\alpha_*(\mu_{Z_v}) = \mu_{Y_v}$. Note in particular that $\alpha(S_{Z_v}) = S_{Y_v}$, where S_{Z_v} and S_{Y_v} are the canonical subsets.

We take the Mumford models of A_v^N and of A_v^{N-1} associated to rational polytopal decompositions of $(\mathbb{R}^n/\Lambda)^N$ and $(\mathbb{R}^n/\Lambda)^{N-1}$, respectively. Subdividing the rational polytopal decomposition of $(\mathbb{R}^n/\Lambda)^N$ if necessary, we may and do assume that the map $\alpha : A_v^N \to A_v^{N-1}$ extends to a morphism between the Mumford models (cf. § 3.3). Let $\mathscr{X}$ and $\mathscr{Y}$ be the closure of Z_v and Y_v in these Mumford models of A_v^N and of A_v^{N-1}, respectively. Note that we have a morphism $\mathscr{X}\to\mathscr{Y}$. It follows from Lemma 5.3 that there exists a semistable alteration $\varphi : \mathscr{X}'\to\mathscr{X}$ for $\mathscr{X}$ as in Remark 5.4. Note that the composite $\psi : \mathscr{X}'\to\mathscr{X}\to\mathscr{Y}$ is a semistable alteration for $\mathscr{Y}$ as in Remark 5.4. Let g be the composite $(\mathscr{X}')^{\mathrm{an}}\to Z_v\hookrightarrow A_v^N$ and put $h := \alpha\circ g : (\mathscr{X}')^{\mathrm{an}}\to A_v^{N-1}$.

We note that $\alpha|_{S_{Z_v}} : S_{Z_v}\to S_{Z_v}$ is a piecewise linear map. Indeed, we consider the diagram

$$\begin{tikzcd} S_{Z_v} \arrow[r, "\alpha|_{S_{Z_v}}"] \arrow[d, "\overline{\mathrm{val}}|_{S_{Z_v}}"'] & S_{Y_v} \arrow[d, "\overline{\mathrm{val}}|_{S_{Y_v}}"] \\ \overline{\mathrm{val}}(S_{Z_v}) \arrow[r, "\overline{\alpha}_{\mathrm{aff}}|_{\overline{\mathrm{val}}(S_{Z_v})}"'] & \overline{\mathrm{val}}(S_{Y_v}) \end{tikzcd}$$

where $\overline{\alpha}_{\mathrm{aff}}$ is the affine map associated to α (cf. § 3.2). Then $\overline{\alpha}_{\mathrm{aff}}|_{\overline{\mathrm{val}}(S_{Z_v})}$ is a piecewise linear map. Since the columns are finite piecewise linear maps (cf. Remark 5.5), it follows that the continuous map $\alpha|_{S_{Z_v}}$ is a piecewise linear map.

By Lemma 5.8, there exist a φ-subdivisional rational polytopal decomposition Σ_{Z_v} of S_{Z_v} and a ψ-subdivisional rational polytopal decomposition Σ_{Y_v} of S_{Y_v}. Taking a subdivision of Σ_{Z_v} if necessary, we may assume that, for any $\sigma'\in\Sigma_{Z_v}$, there exists a unique $\sigma''\in\Sigma_{Y_v}$ such that $\mathrm{relin}(\alpha(\sigma'))\subset\mathrm{relin}(\sigma'')$ and that $\alpha|_{\sigma'}$ is an affine map. In particular, for any $\sigma'\in\Sigma_{Z_v}$, $\alpha(\sigma')$ is a polytope contained in some polytope in Σ_{Y_v}.

By [Gub10, Theorem 1.1], $\overline{\mathrm{val}}(X_v)$ coincides with the support of $\overline{\mathrm{val}}_*(\mu_{X_v})$. Since it contains a polytope of positive dimension by our assumption at the beginning, there exists a positive-dimensional rational polytope P in $\overline{\mathrm{val}}(X_v)$ such that $\overline{\mathrm{val}}_*(\mu_{X_v}) - \epsilon\delta_P$ is semipositive for a small $\epsilon>0$. By [Yam13, Lemma 4.1 and Proposition 4.5], $\overline{\mathrm{val}}_*^N(\mu_{Z_v})$ is the product measure of N copies of $\overline{\mathrm{val}}_*(\mu_{X_v})$. Thus $\overline{\mathrm{val}}_*^N(\mu_{Z_v}) - \epsilon\delta_{P^N}$ is semipositive for a small $\epsilon>0$, where we recall that $\overline{\mathrm{val}}^m : A_v^m\to(\mathbb{R}^n/\Lambda)^m$ is the valuation map for A_v^m for $m\in\mathbb{N}$ (cf. § 3.2). Since Σ_{Z_v} is φ-subdivisional, we have an expression

$$\overline{\mathrm{val}}_*^N(\mu_{Z_v}) = \sum_{\sigma'} r_{\sigma'}\overline{\mathrm{val}}_*^N(\delta_{\sigma'}) \quad (r_{\sigma'}>0),$$

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where σ' runs through the polytopes in Σ_{Z_v} that are strict supports of μ_{Z_v}. It follows that there exists a strict support $\sigma \in \Sigma_{Z_v}$ of μ_{Z_v} such that $\mathrm{relin}(\overline{\mathrm{val}}^N(\sigma)) \cap \mathrm{relin}(P^N) \neq \emptyset$, where we remark that $\overline{\mathrm{val}}^N(\sigma)$ is a polytope. Note that $\dim(\overline{\mathrm{val}}^N(\sigma)) = \dim P^N$.

Consider the affine map $\overline{\alpha}_{\mathrm{aff}} : (\mathbb{R}^n/\Lambda)^N \to (\mathbb{R}^n/\Lambda)^{N-1}$. Since $\overline{\alpha}_{\mathrm{aff}}|_{P^N}$ contracts the diagonal of P^N to a point, it follows that $\dim(\overline{\alpha}_{\mathrm{aff}}(P^N)) < \dim(P^N)$, and thus we have $\dim(\overline{\mathrm{val}}^{N-1}(\alpha(\sigma))) < \dim(\overline{\mathrm{val}}^N(\sigma))$. This inequality, together with Remark 5.5, leads to

$$\dim \alpha(\sigma) < \dim \sigma. \tag{6.1}$$

We take a polytope $\tau \in \Sigma_{Y_v}$ with $\mathrm{relin}(\alpha(\sigma)) \subset \mathrm{relin}(\tau)$. By our assumption on Σ_{Z_v} and Σ_{Y_v}, such a polytope τ uniquely exists and is characterized by the condition that $\mathrm{relin}(\alpha(\sigma)) \cap \mathrm{relin}\,\tau \neq \emptyset$. We claim that $\dim \tau = \dim \alpha(\sigma)$ and that τ is a strict support of μ_{Y_v}. Indeed, we take a compact subset $V \subset \mathrm{relin}(\alpha(\sigma))$ such that V is a polytope in $\alpha(\sigma)$ with $\dim V = \dim \alpha(\sigma)$. Then there is an $\epsilon' > 0$ such that $\alpha_* \delta_\sigma - \epsilon' \delta_V \geqslant 0$. Since σ is a strict support of μ_{Z_v}, there exists an $\epsilon'' > 0$ such that $\mu_{Z_v} - \epsilon'' \delta_\sigma \geqslant 0$. Putting $\epsilon := \epsilon' \epsilon''$, we obtain

$$\mu_{Y_v} - \epsilon \delta_V \geqslant \alpha_* \mu_{Z_v} - \epsilon'' \alpha_* \delta_\sigma = \alpha_*(\mu_{Z_v} - \epsilon'' \delta_\sigma) \geqslant 0.$$

Since we can write $\mu_{Y_v} = \sum_{\sigma' \in \Sigma_{Y_v}} r_{\sigma'} \delta_{\sigma'}$ with $r_{\sigma'} \geqslant 0$ (cf. (5.3)), we find a strict support $\tau' \in \Sigma_{Y_v}$ of μ_{Y_v} such that $\mathrm{relin}(V) \cap \mathrm{relin}(\tau') \neq \emptyset$ and $\dim V = \dim \tau'$. Note that $\mathrm{relin}(\alpha(\sigma)) \cap \mathrm{relin}(\tau') \neq \emptyset$ in particular, for $\mathrm{relin}(V) \subset \mathrm{relin}(\alpha(\sigma))$. Since τ is the unique polytope in Σ_{Y_v} with $\mathrm{relin}(\alpha(\sigma)) \cap \mathrm{relin}(\tau) \neq \emptyset$, it follows that $\tau = \tau'$. Thus τ is a strict support of μ_{Y_v}. Further, we have $\dim \tau = \dim V = \dim \alpha(\sigma)$.

Since σ is a strict support of μ_{Z_v}, Lemma 5.10 gives us a canonical simplex Δ_S of $\underline{S}(\mathscr{X})$ non-degenerate with respect to g such that $g(\Delta_S) \supset \sigma$ and $\dim \Delta_S = \dim \sigma$. Note that $\dim \overline{h}_{\mathrm{aff}}(\Delta_S) = \dim \tau$; indeed, we see that

$$\dim \overline{h}_{\mathrm{aff}}(\Delta_S) = \dim \overline{\alpha}_{\mathrm{aff}}(\overline{g}_{\mathrm{aff}}(\Delta_S)) = \dim \overline{\alpha}_{\mathrm{aff}}(\overline{\mathrm{val}}(\sigma)) = \dim \overline{\mathrm{val}}(\alpha(\sigma)) = \dim \alpha(\sigma) = \dim \tau.$$

Since $g(\Delta_S) \supset \sigma$, we have $h(\Delta_S) \supset \alpha(\sigma)$. Since $\mathrm{relin}(\tau) \supset \mathrm{relin}(\alpha(\sigma))$, it follows that $h(\Delta_S) \cap \mathrm{relin}(\tau) \neq \emptyset$. By the assumption that polytopal decomposition Σ_{Y_v} is ψ-subdivisional, this shows that $h(\Delta_S) \supset \tau$. Since τ is a strict support of μ_{Y_v}, it follows from Proposition 5.12 with Lemma 5.13 that $\dim \overline{h}_{\mathrm{aff}}(\Delta_S) = \dim \Delta_S$. However, inequality (6.1) shows that

$$\dim \overline{h}_{\mathrm{aff}}(\Delta_S) = \dim \alpha(\sigma) < \dim \sigma = \dim \Delta_S.$$

That is a contradiction. Thus we complete the proof of the theorem. □

7. Main results

In this section, after recalling the geometric Bogomolov conjecture for abelian varieties, we prove Theorems F and E with the use of Theorem 6.2.

7.1 Special subvarieties and the geometric Bogomolov conjecture

Let A be an abelian variety over $\overline{K}$. Let $(A^{\overline{K}/k}, \mathrm{Tr}_A)$ be the $\overline{K}/k$-trace of A, that is, the pair of an abelian variety $A^{\overline{K}/k}$ over k and a homomorphism $\mathrm{Tr}_A : (A^{\overline{K}/k})_{\overline{K}} \to A$ with the following universal property: for any abelian variety A' over k and for any homomorphism $\phi : A'_{\overline{K}} \to A$, there exists a unique homomorphism $\mathrm{Tr}(\phi) : A' \to A^{\overline{K}/k}$ over k such that $\mathrm{Tr}_A \circ \mathrm{Tr}(\phi)_{\overline{K}} = \phi$. We refer to [Lan83a, Lan83b] for details.

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We recall the notion of special subvarieties introduced in [Yam13]. Let X be a closed subvariety of A. We say that X is *special* if there exist a torsion point $\tau \in A(\overline{K})$ and a closed subvariety $X' \subset A^{\overline{K}/k}$ over k such that $X = G_X + \mathrm{Tr}_A(X'_{\overline{K}}) + \tau$, where G_X is the stabilizer of X. We say that a point $x \in A$ is a *special point* if $\{x\}$ is a special subvariety. In the definition of special subvariety, we may replace the condition for τ being torsion by that for τ being special (cf. [Yam13, Remark 2.6]).

We gather together some properties of special subvarieties in the following lemma. Note that the equivalence between (a) and (b) in (1) below shows that the special subvariety in the introduction is the same as the one recalled above. Note also that [Yam13, Proposition 2.11] shows that X is a special subvariety of A if and only if X/G_X is a special subvariety of A/G_X, and Lemma 7.1(1) generalizes this equivalence.

Lemma 7.1. *Let A be an abelian variety over $\overline{K}$ and let X be a closed subvariety of A. Further, let G' be a closed algebraic subgroup of G_X, where G_X is the stabilizer of X.*

(1) *The following are equivalent.*
   (a) *X is a special subvariety.*
   (b) *There exist an abelian variety B over k, a closed subvariety $Y \subset B$, a homomorphism $\phi : B_{\overline{K}} \to A$, a torsion point $\tau \in A(\overline{K})$, and an abelian subvariety A' of A such that $X = A' + \phi(Y_{\overline{K}}) + \tau$.*
   (c) *The quotient X/G' is a special subvariety of A/G'.*

(2) *Let A_1 be an abelian variety over $\overline{K}$ and let $\psi : A \to A_1$ be a homomorphism. Suppose that X is a special subvariety of A. Then $\psi(X)$ is a special subvariety of A_1.*

Proof. In this proof, we first show the equivalence between (a) and (b), next show that (2) holds, and finally show the equivalence between (a) and (c).

We begin by showing that (a) implies (b). Suppose that X is a special subvariety. Then by definition, $X = G_X + \mathrm{Tr}_A(X'_{\overline{K}}) + \tau$ for some closed variety $X' \subset A^{\overline{K}/k}$ and some torsion point τ. Let G_X° be the connected component of G_X with $0 \in G_X$. Since X is irreducible, we then have $X = G_X^{\circ} + \mathrm{Tr}_A(X'_{\overline{K}}) + \tau$, which shows (b).

To complete the proof of the equivalence between (a) and (b), we next show that (b) implies (a). Write $X = A' + \phi(Y_{\overline{K}}) + \tau$ as in (b). Then $A' \subset G_X$, and hence $X = G_X + \phi(Y_{\overline{K}}) + \tau$. Further, by the universality of the trace, there exists a homomorphism $\mathrm{Tr}(\phi) : B \to A^{\overline{K}/k}$ such that $\phi = \mathrm{Tr}_A \circ \mathrm{Tr}(\phi)_{\overline{K}}$. Putting $X' := \mathrm{Tr}(\phi)(Y)$, we have $X = G_X + \mathrm{Tr}_A(X'_{\overline{K}}) + \tau$, which shows (a).

Now, using the equivalence between (a) and (b) shown above, let us prove (2). Since X is special, we write $X = A' + \phi(Y_{\overline{K}}) + \tau$ as in (b). Then we have $\psi(X) = \psi(A') + \psi \circ \phi(Y_{\overline{K}}) + \psi(\tau)$, which is again an expression as in (b). This shows that $\psi(X)$ is special in A_1.

Let us return to (1) and prove the remaining equivalences. It follows immediately from (2) that (a) implies (c). Suppose that (c) holds. Then by (2), X/G_X is also special, and hence [Yam13, Proposition 2.11] tells us that X is special. $\square$

Remark 7.2. We consider the case where k is an algebraic closure of a finite field. In [Sca05, Definition 2.1], Scanlon defined 'special' subvarieties, from which our special subvarieties are a little different. Let X be a closed subvariety of A. Then, in the setting of this section, one sees that X is special in the sense of Scanlon if and only if there exist a closed subvariety X' of $A^{\overline{K}/k}$

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and a (not necessarily torsion) point $a \in A(\overline{K})$ such that $X = G_X + \mathrm{Tr}_A(X'_{\overline{K}}) + a$. Thus X is special in our sense if and only if it is special in the sense of Scanlon and the point a above can be taken to be a torsion point.

As remarked in the introduction, it is known that any special subvariety has dense small points. The geometric Bogomolov conjecture claims that the converse should also hold.

CONJECTURE 7.3 (Geometric Bogomolov conjecture for abelian varieties). Let A be an abelian variety over $\overline{K}$. Let $X \subset A$ be a closed subvariety. If X has dense small points, then X is a special subvariety.

Remark 7.4. A point is special if and only if it is of height zero (cf. [Yam13, (2.5.4)]), and hence $\{x\}$ has dense small points if and only if x is a special point. This means that Conjecture 7.3 holds if $\dim X = 0$.

We make a remark on the relationship between special subvarieties and tropically trivial subvarieties. Since a special subvariety has dense small points, it follows from Theorem 6.2 that, if X is a special subvariety of A, then X/G_X is tropically trivial. This assertion itself can be shown directly, rather than as a corollary of Theorem 6.2. Indeed, taking the quotient by G_X, we may assume that $G_X = 0$. Since X is a special subvariety, we translate X by a special point to take an abelian variety A' over k, a closed subvariety $Y \subset A'$ and a homomorphism $\alpha : A'_{\overline{K}} \to A$ such that $\alpha(Y_{\overline{K}}) = X$. Since $A'_{\overline{K}}$ has torus rank 0 at any place, the subvariety $Y_{\overline{K}}$ is tropically trivial, and hence its image X is also tropically trivial (cf. (3.8)).

7.2 Isogeny and the conjecture

Let $\phi : A \to B$ be an isogeny of abelian varieties over $\overline{K}$ and let $X \subset A$ be a closed subvariety. By [Yam13, Lemma 2.3], X has dense small points if and only if the same property holds for $\phi(X)$. This suggests that, if our formulation of the geometric Bogomolov conjecture is correct, then X being special should be equivalent to $\phi(X)$ being special. In fact, this holds true.

PROPOSITION 7.5. *Let $\phi : A \to B$ be an isogeny of abelian varieties over $\overline{K}$ and let $X \subset A$ be a closed subvariety. Then X is a special subvariety if and only if $Y := \phi(X)$ is a special subvariety.*

Proof. It follows from Lemma 7.1(2) that if X is special, Y is also special. Let us show the other implication. We note that $\phi(G_X) \subset G_Y$, so that we have a homomorphism $\phi' : A/G_X \to B/G_Y$. Furthermore, we see that ϕ' is an isogeny. Indeed, let $a \in A(\overline{K})$ be a point with $\phi(a) \in G_Y$. Then for any $m \in \mathbb{Z}$, we have $\phi(ma + X) = m\phi(a) + Y = Y$, and thus $ma + X \subset \phi^{-1}(Y)$. Since ϕ is an isogeny, the subset $ma + X$ is an irreducible component of $\phi^{-1}(Y)$ for any $m \in \mathbb{Z}$. Since the number of irreducible components of $\phi^{-1}(Y)$ is at most $\deg \phi$, it follows that there exists a positive integer m_0 with $m_0 \leqslant \deg \phi$ such that $m_0 a + X = X$, i.e. $m_0 a \in G_X$. This shows that any element of the kernel of $\phi' : A/G_X \to B/G_Y$ is a torsion point of order at most $\deg \phi$, leading to the conclusion that ϕ' is an isogeny.

Now suppose that Y is special. By the equivalence between (a) and (c) in Lemma 7.1(1) or by [Yam13, Proposition 2.11], we see that Y/G_Y is special and that the specialness of X should follow from that of X/G_X. Since $\phi' : A/G_X \to B/G_Y$ is an isogeny, we may and do thus assume that G_X and G_Y are trivial. Further, taking the translation by a torsion point, we may assume that $Y = \mathrm{Tr}_B(Y'_{\overline{K}})$ for some closed subvariety $Y' \subset B^{\overline{K}/k}$, where $(B^{\overline{K}/k}, \mathrm{Tr}_B)$ is the $\overline{K}/k$-trace of B. It follows from the universality of the $\overline{K}/k$-trace that there is a unique homomorphism

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$\mathrm{Tr}(\phi) : A^{\overline{K}/k} \to B^{\overline{K}/k}$ characterized by $\mathrm{Tr}_B \circ \mathrm{Tr}(\phi)_{\overline{K}} = \phi \circ \mathrm{Tr}_A$, where $(A^{\overline{K}/k}, \mathrm{Tr}_A)$ is the $\overline{K}/k$-trace of A. Since the homomorphism $\mathrm{Tr}(\phi)$ is surjective by [Yam13, Lemma 1.5], we have $\mathrm{Tr}(\phi)(\mathrm{Tr}(\phi)^{-1}(Y')) = Y'$. Note that $\mathrm{Tr}(\phi)^{-1}(Y')_{\overline{K}} = (\mathrm{Tr}(\phi)_{\overline{K}})^{-1}(Y'_{\overline{K}})$. Then we have

$$\phi(\mathrm{Tr}_A(\mathrm{Tr}(\phi)^{-1}(Y')_{\overline{K}})) = \mathrm{Tr}_B(\mathrm{Tr}(\phi)_{\overline{K}}((\mathrm{Tr}(\phi)_{\overline{K}})^{-1}(Y'_{\overline{K}}))) = \mathrm{Tr}_B(Y'_{\overline{K}}) = \phi(X).$$

Since ϕ is an isogeny and X is irreducible, it follows that there exists a torsion point $\tau \in A(\overline{K})$ such that $Z := X - \tau$ is an irreducible component of $\mathrm{Tr}_A(\mathrm{Tr}(\phi)^{-1}(Y')_{\overline{K}})$. Thus there exists an irreducible component W of $\mathrm{Tr}(\phi)^{-1}(Y')_{\overline{K}}$ such that $\mathrm{Tr}_A(W) = Z$. Since k is algebraically closed, there exists an irreducible component W' of $\mathrm{Tr}(\phi)^{-1}(Y')$ such that $W'_{\overline{K}} = W$. Then $\mathrm{Tr}_A(W'_{\overline{K}}) + \tau = Z + \tau = X$, and we conclude that X is a special subvariety. □

Corollary 7.6. *Let $\phi : A \to B$ be an isogeny of abelian varieties over $\overline{K}$. Then the geometric Bogomolov conjecture holds for A if and only if it holds for B.*

Proof. This assertion follows from [Yam13, Lemma 2.3] and Proposition 7.5 above. □

The following lemma will be used in the proof of Corollary 7.22.

Lemma 7.7. *Let $\phi : A \to B$ be a surjective homomorphism of abelian varieties over $\overline{K}$. If the geometric Bogomolov conjecture holds for A, then it holds for B.*

Proof. By Poincaré's complete reducibility theorem (cf. [Mil86, Proposition 12.1] or [Mum08, § 19, Theorem 1]), we take an abelian subvariety $A' \subset A$ such that $\phi|_{A'} : A' \to B$ is an isogeny. Note that the geometric Bogomolov conjecture for A implies that for A'. Furthermore, Corollary 7.6 shows that the geometric Bogomolov conjecture for A' implies that for B. Thus we conclude that the conjecture for A implies that for B. □

7.3 Maximal nowhere degenerate abelian subvariety and nowhere-degeneracy rank

In this subsection, we define the notion of maximal nowhere degenerate abelian subvarieties and that of nowhere-degeneracy rank, and we describe their properties. In our main results on the geometric Bogomolov conjecture, they play key roles.

An abelian variety A over $\overline{K}$ is said to be *somewhere degenerate* if A is degenerate at some place of $\overline{K}$, and is said to be *nowhere degenerate* if A_v is non-degenerate for all $v \in M_{\overline{K}}$ (cf. § 3.1).

Lemma 7.8. *Let A_1, A_2 and A be abelian varieties over $\overline{K}$.*

(1) *Let B_1 and B_2 be nowhere degenerate abelian subvarieties of A_1 and A_2, respectively. Then $B_1 \times B_2$ is a nowhere degenerate abelian subvariety of $A_1 \times A_2$.*
(2) *Let $\phi : A_1 \to A_2$ be a homomorphism of abelian varieties. Let B_1 be a nowhere degenerate abelian subvariety of A_1. Then $\phi(B_1)$ is a nowhere degenerate abelian subvariety of A_2.*
(3) *If B and B' are nowhere degenerate abelian subvarieties of A, then so is $B + B'$.*
(4) *Let B be an abelian subvariety of A. Suppose that A is nowhere degenerate. Then B is nowhere degenerate.*

Proof. For an abelian variety A' over $\overline{K}$ and for a place $v \in M_{\overline{K}}$, let $n((A')_v)$ denote the torus rank of $(A')_v$.

(1) For any $v \in M_{\overline{K}}$, we have $n((B_1 \times B_2)_v) = n((B_1)_v) + n((B_2)_v)$ by Proposition 3.3, and the right-hand side equals 0 by the assumption. This shows that $B_1 \times B_2$ is nowhere degenerate.

(2) For any $v \in M_{\overline{K}}$, we have $n(\phi(B_1)_v) \leqslant n((B_1)_v)$ by Proposition 3.3. Since $n((B_1)_v) = 0$, we have $n(\phi(B_1)_v) = 0$, which shows that $\phi(B_1)$ is nowhere degenerate.

(3) Let $\phi : B \times B' \to A$ be the homomorphism given by $\phi(b, b') = b + b'$. Then $\phi(B \times B') = B + B'$. It follows from (1) and (2) that, if B and B are nowhere degenerate, then $B + B'$ is nowhere degenerate.

(4) For any $v \in M_{\overline{K}}$, we have $n(B_v) \leqslant n(A_v)$ by Proposition 3.3. Since $n(A_v) = 0$, we have $n(B_v) = 0$, which shows that B is nowhere degenerate. □

The following lemma is used to make the key definitions.

LEMMA 7.9. *Let A be an abelian variety over $\overline{K}$. Then there exists a unique nowhere degenerate abelian subvariety $\mathfrak{m}$ of A such that, for any nowhere degenerate abelian subvariety B' of A, we have $B' \subset \mathfrak{m}$.*

Proof. The uniqueness follows from the condition. We show the existence. Let $\mathfrak{m}$ be a nowhere degenerate abelian subvariety of A of maximal dimension. For any nowhere degenerate abelian subvariety B', the sum $\mathfrak{m} + B'$ is nowhere degenerate by Lemma 7.8(3). Since $\mathfrak{m}$ has maximal dimension among the nowhere degenerate abelian subvarieties, we have $\dim(\mathfrak{m} + B') = \dim \mathfrak{m}$. Thus we have $\mathfrak{m} + B' = \mathfrak{m}$ and hence $B' \subset \mathfrak{m}$. □

DEFINITION 7.10. Let A be an abelian variety over $\overline{K}$.

(1) The unique abelian subvariety $\mathfrak{m}$ of A in Lemma 7.9 is called the *maximal nowhere degenerate abelian subvariety of A*.
(2) Let $\mathfrak{m}$ be the maximal nowhere degenerate abelian subvariety of A. The *nowhere-degeneracy rank* of A is defined to be $\text{nd-rk}(A) := \dim \mathfrak{m}$.

Note that $\mathfrak{m} = 0$ if and only if $\text{nd-rk}(A) = 0$.

PROPOSITION 7.11. *Let A_1 and A_2 be abelian varieties over $\overline{K}$. Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be the maximal nowhere degenerate abelian subvarieties of A_1 and A_2, respectively. Let $\phi : A_1 \to A_2$ be a homomorphism. Suppose that ϕ is surjective. Then $\phi(\mathfrak{m}_1) = \mathfrak{m}_2$.*

Proof. It follows from Lemma 7.8(2) that $\phi(\mathfrak{m}_1)$ is nowhere degenerate. By the maximality of $\mathfrak{m}_2$, we obtain $\phi(\mathfrak{m}_1) \subset \mathfrak{m}_2$.

Let us prove the other inclusion. By considering the homomorphism $A_1/\mathfrak{m}_1 \to A_2/\phi(\mathfrak{m}_1)$ between the quotients instead of $\phi : A_1 \to A_2$, we may and do assume that $\mathfrak{m}_1 = 0$. Then our goal is to show that $\mathfrak{m}_2 = 0$. By the Poincaré complete reducibility theorem and Mumford's remark [Mum08, p. 157], there exists an abelian subvariety A_2' of A_2 with an isogeny $\alpha : A_2 \to A_2' \times \mathfrak{m}_2$. Let $p : A_2' \times \mathfrak{m}_2 \to \mathfrak{m}_2$ be the second projection. Let $\psi : A_1 \to \mathfrak{m}_2$ be the composite of homomorphisms

$$A_1 \xrightarrow{\phi} A_2 \xrightarrow{\alpha} A_2' \times \mathfrak{m}_2 \xrightarrow{p} \mathfrak{m}_2.$$

Then ψ is surjective, and thus we are reduced to showing $\psi = 0$.

If $A_1 = 0$, then $\psi = 0$. Therefore we consider the case where $A_1 \neq 0$. We recall that there exist non-trivial simple abelian varieties $B_1, \ldots, B_s$ and an isogeny $\gamma : B_1 \times \cdots \times B_s \to A_1$. Let $\mathfrak{n}$ be the maximal nowhere degenerate abelian subvariety of $B_1 \times \cdots \times B_s$. We have $\gamma(\mathfrak{n}) \subset \mathfrak{m}_1$ by Lemma 7.8(2). Since $\mathfrak{m}_1 = 0$, it follows that $\gamma(\mathfrak{n}) = 0$. Since γ is finite, we obtain $\mathfrak{n} = 0$. It follows from Lemma 7.8(1) that the maximal nowhere degenerate abelian subvariety of each B_i is trivial. Thus each B_i is somewhere degenerate.

Let $\iota_i : B_i \to B_1 \times \cdots \times B_s$ be the canonical injection, and set $\varphi_i := \psi \circ \gamma \circ \iota_i : B_i \to \mathfrak{m}_2$. Note that $\psi = 0$ if $\psi \circ \gamma = 0$, and that $\psi \circ \gamma = 0$ if $\varphi_i = 0$ for any $i = 1, \ldots, s$. Thus we only have to show that $\varphi_i = 0$ for any $i = 1, \ldots, s$.

Claim. *Let A be a somewhere degenerate simple abelian variety and let B be a nowhere degenerate abelian variety. Then any homomorphism $\varphi : A \to B$ is trivial.*

Proof. Set $B' := \varphi(A)$. By Lemma 7.8(4), it is nowhere degenerate. To argue by contradiction, we suppose that B' is non-trivial. Then, since A is simple, the morphism $\varphi : A \to B'$ is an isogeny. Therefore we have an isogeny $\varphi' : B' \to A$. By Lemma 7.8(2), this shows that A is nowhere degenerate, which is a contradiction. Thus $B' = \varphi(A)$ is trivial. □

Applying the claim to $\varphi_i : B_i \to \mathfrak{m}_2$ shows that φ_i is trivial for each $i = 1, \ldots, s$. This completes the proof of Proposition 7.11. □

Corollary 7.12. *Let $\phi : A_1 \to A_2$ be a homomorphism of abelian varieties. If ϕ is surjective, then* $\text{nd-rk}(A_1) \geqslant \text{nd-rk}(A_2)$.

Proof. Let $\mathfrak{m}_i$ be the maximal nowhere degenerate abelian subvariety of A_i for $i = 1, 2$. It follows from Proposition 7.11 that $\dim \mathfrak{m}_1 \geqslant \dim \mathfrak{m}_2$. Thus $\text{nd-rk}(A_1) \geqslant \text{nd-rk}(A_2)$. □

Proposition 7.13. *Let $B_1, \ldots, B_s$ and A be abelian varieties over $\overline{K}$. Suppose that A is isogenous to $B_1 \times \cdots \times B_s$. Then* $\text{nd-rk}(A) = \text{nd-rk}(B_1) + \cdots + \text{nd-rk}(B_s)$.

Proof. We set $B := B_1 \times \cdots \times B_s$. Since there are surjective homomorphisms $A \to B$ and $B \to A$, it follows from Corollary 7.12 that $\text{nd-rk}(A) = \text{nd-rk}(B)$. It remains to show that $\text{nd-rk}(B) = \text{nd-rk}(B_1) + \cdots + \text{nd-rk}(B_s)$. Let $\mathfrak{n}$ be the maximal nowhere degenerate abelian subvariety of B and $p_i : B \to B_i$ be the canonical projection. Let $\mathfrak{n}_i$ be the maximal nowhere degenerate abelian subvariety of B_i, for $i = 1, \ldots, s$. We prove $\text{nd-rk}(B) = \text{nd-rk}(B_1) + \cdots + \text{nd-rk}(B_s)$ by showing $\mathfrak{n} = \mathfrak{n}_1 \times \cdots \times \mathfrak{n}_s$. By Lemma 7.8(1), $\mathfrak{n}_1 \times \cdots \times \mathfrak{n}_s$ is a nowhere degenerate abelian subvariety of B. Then, by the maximality of $\mathfrak{n}$, we have $\mathfrak{n} \supset \mathfrak{n}_1 \times \cdots \times \mathfrak{n}_s$. On the other hand, Proposition 7.11 gives us $p_i(\mathfrak{n}) = \mathfrak{n}_i$ for all $i = 1, \ldots, s$, which shows $\mathfrak{n} \subset \mathfrak{n}_1 \times \cdots \times \mathfrak{n}_s$. Thus we obtain $\mathfrak{n} = \mathfrak{n}_1 \times \cdots \times \mathfrak{n}_s$ and hence the proposition. □

7.4 Proof of Theorem F

In this subsection, we establish Theorem F as a consequence of Theorem 7.17. This theorem shows that if a closed subvariety of an abelian variety has dense small points, then it is contained in the translate of the sum of the maximal nowhere degenerate abelian subvariety and the identity component of its stabilizer by a special point.

We begin with two lemmas. For a subvariety X of an abelian variety A, let $\langle X \rangle$ denote the abelian subvariety of A generated by X, that is, the smallest abelian subvariety containing X.

Lemma 7.14. *Let X be a closed subvariety of an abelian variety A over $\overline{K}$. Assume that $0 \in X$. Let v be a place of $\overline{K}$. Then $\langle X \rangle_v$ is the smallest analytic subgroup of A_v containing X_v.*

Proof. Let us consider, for each $l \in \mathbb{N}$, a morphism $X^{2l} \to A$ given by

$$(x_1, x_2, \ldots, x_{2l-1}, x_{2l}) \mapsto (x_1 - x_2) + \cdots + (x_{2l-1} - x_{2l}). \tag{7.2}$$

Let X_l be the image of this morphism. It is a closed subvariety of A. Note that $X_l \subset \langle X \rangle$. Taking into account that $0 \in X$, we find $X \subset X_1 \subset X_2 \subset \cdots \subset X_l \subset \cdots$. Since each X_l is an irreducible closed subset, there exists l_0 such that $X_l = X_{l_0}$ for all $l \geqslant l_0$, and thus $\bigcup_{l \in \mathbb{N}} X_l = X_{l_0}$. We have $X_l + X_m \subset X_{l+m}$, $0 \in X_l$, and $-X_l = X_l$ for all $l, m \in \mathbb{N}$ by their definitions, which tells us that $\bigcup_{l \in \mathbb{N}} X_l = X_{l_0}$ is a subgroup scheme. Since X_{l_0} is irreducible, it follows that X_{l_0} is an abelian subvariety. Since $X_{l_0} \subset \langle X \rangle$ and since $\langle X \rangle$ is the smallest abelian subvariety containing X, we obtain $\langle X \rangle = X_{l_0}$.

Let B be an analytic subgroup of A_v containing X_v. We then have $B \supset (X_{l_0})_v$ by the definition of X_{l_0} and hence $B \supset \langle X \rangle_v$. This shows that $\langle X \rangle_v$ is the smallest analytic subgroup containing X_v. □

Lemma 7.15. *Let A be an abelian variety over $\overline{K}$ and let X be a closed subvariety of A. Let v be a place of $\overline{K}$. Suppose that A is simple and is degenerate at v. Then if $\overline{\mathrm{val}}(X_v)$ consists of a single point, then the same holds for X, where $\overline{\mathrm{val}}$ is the valuation map for A_v.*

Proof. Taking the translation of X by a point in $A(\overline{K})$, we may assume that $0 \in X$. Then $\overline{\mathrm{val}}(X_v) = \{\mathbf{\overline{0}}\}$ by assumption, and hence $X_v \subset \overline{\mathrm{val}}^{-1}(\mathbf{\overline{0}})$. Since $\overline{\mathrm{val}}^{-1}(\mathbf{\overline{0}})$ is an analytic subgroup of A_v° (cf. § 3.1), it follows from Lemma 7.14 that $\langle X \rangle_v \subsetneq A_v$, and thus $\langle X \rangle \subsetneq A$. Since A is simple, it follows that $\langle X \rangle = \{0\}$, which shows the lemma. □

We now show the following proposition.

Proposition 7.16. *Let X be a closed subvariety of an abelian variety A over $\overline{K}$. Then the following statements are equivalent.*

(a) *The subvariety X is tropically trivial.*
(b) *There exists a point $a \in A(\overline{K})$ such that $X \subset a + \mathfrak{m}$, where $\mathfrak{m}$ is the maximal nowhere degenerate abelian subvariety of A.*

Proof. We first show that (a) implies (b). We may assume that $\mathfrak{m} \neq A$. By Poincaré's reducibility theorem and Mumford's remark [Mum08, p. 157], there exist an abelian subvariety A' of A and an isogeny $\beta : A \to A' \times \mathfrak{m}$. Note that $\text{nd-rk}(A') = 0$ by Proposition 7.13. Let $p : A' \times \mathfrak{m} \to A'$ be the canonical projection and put $\psi := p \circ \beta : A \to A'$.

By the Poincaré complete reducibility theorem, there exists an isogeny $\nu : A' \to B_1' \times \cdots \times B_s'$, where each B_i' is a simple abelian variety. Since $\text{nd-rk}(A') = 0$, it follows from Proposition 7.13 that $\text{nd-rk}(B_i') = 0$ for any $i = 1, \ldots, s$, and thus any B_i' is degenerate at some place v_i.

Let $p_i : B_1' \times \cdots \times B_s' \to B_i'$ be the canonical projection for each i, and put $\psi_i := p_i \circ \nu \circ \psi : A \to B_i'$. Since $\overline{\mathrm{val}}(X_{v_i})$ is a single point by our assumption, so is $\overline{(\psi_i)}_{\mathrm{aff}}(\overline{\mathrm{val}}(X_{v_i}))$, where $\overline{(\psi_i)}_{\mathrm{aff}}$ is the affine map associated to ψ_i (cf. § 3.2). Since $\overline{\mathrm{val}}(\psi_i(X)_{v_i}) = \overline{(\psi_i)}_{\mathrm{aff}}(\overline{\mathrm{val}}(X_{v_i}))$ (cf. (3.8)), we see that $\overline{\mathrm{val}}(\psi_i(X)_{v_i})$ is a single point. Since B_i' is simple and degenerate at v_i, it follows from Lemma 7.15 that $\psi_i(X)$ is a single point for any i, which implies that $\nu(\psi(X))$ is a single point. Since ν is finite and $\psi(X)$ is connected, we see that $\psi(X)$ is a single point, and thus we write $\psi(X) = \{b'\}$ for some $b' \in B'(\overline{K})$.

We then have $p^{-1}(\psi(X)) = \{b'\} \times \mathfrak{m}$, and hence

$$\psi^{-1}(\psi(X)) = \beta^{-1}(\{b'\} \times \mathfrak{m}) = \bigcup_{\sigma \in \beta^{-1}(b',0)} (\sigma + \mathfrak{m}),$$

which is a finite union of subvarieties. Since $X \subset \bigcup_{\sigma \in \beta^{-1}(b',0)} (\sigma + \mathfrak{m})$ and X is irreducible, there exists a point $a \in \beta^{-1}(b', 0)$ with $X \subset a + \mathfrak{m}$. Thus we obtain (b).

Next we show that (b) implies (a). Since $\mathfrak{m}$ is nowhere degenerate, the valuation map for $\mathfrak{m}$ is trivial. Take any $v \in M_{\overline{K}}$. By the compatibility of valuation maps with homomorphisms (cf. (3.8)), we have $\overline{\mathrm{val}}(\mathfrak{m}_v) = \{\overline{\mathbf{0}}\}$, where $\overline{\mathrm{val}} : A_v \to \mathbb{R}^n/\Lambda$ is the valuation map for A_v. Then $\overline{\mathrm{val}}(X_v) \subset \{\overline{\mathrm{val}}(a)\}$, and thus $\overline{\mathrm{val}}(X_v)$ is a single point. Since v is arbitrary, we conclude that X is tropically trivial. $\square$

Using Theorem 6.2 with Proposition 7.16, we show the following theorem. Here, let G_X° denote the connected component of the stabilizer G_X with $0 \in G_X^{\circ}$.

THEOREM 7.17. *Let A be an abelian variety over $\overline{K}$ with maximal nowhere degenerate abelian subvariety $\mathfrak{m}$ and let X be a closed subvariety of A. Suppose that X has dense small points. Then there exists a special point a_0 of A such that $X \subset a_0 + G_X^{\circ} + \mathfrak{m}$.*

Proof. Let $\phi : A \to A/G_X$ be the quotient homomorphism. Then $\phi(X) = X/G_X$ has trivial stabilizer. Since X has dense small points, it follows that $\phi(X)$ is tropically trivial by Theorem 6.2. Note that by Proposition 7.11, $\phi(\mathfrak{m})$ is the maximal nowhere degenerate abelian subvariety of A/G_X. By Proposition 7.16, it follows that there exists a point $a_0' \in A(\overline{K})$ such that $\phi(X) \subset \phi(a_0') + \phi(\mathfrak{m})$, and hence $X \subset a_0' + G_X + \mathfrak{m}$. Since X is irreducible, $X \subset a_0' + G_X' + \mathfrak{m}$ for some connected component G_X' of G_X. Note that $G_X' = b + G_X^{\circ}$ for some torsion point $b \in A(\overline{K})$. Setting $a_0 := a_0' + b$, we then have $X \subset a_0 + G_X^{\circ} + \mathfrak{m}$.

Now it suffices to show that a_0 can be chosen to be a special point. Let $\psi : A \to A/(G_X^{\circ} + \mathfrak{m})$ be the quotient. Then $\psi(X) = \{\psi(a_0)\}$ has dense small points by [Yam13, Lemma 2.1], and thus $\psi(a)$ is a special point of $A/(G_X^{\circ} + \mathfrak{m})$ (cf. Remark 7.4). By [Yam13, Lemma 2.10], there exists a special point a_0'' of A with $\psi(a_0'') = \psi(a_0)$. Replacing a_0 with a_0'', we have $X \subset a_0 + G_X^{\circ} + \mathfrak{m}$ with a_0 special. Thus, the proof of the theorem is complete. $\square$

As a consequence, we obtain the following two corollaries. The first gives us a sufficient condition for a closed subvariety having dense small points to be the translate of an abelian subvariety by a special point.

COROLLARY 7.18. *Let A be an abelian variety over $\overline{K}$ and let X be a closed subvariety of A. Let G_X be the stabilizer of X in A. Assume that $\dim X/G_X \geqslant \text{nd-rk}(A/G_X)$. Then if X has dense small points, then there exists a special point $x_0 \in X(\overline{K})$ such that $X = x_0 + G_X$.*

Proof. Let $\phi : A \to A/G_X$ be the quotient homomorphism and let $\mathfrak{m}$ be the maximal nowhere degenerate abelian subvariety of A. Then $\phi(\mathfrak{m})$ is the maximal nowhere degenerate abelian subvarieties of A/G_X by Proposition 7.11. Suppose that X has dense small points. By Theorem 7.17, there exists a special point x_0 of A such that $X/G_X = \phi(X) \subset \phi(x_0) + \phi(\mathfrak{m})$. Since $\dim(X/G_X) \geqslant \text{nd-rk}(A/G_X) = \dim(\phi(\mathfrak{m}))$, it follows that $X/G_X = \phi(x_0) + \phi(\mathfrak{m})$. Since X/G_X has trivial stabilizer, we find that $X/G_X = \{\phi(x_0)\}$, which shows that $X = x_0 + G_X$. Further, this equality also shows that $x_0 \in X(\overline{K})$. Thus we obtain the corollary. $\square$

The second corollary shows Theorem F, which gives us a partial answer to the geometric Bogomolov conjecture.

Corollary 7.19 (cf. Theorem F). *Let A be an abelian variety over $\overline{K}$ with* $\text{nd-rk}(A) \leqslant 1$ *and let X be a closed subvariety of A. Suppose that X has dense small points. Then X is the translate of an abelian subvariety by a special point. In particular, it is a special subvariety.*

Proof. Let $\phi : A \to A/G_X$ be the quotient. Since X has dense small points, Theorem 7.17 gives us a special point $x_0 \in X(\overline{K})$ such that $X \subset x_0 + G_X + \mathfrak{m}$, where $\mathfrak{m}$ is the maximal nowhere degenerate abelian subvariety of A. We then have $X/G_X \subset \phi(x_0) + \phi(\mathfrak{m})$.

Note that if we show $\dim X/G_X = 0$, then $X/G_X = \{\phi(x_0)\}$ and hence $X = x_0 + G_X$. Further, since X is irreducible, G_X is an abelian subvariety. Therefore it suffices to claim that $\dim X/G_X = 0$. Since $\text{nd-rk}(A) \leqslant 1$, $\phi(\mathfrak{m})$ has dimension 0 or 1. If $\dim \phi(\mathfrak{m}) = 0$, then we are done. Suppose that $\dim \phi(\mathfrak{m}) = 1$. Since X/G_X has trivial stabilizer, we find that $X/G_X \subsetneq \phi(x_0) + \phi(\mathfrak{m})$, which implies $\dim X/G_X = 0$. Thus we obtain the corollary. ◻

Remark 7.20. Let A be an abelian variety over $\overline{K}$.

(1) Suppose that $b(A_v) \leqslant 1$ for some place v, where $b(A_v)$ is the abelian rank of A_v. Since $b(A_v) \geqslant \text{nd-rk}(A)$ in general, Corollary 7.19 generalizes Theorem D and hence Theorem B.
(2) Suppose that A is simple and somewhere degenerate. Then $\text{nd-rk}(A) = 0$ and hence the geometric Bogomolov conjecture holds for A.

7.5 Reduction to the nowhere degenerate case

In this subsection, we reduce the geometric Bogomolov conjecture for an abelian variety to the conjecture for its maximal nowhere degenerate abelian subvariety (Theorem E).

We first establish the following theorem.

Theorem 7.21. *Let A be an abelian variety over $\overline{K}$, B a nowhere degenerate abelian variety over $\overline{K}$, and let $\phi : A \to B$ be a surjective homomorphism with* $\dim B = \text{nd-rk}(A)$. *Let X be a closed subvariety of A. Suppose that X has dense small points and that $\phi(X)$ is a special subvariety. Then X is a special subvariety.*

Proof. Let G_X° be the connected component of the stabilizer G_X of X with $0 \in G_X^\circ$. Let $\mathfrak{m}$ be the maximal nowhere degenerate abelian subvariety of A. Note that the restriction $\phi|_{\mathfrak{m}} : \mathfrak{m} \to B$ is an isogeny. Indeed, we have $\dim \mathfrak{m} = \dim B$ by assumption, and $\phi(\mathfrak{m}) = B$ by Proposition 7.11.

Step 1. First we show the assertion under the assumption that $G_X^\circ = 0$. Under this assumption, Theorem 7.17 gives us a special point a_0 of A such that $X \subset a_0 + \mathfrak{m}$. We put $X' = X - a_0$. Then $X' \subset \mathfrak{m}$, and X' has dense small points. On the other hand, since $\phi(a_0)$ is a special point of B, $\phi|_{\mathfrak{m}}(X') = \phi(X) - \phi(a_0)$ is also special (cf. [Yam13, Remark 2.6]). By Proposition 7.5, it follows that X' is special. Since a_0 is a special point, $X = a_0 + X'$ is special (cf. [Yam13, Remark 2.6]).

Step 2. Next we conclude the theorem by reducing the general case to Step 1. The quotient $B/\phi(G_X^\circ)$ is nowhere degenerate by Lemma 7.8(2). We would like to show that $\text{nd-rk}(A/G_X^\circ) = \dim(B/\phi(G_X^\circ))$.

First, we note that

$$\dim(G_X^\circ \cap \mathfrak{m}) = \dim \phi(G_X^\circ). \tag{7.3}$$

Indeed, let $\mathfrak{n}$ be the maximal nowhere degenerate abelian subvariety of G_X° and let $(G_X^\circ \cap \mathfrak{m})^\circ$ be the connected component of $G_X^\circ \cap \mathfrak{m}$ with $0 \in (G_X^\circ \cap \mathfrak{m})^\circ$. By Lemma 7.8(4), we then have

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$(G_X^\circ \cap \mathfrak{m})^\circ \subset \mathfrak{n}$. Further, the maximality of $\mathfrak{m}$ tells us that $\mathfrak{n} \subset \mathfrak{m}$ and hence $\mathfrak{n} \subset (G_X^\circ \cap \mathfrak{m})^\circ$. Thus $(G_X^\circ \cap \mathfrak{m})^\circ = \mathfrak{n}$. On the other hand, $\phi(G_X^\circ)$ is nowhere degenerate by Lemma 7.8(4). Therefore, by Proposition 7.11, we have $\phi|_{\mathfrak{m}}((G_X^\circ \cap \mathfrak{m})^\circ) = \phi|_{\mathfrak{m}}(\mathfrak{n}) = \phi(G_X^\circ)$. Since $\phi|_{\mathfrak{m}}$ is a finite morphism, this shows (7.3).

Let $\alpha : A \to A/G_X^\circ$ and $\beta : B \to B/\phi(G_X^\circ)$ denote the quotient homomorphisms and let $\phi' : A/G_X^\circ \to B/\phi(G_X^\circ)$ be the homomorphism induced from ϕ. Remark that Proposition 7.11 tells us that $\alpha(\mathfrak{m})$ is the maximal nowhere degenerate abelian subvariety of A/G_X°, and hence $\operatorname{nd-rk}(A/G_X^\circ) = \dim \alpha(\mathfrak{m})$. We consider the homomorphism $\phi'|_{\alpha(\mathfrak{m})} : \alpha(\mathfrak{m}) \to B/\phi(G_X^\circ)$. Since $\alpha(\mathfrak{m}) \cong \mathfrak{m}/(G_X^\circ \cap \mathfrak{m})$, we have $\dim \alpha(\mathfrak{m}) = \operatorname{nd-rk}(A) - \dim(G_X^\circ \cap \mathfrak{m})$. Since $\operatorname{nd-rk}(A) = \dim B$ and since we have (7.3), that equals $\dim B - \dim \phi(G_X^\circ)$. Therefore $\dim \alpha(\mathfrak{m}) = \dim B/\phi(G_X^\circ)$, and thus $\operatorname{nd-rk}(A/G_X^\circ) = \dim(B/\phi(G_X^\circ))$.

Now we can apply Step 1 to $\phi' : A/G_X^\circ \to B/\phi(G_X^\circ)$. Indeed, we know that $\operatorname{nd-rk}(A/G_X^\circ) = \dim(B/\phi(G_X^\circ))$. Since X has dense small points, so does $\alpha(X)$. Since $\phi(X)$ is special, $\phi'(\alpha(X)) = \beta(\phi(X))$ is also special by Lemma 7.1(2). Further, $\alpha(X)$ has stabilizer G_X/G_X°, which has dimension 0. Now applying Step 1, we see that $X/G_X^\circ = \alpha(X)$ is a special subvariety. By Lemma 7.1(1), we conclude that X is a special subvariety. □

As a consequence, we obtain the following corollary (Theorem E).

Corollary 7.22 (Theorem E). *Let A be an abelian variety and let $\mathfrak{m}$ be the maximal nowhere degenerate abelian subvariety of A. Then the following are equivalent.*

(a) *The geometric Bogomolov conjecture holds for A.*
(b) *The geometric Bogomolov conjecture holds for $\mathfrak{m}$.*

Proof. By the Poincaré complete reducibility theorem and Mumford's remark [Mum08, p. 157], we have a surjective homomorphism $\phi : A \to \mathfrak{m}$. Then (a) implies (b) by Lemma 7.7. To see the other direction, suppose (b) and let X be a closed subvariety of A having dense small points. Then $\phi(X)$ has dense small points. Since the conjecture holds for $\mathfrak{m}$, $\phi(X)$ is a special subvariety of $\mathfrak{m}$. By Theorem 7.21, we conclude that X is special. Thus (b) implies (a). □

Further, we have the following corollary.

Corollary 7.23. *For any non-negative integer s, the following statements are equivalent.*

(a) *Conjecture 7.3 holds for any abelian variety A with* $\operatorname{nd-rk}(A) \leqslant s$.
(b) *Conjecture 7.3 holds for any nowhere degenerate abelian variety B with* $\dim B \leqslant s$.

Proof. It is trivial that (a) implies (b). Suppose that (b) holds. Let A be an abelian variety over $\overline{K}$ with $\operatorname{nd-rk}(A) \leqslant s$. Let $\mathfrak{m}$ be the maximal nowhere degenerate abelian subvariety of A. Then $\dim \mathfrak{m} \leqslant s$, so that Conjecture 7.3 holds for $\mathfrak{m}$ by assumption. It follows from Corollary 7.22 that Conjecture 7.3 holds for A. □

We remark that Theorem F follows also from Corollary 7.23, since the geometric Bogomolov conjecture holds for elliptic curves.

Finally in this section, we remark that by Corollary 7.23, Conjecture 7.3 is equivalent to the following conjecture.

Conjecture 7.24 (Geometric Bogomolov conjecture for nowhere degenerate abelian varieties). Let A be a nowhere degenerate abelian variety over $\overline{K}$ and let X be a closed subvariety of A. Then if X has dense small points, then it should be a special subvariety.

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8. Geometric Bogomolov conjecture for curves

In this section we consider the geometric Bogomolov conjecture for curves. This conjecture claims that the set of $\overline{K}$-points of a non-isotrivial smooth projective curve of genus at least 2 is 'discrete' in its Jacobian with respect to the Néron–Tate seminorm. To be precise, let C be a smooth projective curve over $\overline{K}$ of genus $g \geqslant 2$ and let J_C be its Jacobian variety. For a divisor D of degree 1 on C, let $j_D : C \to J_C$ be the embedding given by $x \mapsto [x - D]$, where $[x - D]$ denotes the divisor class of $x - D$. Let $\| \cdot \|_{NT}$ be the canonical Néron–Tate seminorm arising from the canonical Néron–Tate pairing on J_C. The following assertion is called the geometric Bogomolov conjecture for curves. Here C is said to be *isotrivial* if there is a curve C' over k such that $C'_{\overline{K}} \cong C$.

Conjecture 8.1. Assume that C is non-isotrivial. For any divisor D of degree 1 on C and for any $P \in J_C(\overline{K})$, there should exist an $\epsilon > 0$ such that

$$\{x \in C(\overline{K}) \mid \|j_D(x) - P\|_{NT} \leqslant \epsilon\}$$

is a finite set.

A stronger version of this conjecture is also well known as the effective geometric Bogomolov conjecture for curves.

Conjecture 8.2. Assume that C is non-isotrivial. Then there should exist an $\epsilon > 0$ such that $\{x \in C(\overline{K}) \mid \|j_D(x) - P\|_{NT} \leqslant \epsilon\}$ is a finite set for any divisor D of degree 1 on C and any $P \in J_C(\overline{K})$. Moreover, if C has a stable model over $\mathfrak{B}$, then we can describe such an ϵ effectively in terms of geometric information of the stable model.

There are some results on Conjecture 8.2 in the setting where $\mathfrak{B}$ is a curve, i.e. K is a function field of one variable. In char $K = 0$, after partial results by Zhang [Zha93], Moriwaki [Mor96, Mor97, Mor98], the author [Yam02, Yam08], and Faber [Fab09], Cinkir proved Conjecture 8.2 in [Cin11]. In the positive characteristic case, Conjecture 8.1 as well as Conjecture 8.2 are unsolved in full generality, and there are only some partial results. Conjecture 8.2 is solved in the case where:

- the stable model of C has only irreducible fibers [Mor98];
- C is a curve of genus 2 [Mor96];
- C is a hyperelliptic curve [Yam08]; or
- C is non-hyperelliptic and $g = 3$ [Yam02].

The geometric Bogomolov conjecture for abelian varieties implies Conjecture 8.1.[15] Indeed, let $\hat{h}$ be the Néron–Tate height such that $\hat{h}(x) = \|x\|^2_{NT}$. Then we have $\hat{h}(j_{D+P}(x)) = \|j_D(x) - P\|_{NT}$ for all $P \in J_C(\overline{K})$. Thus Conjecture 8.1 is equivalent to saying that, if C is non-isotrivial, then $j_D(C)$ does not have dense small points for any divisor D on C of degree 1. To show its contraposition, we assume that $j_D(C)$ has dense small points for some D. Since Conjecture 7.3 is assumed to be true, it follows that $j_D(C)$ is a special subvariety of J_C. Since j_D is an embedding and since the trace homomorphism is a purely inseparable finite morphism (cf. [Lan83b, VIII, § 3, Corollary 2] or [Yam13, Lemma 1.4]), there exist a smooth projective curve C_0 over k and a purely inseparable finite morphism $\phi : (C_0)_{\overline{K}} \to C$. Let

[15] The argument from here to the end of the proof of Theorem 8.3 works well even if $\mathfrak{B}$ is a higher-dimensional variety.

$C_0 \to C_0^{(q)}$ be the qth relative Frobenius morphism, where q is the degree of ϕ. Then its base-change to $\overline{K}$ is also the qth relative Frobenius morphism. It follows from [Sil86, Corollary II 2.12] that $(C_0^{(q)})_{\overline{K}} \cong C$. We conclude that C is isotrivial.

The following result is obtained as a consequence of Theorem 6.2 without assumption on the characteristic. Recall that a curve C over $\overline{K}$ is of *compact type at* v if the special fiber of the stable model of C_v is a tree of smooth irreducible components. Further, C is said to be of *non-compact type at* v if it is not of compact type at v. It is well known that C is of compact type at v if and only if C_v has non-degenerate Jacobian variety (cf. [BLR90, ch. 9]).

Theorem 8.3. *Suppose that there exists a place at which C is of non-compact type. Then, for any divisor D of C of degree 1 and for any $P \in J_C(\overline{K})$, there exists an $\epsilon > 0$ such that $\{x \in C(\overline{K}) \mid \|j_D(x) - P\|_{NT} \leqslant \epsilon\}$ is finite.*

Proof. Let v be a place at which C is of non-compact type. Then the Jacobian J_C is degenerate at v. It follows from the Poincaré complete reducibility theorem and Proposition 7.13 that there is a non-trivial simple abelian variety A' over $\overline{K}$ degenerate at v with a surjective homomorphism $\phi : J_C \to A'$.

Let D be any divisor on C of degree 1. Let $x_0 \in j_D(C(\overline{K}))$ be a point. Note that J_C itself is the smallest abelian subvariety of J_C containing $j_D(C) - x_0$. Since ϕ is surjective and since $\dim A' > 0$, it follows that the image of $j_D(C) - x_0$ by ϕ cannot be a point. By Lemma 7.15, we see that $j_D(C)$ is tropically non-trivial. Since $j_D(C)$ has at most finite stabilizer in J_C, it follows from Theorem 6.2 that $j_D(C)$ does not have dense small points. Thus the assertion holds for C, as is noted above. □

We end with a couple of remarks. In characteristic zero, Conjecture 8.1 is deduced from the combination of [Mor98, Theorem E] and our Theorem 8.3. This suggests that we can avoid the hard analysis on metric graphs carried out in [Cin11] inasmuch as we consider the non-effective version only. If the inequality of [Mor98, Theorem D] also holds in positive characteristic, then the same proof as for [Mor98, Theorem E] works, and hence we obtain Conjecture 8.1. Thus our argument makes a contribution to Conjecture 8.1. We should also note, however, that our approach does not say anything on Conjecture 8.2, the effective version of Conjecture 8.1.

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