SINGULAR PROBLEMS MODELLING PHENOMENA IN THE
THEORY OF PSEUDOPLASTIC FLUIDS

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Abstract

Existence criteria are presented for nonlinear singular initial and boundary value problems. In particular our theory includes a problem arising in the theory of pseudoplastic fluids.

1. Introduction

This paper is motivated by the boundary value problem

\[ \begin{cases}
  y^{1/n}y'' + nt = 0, & 0 < t < 1 \\
  y'(0) = y(1) = 0
\end{cases} \]

which arises in the theory of pseudoplastic fluids. In particular we present existence theory for the mixed boundary value problem

\[ \begin{cases}
  \frac{1}{p} (py')' + q(t)f (t, y) = 0, & 0 < t < 1 \\
  \lim_{t \to 0^+} p(t)y'(t) = y(1) = 0
\end{cases} \]

where \( f : [0, 1] \times (0, \infty) \to \mathbb{R} \) is continuous. Notice \( f \) may be singular at \( y = 0 \). Problems of the above form have been discussed extensively in the literature (see [2–11]) usually when \( f \) is positone, that is, \( f : (0, 1) \times (0, \infty) \to (0, \infty) \). Only a handful of papers (see [3–5] and the references therein) have appeared where the nonlinearity \( f \) is allowed to change sign. This paper presents a new theory, with the idea being to approximate the singular problem by a sequence of nonsingular

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problems each of which has a lower solution $\alpha_m$ and an upper solution $\beta$, and then use a limiting argument. This seems to be more natural and more general than the theory presented in [3–5] since the study of lower solutions to nonsingular problems is well documented. Also in this paper we discuss the singular initial value problem

$$\begin{cases}
y' = q(t)f(t, y), & 0 < t < T(< \infty) \\
y(0) = 0.
\end{cases}$$

For the remainder of this section we describe the physical problem which motivates our study. The boundary layer equations for steady flow over a semi-infinite plate [1] are

$$\begin{align*}
U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} &= \frac{1}{\rho} \frac{\partial \tau_{XY}}{\partial Y}, \\
\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} &= 0,
\end{align*}$$

where the $X$ and $Y$ axes are taken along and perpendicular to the plate, $\rho$ is the density, $U$ and $V$ are the velocity components parallel and normal to the plate and the shear stress $\tau_{XY} = K (\partial U / \partial Y)^n$. The case $n = 1$ corresponds to a Newtonian fluid and for $0 < n < 1$ the power law relation between shear stress and rate of strain describes pseudoplastic non-Newtonian fluids. The fluid has zero velocity on the plate and the flow approaches stream conditions far from the plate, that is,

$$U(X, 0) = V(X, 0) = 0, \quad U(X, \infty) = U_\infty,$$

where $U_\infty$ is the uniform potential flow. The above results (if we use stream function-similarity variables) [1,9] in a third-order infinite interval problem

$$F''' + F(F'')^{2-n} = 0, \quad F(0) = F'(0) = 0, \quad F'(\infty) = 1.$$

Now use the Crocco-type transformation $u = F'$ and $G = F''$ to obtain

$$G'' + (n-1)G^{-1}G' + u = 0, \quad G'(0) = 0, \quad G(1) = 0.$$
2. Mixed boundary value problems

Motivated by the example in Section 1 concerning non-Newtonian fluids, we consider the mixed boundary value problem

\[
\begin{align*}
\frac{1}{p} (py') + q(t)f(t, y) &= 0, \\
\lim_{t \to 0^+} p(t)y'(t) &= y(1) = 0.
\end{align*}
\] (2.1)

We note also that we do not assume \( \int_0^1 ds/p(s) < \infty \). For our first result in this section we will assume the following conditions are satisfied:

\( p \in C[0, 1] \cap C^1(0, 1) \) with \( p > 0 \) on \( (0, 1) \) \hspace{1cm} (2.2)
\( q \in C(0, 1) \) with \( q > 0 \) on \( (0, 1) \) \hspace{1cm} (2.3)
\( \int_0^1 p(s)q(s)ds < \infty \) and \( \int_0^1 \frac{1}{p(t)} \int_0^t p(s)q(s)ds \, dt < \infty \) \hspace{1cm} (2.4)
\( f : [0, 1] \times (0, \infty) \to \mathbb{R} \) is continuous \hspace{1cm} (2.5)
\( \exists n_0 \in \{1, 2, \ldots \} \) and associated with each \( m \in N_0 = \{n_0, n_0 + 1, \ldots \} \),
\( \exists \alpha_m \in C[0, 1] \cap C^2(0, 1) \), \( p\alpha_m' \in AC[0, 1] \),
with \( p(t)q(t)f(t, \alpha_m(t)) + (p(t)\alpha_m'(t))' \geq 0 \) for \( t \in (0, 1) \), \hspace{1cm} (2.6)
\( \lim_{t \to 0^+} p(t)\alpha_m'(t) \geq 0 \) and \( 0 < \alpha_m(1) \leq 1/m \)
\( \exists \alpha \in C[0, 1], \alpha > 0 \) on \( [0, 1] \) and \( \alpha(t) \leq \alpha_m(t) \), \hspace{1cm} (2.7)
\( \exists \beta \in C[0, 1] \cap C^2(0, 1) \), \( p\beta' \in AC[0, 1] \) with
\( p(t)q(t)f(t, \beta(t)) + (p(t)\beta'(t))' \leq 0 \) for \( t \in (0, 1) \), \hspace{1cm} (2.8)
\( \lim_{t \to 0^+} p(t)\beta'(t) \leq 0 \) and \( \beta(1) \geq \beta_0 > 0 \)
and
\( \alpha_m(t) \leq \beta(t), \quad t \in [0, 1] \) for each \( m \in N_0 \). \hspace{1cm} (2.9)

**Theorem 2.1.** \( (I) \) Suppose (2.2)–(2.9) hold and in addition assume the following condition is satisfied:

\[
\begin{align*}
0 \leq f(t, y) \leq g(y) &\quad \text{on } [0, 1] \times (0, a_0] \quad \text{with } g > 0 \\
\text{continuous and nonincreasing on } (0, \infty); \hspace{1cm} (2.10)
\end{align*}
\]

Here \( a_0 = \sup_{t \in [0, 1]} \beta(t) \). Then (2.1) has a solution \( y \in C[0, 1] \cap C^2(0, 1) \) with \( y(t) \geq \alpha(t) \) for \( t \in [0, 1] \).
(II) Suppose (2.2)–(2.9) hold and in addition assume the following condition is satisfied:

\[ f(t, x) - f(t, y) > 0 \text{ for } 0 < x < y, \text{ for each fixed } t \in (0, 1). \]  

(2.11)

Then (2.1) has a solution \( y \in C[0, 1] \cap C^2(0, 1) \) with \( y(t) \geq \alpha(t) \) for \( t \in [0, 1] \).

PROOF. Without loss of generality assume \( \beta_0 \geq 1/n_0 \). Fix \( m \in N_0 \) and consider the boundary value problem

\[
\begin{align*}
(p y')' + p q f_m^*(t, y) &= 0, & 0 < t < 1 \\
\lim_{t \to 0^+} p(t) y'(t) &= 0 \\
y(1) &= 1/m,
\end{align*}
\]

(2.12)

where

\[
f_m^*(t, y) = \begin{cases} 
  f(t, \beta(t)) + r(\beta(t) - y), & y > \beta(t) \\
  f(t, y), & \alpha_m(t) \leq y \leq \beta(t) \\
  f(t, \alpha_m(t)) + r(\alpha_m(t) - y), & y < \alpha_m(t)
\end{cases}
\]

with \( r : \mathbb{R} \to [-1, 1] \) the radial retraction defined by

\[
r(u) = \begin{cases} 
  u, & |u| \leq 1 \\
  u/|u|, & |u| > 1
\end{cases}
\]

It is immediate from Schauder’s fixed point theorem (see [10]) that (2.12) has a solution \( y_m \in C[0, 1] \) (in fact \( y_m \in C[0, 1] \cap C^2(0, 1) \) with \( p y_m' \in AC[0, 1] \)). A standard argument (see [10, Chapter 5]; note \( f_m^* : [0, 1] \times \mathbb{R} \to \mathbb{R} \) is continuous) guarantees that

\[
\alpha_m(t) \leq y_m(t) \leq \beta(t) \quad \text{for } t \in [0, 1].
\]

(2.13)

As a result \( y_m \) is a solution of

\[
\begin{align*}
(p y')' + p q f^*(t, y) &= 0, & 0 < t < 1 \\
\lim_{t \to 0^+} p(t) y'(t) &= 0 \\
y(1) &= 1/m.
\end{align*}
\]

(2.14)

In addition (2.7) guarantees that

\[
\alpha(t) \leq \alpha_m(t) \leq y_m(t) \leq \beta(t) \quad \text{for } t \in [0, 1].
\]

(2.15)

The proof is now broken into two cases.
Case (A). Suppose (2.10) holds.

We first show

\[ \{y_m\}_{m \in \mathbb{N}_0} \text{ is a bounded, equicontinuous family on } [0, 1]. \] (2.16)

First notice from (2.10) that \((py_m')' \leq 0\) on \((0, 1)\), so \(py_m' \leq 0\) on \((0, 1)\). In addition

\[-(p(t)y_m'(t))' \leq p(t)q(t)g(y_m(t)) \text{ for } t \in (0, 1),\]

so integration from 0 to \(t\) yields

\[-p(t)y_m'(t) \leq g(y_m(t)) \int_0^t p(s)q(s) \, ds \text{ for } t \in (0, 1).\]

As a result

\[ 0 \leq \frac{-y_m'(t)}{g(y_m(t))} \leq \frac{1}{p(t)} \int_0^t p(s)q(s) \, ds \text{ for } t \in (0, 1).\]

Now consider \(I(z) = \int_0^z \frac{1}{g(u)} \, du\). For \(t, s \in [0, 1]\) we have

\[ |I(y_m(t)) - I(y_m(s))| = \left| \int_s^t \frac{y_m'(x)}{g(y_m(x))} \, dx \right| \leq \left| \int_s^t \frac{1}{p(x)} \int_0^x p(z)q(z) \, dz \, dx \right|, \]

so

\[ \{I(y_m)\}_{m \in \mathbb{N}_0} \text{ is a bounded, equicontinuous family on } [0, 1]. \] (2.17)

The uniform continuity of \(I^{-1}\) on \([0, I(a_0)]\) together with (2.17) and

\[ |y_m(t) - y_m(s)| = |I^{-1}(I(y_m(t))) - I^{-1}(I(y_m(s)))| \]

guarantees (2.16). A standard argument \([2, \text{ page } 90]\) using the Arzelà-Ascoli theorem (and (2.15)) completes the proof.

Case (B). Suppose (2.11) holds.

We begin by showing

\[ y_{m+1}(t) \leq y_m(t) \text{ for } t \in [0, 1] \text{ for each } m \in \mathbb{N}_0. \] (2.18)

Suppose (2.18) is false. Then for some \(m \in \mathbb{N}_0\), \(y_{m+1} - y_m\) would have a positive absolute maximum at say \(\tau_0 \in [0, 1]\). Suppose to begin with \(\tau_0 \in (0, 1)\), so \((y_{m+1} - y_m)'(\tau_0) = 0\) and \((p(y_{m+1} - y_m))'(\tau_0) \leq 0\). On the other hand, (2.11) implies

\[ (p(y_{m+1} - y_m))'(\tau_0) = -p(\tau_0)q(\tau_0)[f(\tau_0, y_{m+1}(\tau_0)) - f(\tau_0, y_m(\tau_0))] > 0, \]

a contradiction. If \(\tau_0 = 0\) then \(\lim_{r \to 0^+} p(t)[y_{m+1} - y_m]'(t) = 0\) and there exists \(\mu > 0\) with \(y_{m+1}(s) - y_m(s) > 0\) for \(s \in (0, \mu)\). Thus for \(t \in (0, \mu)\) we have from (2.11) that

\[ p(y_{m+1} - y_m)'(t) = \int_0^t p(s)q(s)[f(s, y_m(s)) - f(s, y_{m+1}(s))] \, ds > 0, \]
a contradiction since \( y_{m+1} - y_m \) has a positive absolute maximum at 0. As a result (2.18) holds.

Let's look at the interval \([0, 1 - 1/n_0]\). Let

\[
R_{n_0} = \sup \{ |f(t, y)| : t \in [0, 1 - 1/n_0] \text{ and } \alpha(t) \leq y \leq a_0 \};
\]

(2.19)

here \( a_0 = \sup_{t \in [0, 1]} \beta(t) \). In addition

\[
|y'_m(t)| \leq \frac{R_{n_0}}{p(t)} \int_0^t p(s)q(s)\,ds \quad \text{for } t \in (0, 1 - 1/n_0).
\]

Thus \( \{y_m\}_{m \in N_0} \) is a bounded, equicontinuous family on \([0, 1 - 1/n_0]\). The Arzela-Ascoli theorem guarantees the existence of a subsequence \( N_{n_0} \) of \( N_0 \) and a function \( z_{n_0} \in C[0, 1 - 1/n_0] \) with \( y_m \) converging uniformly on \([0, 1 - 1/n_0]\) to \( z_{n_0} \) as \( m \to \infty \) through \( N_{n_0} \). Proceed inductively to obtain subsequences of integers

\[
N_{n_0} \supseteq N_{n_0 + 1} \supseteq \cdots \supseteq N_k \supseteq \cdots
\]

and functions \( z_k \in C[0, 1 - 1/k] \) with \( y_m \) converging uniformly on \([0, 1 - 1/k]\) to \( z_k \) as \( m \to \infty \) through \( N_k \), and \( z_{k+1} = z_k \) on \([0, 1 - 1/k]\).

Define a function \( y : [0, 1] \to [0, \infty) \) by \( y(x) = z_k(x) \) on \([0, 1 - 1/k]\) and \( y(1) = 0 \). Notice \( y \) is well-defined and \( \alpha(t) \leq y(t) \leq a_0 \) for \( t \in [0, 1) \). Next fix \( t \in (0, 1) \) and let \( k \in \{n_0, n_0 + 1, \ldots\} \) be such that \( 0 < t < 1 - 1/k \). Let \( N_k^* = \{n \in N_k : n \geq k\} \). Now \( y_m, m \in N_k^* \), satisfies

\[
y_m(t) = y_m(0) - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)f(x, y_m(x))\,dx\,ds.
\]

Let \( m \to \infty \) through \( N_k^* \) to obtain

\[
y(t) = y(0) - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)f(x, y(x))\,dx\,ds.
\]

We can do this argument for each \( t \in (0, 1) \), so \((py')(t) + p(t)q(t)f(t, y(t)) = 0\) for \( t \in (0, 1) \) and \( \lim_{t \to 0^+} p(t)y'(t) = 0 \).

It remains to show \( y \) is continuous at 1. Let \( \epsilon > 0 \) be given. Now since \( \lim_{m \to \infty} y_m(1) = 0 \) there exists \( n_1 \in N_0 \) with \( y_{n_1}(1) < \epsilon/2 \). Also since \( y_{n_1} \in C[0, 1] \) there exists \( \delta_{n_1} > 0 \) with \( y_{n_1}(t) < \epsilon/2 \) for \( t \in [1 - \delta_{n_1}, 1] \). From (2.18) for \( m \geq n_1 \) we have \( y_m(t) \leq y_{n_1}(t) < \epsilon/2 \) for \( t \in [1 - \delta_{n_1}, 1] \). As a result for \( m \geq n_1 \) we have

\[
0 \leq \alpha(t) \leq y_m(t) < \epsilon/2 \quad \text{for } t \in [1 - \delta_{n_1}, 1].
\]

Consequently

\[
0 \leq \alpha(t) \leq y(t) < \epsilon/2 < \epsilon \quad \text{for } t \in [1 - \delta_{n_1}, 1],
\]

so \( y \) is continuous at 1.
**Remark 2.1.** In Theorem 2.1 (I) we can replace (2.10) with

\[
\begin{aligned}
|f(t, y)| \leq g(y) \text{ on } [0, 1] \times (0, a_0] \text{ with } g > 0 \\
\text{continuous and nonincreasing on } (0, \infty)
\end{aligned}
\]  

and

\[
\int_0^1 \frac{1}{p(s)} \int_0^s p(x)q(x)g(\alpha(x)) \, dx \, ds < \infty;
\]

here \(a_0 = \sup_{t \in [0, 1]} \beta(t)\). Notice we only used (2.10) to show (2.16). If we assume (2.20) and (2.21) then (2.16) is immediate since

\[
\pm (p(t)y_m'(t))' \leq p(t)q(t)g(y_m(t)) \leq p(t)q(t)g(\alpha(t)) \quad \text{for } t \in (0, 1),
\]

so

\[
|y_m'(t)| \leq \frac{1}{p(t)} \int_0^t p(s)q(s)g(\alpha(s)) \, ds \quad \text{for } t \in (0, 1).
\]

We next state and prove a more general result motivated from Theorem 2.1 (II).

**Theorem 2.2.** Suppose (2.2)–(2.7) hold and in addition assume the following conditions are satisfied:

\[
\begin{aligned}
\text{for each } m \in N_0, \exists \beta_m \in C[0, 1] \cap C^2(0, 1), \ p \beta_m' \in A C[0, 1] \\
\text{with } p(t)q(t)f(t, \beta_m(t)) + (p(t)\beta_m'(t))' \leq 0 \text{ for } t \in (0, 1), \\
\lim_{t \to 0^+} p(t)\beta_m'(t) \leq 0 \text{ and } \beta_m(1) \geq 1/m \\
\alpha_m(t) \leq \beta_m(t), \quad t \in [0, 1] \text{ for each } m \in N_0
\end{aligned}
\]

and

\[
\begin{aligned}
\text{for each } t \in [0, 1] \text{ we have that } \{\beta_m(t)\}_{m \in N_0} \text{ is a} \\
\text{nonincreasing sequence and } \lim_{m \to \infty} \beta_m(1) = 0.
\end{aligned}
\]

Then (2.1) has a solution \(y \in C[0, 1] \cap C^2(0, 1)\) with \(y(t) \geq \alpha(t)\) for \(t \in [0, 1]\).

**Proof.** Fix \(m \in N_0\). Proceed as in Theorem 2.1 with \(\beta_m\) replacing \(\beta\) in \(f_m^*\). The same reasoning as in Theorem 2.1 guarantees that there exists a solution \(y_m \in C[0, 1]\) to (2.14) with \(\alpha(t) \leq \alpha_m(t) \leq y_m(t) \leq \beta_m(t)\) for \(t \in [0, 1]\). Also as in Theorem 2.1 (from (2.19) onwards) there exists \(y \in C[0, 1]\) (as described in Theorem 2.1 (II)) with

\[
\alpha(t) \leq y(t) \leq a_0 = \sup_{t \in [0, 1]} \beta_{n_0}(t) \quad \text{for } t \in [0, 1),
\]

with \((py')(t) + p(t)q(t)f(t, y(t)) = 0, \ 0 < t < 1\) and \(\lim_{t \to 0^+} p(t)y'(t) = 0\).
It remains to show \( y \) is continuous at 1. Let \( \epsilon > 0 \) be given. Now since 
\[
\lim_{m \to \infty} \beta_m(1) = 0 \quad \text{there exists} \quad n_1 \in N_0 \quad \text{with} \quad \beta_{n_1}(1) < \epsilon/2, \quad \text{and so there exists} \quad \delta_{n_1} > 0 \quad \text{with} \quad \beta_{n_1}(t) < \epsilon/2 \quad \text{for} \quad t \in [1 - \delta_{n_1}, 1].
\]
From (2.24) for \( m \geq n_1 \) we have
\[
\alpha(t) \leq \alpha_m(t) \leq y_m(t) \leq \beta_m(t) \leq \beta_{n_1}(t) < \epsilon/2 \quad \text{for} \quad t \in [1 - \delta_{n_1}, 1].
\]
That is, for \( m \geq n_1 \) we have \( 0 \leq \alpha(t) \leq y_m(t) < \epsilon/2 \) for \( t \in [1 - \delta_{n_1}, 1] \). Consequently \( 0 \leq \alpha(t) \leq y(t) \leq \epsilon/2 < \epsilon \) for \( t \in [1 - \delta_{n_1}, 1] \), so \( y \) is continuous at 1.

**EXAMPLE (Fluid problem).** Consider the boundary value problem
\[
\begin{align*}
\begin{cases}
y'' + \frac{vt}{y^{1/v}} = 0, & 0 < t < 1 \\
y'(0) = y(1) = 0
\end{cases}
\tag{2.26}
\end{align*}
\]
where \( 0 < v \leq 1 \). We will show using Theorem 2.1 (part (I) or (II)) that (2.26) has a solution.

First we choose \( n_0 \in \{1, 2, \ldots\} \) so that
\[
\frac{v}{6} + \frac{1}{n_0} \leq 1 \quad \text{and} \quad \left( \frac{v}{6} - 1 \right) \frac{1}{v + 1} + \frac{1}{n_0} \leq 0. \tag{2.27}
\]
Let \( p = 1 \), \( q(t) = 2t \) and clearly (2.2)–(2.5) hold. Also let
\[
\begin{align*}
\alpha_m(t) &= v(1 - t^3)/6 + 1/m, \\
\alpha(t) &= v(1 - t^3)/6
\end{align*}
\tag{2.28}
\]
and \( \beta(t) = 1 - vt^3/(v + 1) \). To check (2.6), for \( m \in N_0 = \{n_0, n_0 + 1, \ldots\} \), notice \( \alpha_m(1) = 1/m, \alpha_m'(0) = 0 \) and
\[
\alpha''_m + qf(t, \alpha_m) = -vt + \frac{vt}{[\alpha_m(t)]^{1/v}} \geq -vt + vt = 0 \quad \text{for} \quad t \in (0, 1),
\]
since \( \alpha_m(t) \leq v/6 + 1/n_0 \leq 1, \ t \in [0, 1] \) from (2.27). Thus (2.6) holds and (2.7) is immediate. To check (2.8) notice \( \beta(1) = 1 - v/v + 1 = \beta_0, \beta'(0) = 0 \) and
\[
\begin{align*}
\beta'' + qf(t, \beta) &= \frac{-6vt}{v + 1} + \frac{vt}{[\beta(t)]^{1/v}} \leq \frac{-6vt}{v + 1} + vt(v + 1)^{1/v} \\
&= vt \left\{ \frac{-6}{v + 1} + (v + 1)^{1/v} \right\} \leq 0 \quad \text{for} \quad t \in (0, 1),
\end{align*}
\]
since \( \beta(t) \geq 1/(v + 1) \) for \( t \in [0, 1] \), and \( (v + 1)^{(v+1)/v} \leq 4 \leq 6 \) for \( 0 < v \leq 1 \) (note with \( f(x) = (x + 1)^{(v+1)/x} \) we have \( f(0^+) = e, f(1) = 4 \) and \( f'(x) \geq 0 \) on \( (0, 1) \)).
Thus (2.8) holds. In addition (2.9) is true since (2.27) implies for $m \in N_0$ that
\[
\alpha_m(t) = \frac{v}{6}(1 - t^2) + \frac{1}{m} \leq \frac{v}{6} \left(1 - \frac{v}{v + 1} t^2\right) + \frac{1}{n_0}
\]
\[
= \frac{v}{6} \beta(t) + \frac{1}{n_0} = \beta(t) + \left\{\frac{1}{n_0} + \left(\frac{v}{6} - 1\right) \beta(t)\right\}
\]
\[
\leq \beta(t) + \left\{\frac{1}{n_0} + \left(\frac{v}{6} - 1\right) \frac{1}{v + 1}\right\} \leq \beta(t) \quad \text{for } t \in (0, 1)
\]
since $\frac{v}{(v + 1)} \leq 1$ and $(v/6 - 1)/(v + 1) + 1/n_0 \leq 0$. Finally (2.10) with $g(y) = 1/y^1/v$ (or (2.11) since if $0 < x < y$ then $x^{1/v} < y^{1/v}$) holds. The existence of a solution $y$ to (2.26) follows from Theorem 2.1 (I) (or (II)). Note as well that $y(t) \geq \alpha(t)$ for $t \in [0, 1]$ where $\alpha$ is given in (2.28).

3. Initial value problems

In this section we consider the initial boundary value problem
\[
\begin{align*}
\dot{y} &= qf(t, y), \quad 0 < t < T(< \infty) \\
y(0) &= 0.
\end{align*}
\]
(3.1)

Our results in this section differ from those in [4], that is, instead of assuming the existence of a lower solution to the singular problem (which is difficult to construct in practice) as in [4] we assume only the existence of a lower solution to the "approximating nonsingular problem". For our first result in this section we assume the following conditions are satisfied:

\[ f : [0, T] \times (0, \infty) \rightarrow \mathbb{R} \text{ is continuous} \quad (3.2) \]

\[ q \in C(0, T], \quad q > 0 \text{ on } (0, T) \quad \text{and} \quad \int_0^T q(x) \, dx < \infty \quad (3.3) \]

\[ \exists n_0 \in \{1, 2, \ldots\} \text{ and associated with each } m \in N_0 = \{n_0, n_0 + 1, \ldots\}, \]

\[ \exists \alpha_m \in C[0, T] \cap C^1(0, T) \text{ with} \]

\[ q(t)f\left(t, \alpha_m(t)\right) \geq \alpha'_m(t) \quad \text{for } t \in (0, T) \text{ and } 0 < \alpha_m(0) \leq 1/m \quad (3.4) \]

\[ \exists \alpha \in \mathbb{R} \text{ on } [0, T] \text{ and } \alpha(t) \leq \alpha_m(t), \quad t \in [0, T] \text{ for each } m \in N_0 \quad (3.5) \]

\[ \exists \beta \in C[0, T] \cap C^1(0, T) \text{ with} \]

\[ q(t)f\left(t, \beta(t)\right) \leq \beta'(t) \quad \text{for } t \in (0, T) \text{ and } \beta(0) \geq \beta_0 > 0 \quad (3.6) \]

and

\[ \alpha_m(t) \leq \beta(t), \quad t \in [0, T] \text{ for each } m \in N_0. \quad (3.7) \]
THEOREM 3.1.  

(I) Suppose (3.2)–(3.7) hold and in addition assume the following condition is satisfied:

\[
\begin{align*}
|f(t,y)| &\leq g(y) \text{ on } [0, T] \times (0, a_0] \text{ with } g > 0 \\
\text{continuous and nonincreasing on } (0, \infty);
\end{align*}
\]

where \( a_0 = \sup_{t \in [0, T]} \beta(t) \). Then (3.1) has a solution \( y \in C[0, T] \cap C^1(0, T) \) with \( y(t) \geq \alpha(t) \) for \( t \in [0, T] \).

(II) Suppose (3.2)–(3.7) hold and in addition assume the following condition is satisfied:

\[
f(t, x) - f(t, y) \geq 0 \quad \text{for } 0 < x < y, \text{ for each fixed } t \in (0, T).
\]

Then (3.1) has a solution \( y \in C[0, T] \cap C^1(0, T) \) with \( y(t) \geq \alpha(t) \) for \( t \in [0, T] \).

PROOF. Without loss of generality assume \( \beta_0 \geq 1/n_0 \). Fix \( m \in \mathbb{N}_0 \) and consider

\[
\begin{align*}
y' &= qf_m^*(t, y), \quad 0 < t < T \\
y(0) &= 1/m,
\end{align*}
\]

where

\[
f_m^*(t, y) = \begin{cases} 
  f(t, \beta(t)), & y > \beta(t) \\
  f(t, y), & \alpha_m(t) \leq y \leq \beta(t) \\
  f(t, \alpha_m(t)), & y < \alpha_m(t).
\end{cases}
\]

It is immediate from Schauder’s fixed point theorem (see [10]) that (3.10) has a solution \( y_m \in C[0, T] \). A standard argument (see [11, Chapter 3]; note \( f_m^* : [0, 1] \times \mathbb{R} \to \mathbb{R} \) is continuous) guarantees that

\[
\alpha_m(t) \leq y_m(t) \leq \beta(t) \quad \text{for } t \in [0, T].
\]

As a result \( y_m \) is a solution of

\[
\begin{align*}
y' &= qf(t, y), \quad 0 < t < T \\
y(0) &= 1/m
\end{align*}
\]

with

\[
\alpha(t) \leq \alpha_m(t) \leq y_m(t) \leq \beta(t) \quad \text{for } t \in [0, T].
\]

The proof is now broken into two cases.

Case (A). Suppose (3.8) holds.

We first show

\[
\{y_m\}_{m \in \mathbb{N}_0} \text{ is a bounded, equicontinuous family on } [0, T].
\]
To see this notice (3.8) guarantees that $|y_m'(t)|/g(y_m(t)) \leq q(t)$ for $t \in (0, T)$, and so $\pm v_m'(t) \leq q(t)$ for $t \in (0, T)$; here

$$v_m(t) = \int_0^{y_m(t)} \frac{du}{g(u)} = G(y_m(t)).$$

For $t, s \in [0, T]$ we have

$$|v_m(t) - v_m(s)| = \left| \int_s^t v_m'(\tau) \, d\tau \right| \leq \int_s^t q(\tau) \, d\tau.$$

This together with the uniform continuity of $G^{-1}$ on $[0, G(a_0)]$ and

$$|y_m(t) - y_m(s)| = |G^{-1}(G(y_m(t))) - G^{-1}(G(y_m(s)))|$$


**Case (B).** Suppose (3.9) holds.

We begin by showing

$$y_{m+1}(t) \leq y_m(t) \quad \text{for } t \in [0, T] \text{ for each } m \in N_0. \quad (3.15)$$

Suppose (3.15) is false. Then for some $m \in N_0$ there exists $\tau_1 < \tau_2$ with $y_{m+1}(\tau_1) = y_m(\tau_1), y_{m+1}(\tau_2) > y_m(\tau_2)$ and $y_{m+1}(t) > y_m(t)$ for $t \in (\tau_1, \tau_2)$. As a result from (3.9) we have

$$0 < y_{m+1}(\tau_2) - y_m(\tau_2) = \int_{\tau_1}^{\tau_2} q(s)[f(s, y_{m+1}(s)) - f(s, y_m(s))] \, ds \leq 0,$$

a contradiction. As a result (3.15) holds.

Essentially the same reasoning as in Theorem 2.1 guarantees that there exist subsequences of integers $N_{n_0} \supseteq N_{n_0+1} \supseteq \ldots \supseteq N_k \supseteq \ldots$ and functions $z_k \in C[T/k, T]$ with $y_m$ converging uniformly on $[T/k, T]$ to $z_k$ as $m \to \infty$ through $N_k$, and $z_{k+1} = z_k$ on $[T/k, T]$.

Define a function $y : [0, T] \to [0, \infty)$ by $y(x) = z_k(x)$ on $[T/k, T]$ and $y(0) = 0$. Notice $y$ is well-defined and $\alpha(t) \leq y(t) \leq a_0$ for $t \in (0, T]$. Next fix $t \in (0, T)$ and let $k \in \{n_0, n_0 + 1, \ldots \}$ be such that $T/k < t < T$. Let $N_k^* = \{n \in N_k : n \geq k\}$. Now $y_m, m \in N_k^*$, satisfies

$$y_m(t) = y_m(T) - \int_t^T q(s)f(s, y_m(s)) \, ds.$$

Let $m \to \infty$ through $N_k^*$ to obtain $y(t) = y(T) - \int_t^T q(s)f(s, y(s)) \, ds$. We can do this argument for each $t \in (0, T)$. It remains to show $y$ is continuous at 0. Let $\varepsilon > 0$
be given. Then there exists \( n_1 \in N_0 \) with \( y_{n_1}(0) < \epsilon/2 \), so there exists \( \delta_{n_1} > 0 \) with \( y_{n_1}(t) < \epsilon/2 \) for \( t \in [0, \delta_{n_1}] \). From (3.15) for \( m \geq n_1 \) we have
\[
\alpha(t) \leq y_m(t) \leq y_{n_1}(t) < \epsilon/2 \quad \text{for} \quad t \in [0, \delta_{n_1}].
\]
As a result \( 0 \leq \alpha(t) \leq y(t) \leq \epsilon/2 < \epsilon \) for \( t \in (0, \delta_{n_1}] \), so \( y \) is continuous at 0.

In fact one can obtain a more general result motivated from Theorem 3.1 (II).

**THEOREM 3.2.** Suppose (3.2)–(3.5) hold and in addition assume the following conditions are satisfied:

\[
\begin{align*}
\text{for each} \; m \in N_0, & \exists \beta_m \in C[0, T] \cap C^1(0, T) \text{ with} \\
q(t)f(t, \beta_m(t)) & \leq \beta'_m(t) \quad \text{for} \; t \in (0, T) \quad \text{and} \quad \beta_m(0) \geq 1/m \\
\alpha_m(t) & \leq \beta_m(t), \quad t \in [0, T] \text{ for each} \; m \in N_0
\end{align*}
\] (3.16)

and

\[
\begin{align*}
\text{for each} \; t \in [0, T] \quad \text{we have that} \quad & \{\beta_m(t)\}_{m \in N_0} \quad \text{is a} \\
\text{nonincreasing sequence and} \quad & \lim_{m \to \infty} \beta_m(0) = 0.
\end{align*}
\] (3.17)

Then (3.1) has a solution \( y \in C[0, T] \cap C^1(0, T) \) with \( y(t) \geq \alpha(t) \) for \( t \in [0, T] \).

**Proof.** Fix \( m \in N_0 \). Proceed as in Theorem 3.1 with \( \beta_m \) replacing \( \beta \) in \( f_m^* \). The same reasoning as in Theorem 3.1 guarantees that there exists a solution \( y_m \in C[0, T] \) to (3.12) with \( \alpha(t) \leq \alpha_m(t) \leq y_m(t) \leq \beta_m(t) \) for \( t \in [0, T] \). Also as in Theorem 3.1 there exists \( y \in C(0, T) \) (as described in Theorem 3.1 (II)) with
\[
\alpha(t) \leq y(t) \leq a_0 = \sup_{t \in [0, T]} \beta_m(t) \quad \text{for} \; t \in (0, T],
\]

with \( y' = qf(t, y) \) for \( 0 < t < T \). It is easy to see (using (3.18)) that \( y \) is continuous at 0.

**References**


