Phase transitions for non-singular Bernoulli actions

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Abstract. Inspired by the phase transition results for non-singular Gaussian actions introduced in [AIM19], we prove several phase transition results for non-singular Bernoulli actions. For generalized Bernoulli actions arising from groups acting on trees, we are able to give a very precise description of their ergodic-theoretical properties in terms of the Poincaré exponent of the group.

Key words: non-singular Bernoulli action, phase transition, strong ergodicity, Krieger type
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1. Introduction

When $G$ is a countable infinite group and $(X_0, \mu_0)$ is a non-trivial standard probability space, the probability measure-preserving (pmp) action

$G \actson (X_0, \mu_0)^G : (g \cdot x)_h = x_{g^{-1}h}$

is called a Bernoulli action. Probability measure-preserving Bernoulli actions are among the best-studied objects in ergodic theory and they play an important role in operator algebras [Ioia10, Pop03, Pop06]. When we consider a family of probability measures $(\mu_g)_{g \in G}$ on the base space $X_0$ that need not all be equal, the Bernoulli action

$G \actson (X, \mu) = \prod_{g \in G} (X_0, \mu_g)$

(1.1)

is in general no longer measure-preserving. Instead, we are interested in the case where $G \actson (X, \mu)$ is non-singular, that is, the group $G$ preserves the measure class of $\mu$. By Kakutani’s criterion for equivalence of infinite product measures the Bernoulli action (1.1) is non-singular if and only if $\mu_h \sim \mu_g$ for every $h, g \in G$ and

$$\sum_{h \in G} H^2(\mu_h, \mu_{gh}) < +\infty \quad \text{for every } g \in G.$$  

(1.2)

Here $H^2(\mu_h, \mu_{gh})$ denotes the Hellinger distance between $\mu_h$ and $\mu_{gh}$ (see (2.2)).
It is well known that a pmp Bernoulli action $G \curvearrowright (X_0, \mu_0)^G$ is mixing. In particular, it is ergodic and conservative. However, for non-singular Bernoulli actions, determining conservativeness and ergodicity is much more difficult (see, for instance, [BKV19, Dan18, Kos18, VW17]).

Besides non-singular Bernoulli actions, another interesting class of non-singular group actions comes from the Gaussian construction, as introduced in [AIM19]. If $\pi: G \to O(H)$ is an orthogonal representation of a locally compact second countable (lcsc) group on a real Hilbert space $H$, and if $c: G \to H$ is a 1-cocycle for the representation $\pi$, then the assignment

$$
\alpha_g(\xi) = \pi_g(\xi) + c(g) \quad (1.3)
$$

defines an affine isometric action $\alpha: G \curvearrowright H$. To any affine isometric action $\alpha: G \curvearrowright H$ Arano, Isono and Marrakchi associated a non-singular group action $\hat{\alpha}: G \curvearrowright \hat{H}$, where $\hat{H}$ is the Gaussian probability space associated to $H$. When $\alpha: G \curvearrowright H$ is actually an orthogonal representation, this construction is well established and the resulting Gaussian action is pmp. As explained below [BV20, Theorem D], if $G$ is a countable infinite group and $\pi: G \to \ell^2(G)$ is the left regular representation, the affine isometric representation (1.3) gives rise to a non-singular action that is conjugate with the Bernoulli action $G \curvearrowright \prod_{g \in G}(\mathbb{R}, \nu_F(g))$, where $F: G \to \mathbb{R}$ is such that $c_g(h) = F(g^{-1}h) - F(h)$, and $\nu_F(g)$ denotes the Gaussian probability measure with mean $F(g)$ and variance 1.

By scaling the 1-cocycle $c: G \to H$ with a parameter $t \in [0, +\infty)$ we get a one-parameter family of non-singular actions $\hat{\alpha}_t: G \curvearrowright \hat{H}_t$ associated to the affine isometric actions $\alpha_t: G \curvearrowright H$, given by $\hat{\alpha}_t(\xi) = \pi_g(\xi) + tc(g)$. Arano, Isono and Marrakchi showed that there exists a $t_{\text{diss}} \in [0, +\infty)$ such that $\hat{\alpha}_t$ is dissipative up to compact stabilizers for every $t > t_{\text{diss}}$ and infinitely recurrent for every $t < t_{\text{diss}}$ (see §2 for terminology).

Inspired by the results obtained in [AIM19], we study a similar phase transition framework, but in the setting of non-singular Bernoulli actions. Such a phase transition framework for non-singular Bernoulli actions was already considered by Kosloff and Soo in [KS20]. They showed the following phase transition result for the family of non-singular Bernoulli actions of $G = \mathbb{Z}$ with base space $X_0 = \{0, 1\}$ that was introduced in [VW17, Corollary 6.3]. For every $t \in [0, +\infty)$ consider the family of measures $(\mu_n^t)_{n \in \mathbb{Z}}$ given by

$$
\mu_n^t(0) = \begin{cases} 
1/2 & \text{if } n \leq 4t^2, \\
1/2 + t/\sqrt{n} & \text{if } n > 4t^2.
\end{cases}
$$

Then $\mathbb{Z} \curvearrowright (X, \mu_t) = \prod_{n \in \mathbb{Z}}([0, 1], \mu_n^t)$ is non-singular for every $t \in [0, +\infty)$. Kosloff and Soo showed that there exists a $t_1 \in (1/6, +\infty)$ such that $\mathbb{Z} \curvearrowright (X, \mu_t)$ is conservative for every $t < t_1$ and dissipative for every $t > t_1$ [KS20, Theorem 3]. In [DKR20, Example D] the authors describe a family of non-singular Poisson suspensions for which a similar phase transition occurs. These examples arise from dissipative essentially free actions of $\mathbb{Z}$, and thus they are non-singular Bernoulli actions. We generalize the phase transition result from [KS20] to arbitrary non-singular Bernoulli actions as follows.
Suppose that $G$ is a countable infinite group and let $(\mu_g)_{g \in G}$ be a family of equivalent probability measure on a standard Borel space $X_0$. Let $\nu$ also be a probability measure on $X_0$. For every $t \in [0, 1]$ we consider the family of equivalent probability measures $(\mu^t_g)_{g \in G}$ that are defined by

$$
\mu^t_g = (1-t)\nu + t\mu_g. \tag{1.4}
$$

Our first main result is that in this setting there is a phase transition phenomenon.

**THEOREM A.** Let $G$ be a countable infinite group and assume that the Bernoulli action $G \rhd (X, \mu_1) = \prod_{g \in G} (X_0, \mu_g)$ is non-singular. Let $\nu \sim \mu_e$ be a probability measure on $X_0$ and for every $t \in [0, 1]$ consider the family $(\mu^t_g)_{g \in G}$ of equivalent probability measures given by (1.4). Then the Bernoulli action

$$
G \rhd (X, \mu_t) = \prod_{g \in G} (X_0, \mu^t_g)
$$

is non-singular for every $t \in [0, 1]$ and there exists a $t_1 \in [0, 1]$ such that $G \rhd (X, \mu_t)$ is weakly mixing for every $t < t_1$ and dissipative for every $t > t_1$.

Suppose that $G$ is a non-amenable countable infinite group. Recall that for any standard probability space $(X_0, \mu_0)$, the pmp Bernoulli action $G \rhd (X_0, \mu_0)^G$ is strongly ergodic. Consider again the family of probability measures $(\mu^t_g)_{g \in G}$ given by (1.4). In Theorem B below we prove that for $t$ close enough to 0, the resulting non-singular Bernoulli action is strongly ergodic. This is inspired by [AIM19, Theorem 7.20] and [MV20, Theorem 5.1], which state similar results for non-singular Gaussian actions.

**THEOREM B.** Let $G$ be a countable infinite non-amenable group and suppose that the Bernoulli action $G \rhd (X, \mu_1) = \prod_{g \in G} (X_0, \mu_g)$ is non-singular. Let $\nu \sim \mu_e$ be a probability measure on $X_0$ and for every $t \in [0, 1]$ consider the family $(\mu^t_g)_{g \in G}$ of equivalent probability measures given by (1.4). Then there exists a $t_0 \in (0, 1]$ such that $G \rhd (X, \mu_t) = \prod_{g \in G} (X_0, \mu^t_g)$ is strongly ergodic for every $t < t_0$.

Although we can prove a phase transition result in large generality, it remains very challenging to compute the critical value $t_1$. However, when $G \subset \text{Aut}(T)$, for some locally finite tree $T$, following [AIM19, §10], we can construct generalized Bernoulli actions of which we can determine the conservativeness behaviour very precisely. To put this result into perspective, let us first explain briefly the construction from [AIM19, §10].

For a locally finite tree $T$, let $\Omega(T)$ denote the set of orientations on $T$. Let $p \in (0, 1)$ and fix a root $\rho \in T$. Define a probability measure $\mu_p$ on $\Omega(T)$ by orienting an edge towards $\rho$ with probability $p$ and away from $\rho$ with probability $1-p$. If $G \subset \text{Aut}(T)$ is a subgroup, then we naturally obtain a non-singular action $G \rhd (\Omega(T), \mu_p)$. Up to equivalence of measures, the measure $\mu_p$ does not depend on the choice of root $\rho \in T$. The Poincaré exponent of $G \subset \text{Aut}(T)$ is defined as

$$
\delta(G \rhd T) = \inf \left\{ s > 0 \text{ for which } \sum_{w \in G \cdot v} \exp(-sd(v, w)) < +\infty \right\}, \tag{1.5}
$$
where \( v \in V(T) \) is any vertex of \( T \). In\cite{AIM19, Theorem 10.4} Arano, Isono and Marrakchi showed that if \( G \subset \text{Aut}(T) \) is a closed non-elementary subgroup, the action \( G \acts (\Omega(T), \mu_p) \) is dissipative up to compact stabilizers if \( 2\sqrt{p(1-p)} < \exp(-\delta) \) and weakly mixing if \( 2\sqrt{p(1-p)} > \exp(-\delta) \). This motivates the following similar construction.

Let \( E(T) \subset V(T) \times V(T) \) denote the set of oriented edges, so that vertices \( v \) and \( w \) are adjacent if and only if \( (v, w), (w, v) \in E(T) \). Suppose that \( X_0 \) is a standard Borel space and that \( \mu_0, \mu_1 \) are equivalent probability measures on \( X_0 \). Fix a root \( \rho \in T \) and define a family of probability measures \( (\mu_e)_{e \in E(T)} \) by

\[
\mu_e = \begin{cases} 
\mu_0 & \text{if } e \text{ is oriented towards } \rho, \\
\mu_1 & \text{if } e \text{ is oriented away from } \rho.
\end{cases}
\]

Suppose that \( G \subset \text{Aut}(T) \) is a subgroup. Then the generalized Bernoulli action

\[
G \acts \prod_{e \in E(T)} (X_0, \mu_e) : (g \cdot x)_e = x_{g^{-1} \cdot e}
\]

is non-singular and up to conjugacy it does not depend on the choice of root \( \rho \in T \). In our next main result we generalize\cite[Theorem 10.4]{AIM19} to non-singular actions of the form \eqref{eq:bernoulli}.\[\text{Theorem C. Let } T \text{ be a locally finite tree with root } \rho \in T \text{ and let } G \subset \text{Aut}(T) \text{ be a non-elementary closed subgroup with Poincaré exponent } \delta = \delta(G \acts T). \text{ Let } \mu_0 \text{ and } \mu_1 \text{ be equivalent probability measures on a standard Borel space } X_0 \text{ and define a family of equivalent probability measures } (\mu_e)_{e \in E(T)} \text{ by } \eqref{eq:bernoulli}. \text{ Then the generalized Bernoulli action } \eqref{eq:bernoulli} \text{ is dissipative up to compact stabilizers if } 1 - H^2(\mu_0, \mu_1) < \exp(-\delta/2) \text{ and weakly mixing if } 1 - H^2(\mu_0, \mu_1) > \exp(-\delta/2).\]

2. Preliminaries

2.1. Non-singular group actions. Let \((X, \mu), (Y, \nu)\) be standard measure spaces. A Borel map \( \varphi : X \to Y \) is called non-singular if the pushforward measure \( \varphi_* \mu \) is equivalent to \( \nu \). If in addition there exist countable Borel sets \( X_0 \subset X \) and \( Y_0 \subset Y \) such that \( \varphi : X_0 \to Y_0 \) is a bijection we say that \( \varphi \) is a non-singular isomorphism. We write \( \text{Aut}(X, \mu) \) for the group of all non-singular automorphisms \( \varphi : X \to X \), where we identify two elements if they agree almost everywhere. The group \( \text{Aut}(X, \mu) \) carries a canonical Polish topology.

A non-singular group action \( G \acts (X, \mu) \) of an lcsc group \( G \) on a standard measure space \((X, \mu)\) is a continuous group homomorphism \( G \to \text{Aut}(X, \mu) \). A non-singular group action \( G \acts (X, \mu) \) is called essentially free if the stabilizer subgroup \( G_x = \{ g \in G : g \cdot x = x \} \) is trivial for almost every (a.e.) \( x \in X \). When \( G \) is countable this is the same as the condition that \( \mu((x \in X : g \cdot x = x)) = 0 \) for every \( g \in G \setminus \{ e \} \). We say that \( G \acts (X, \mu) \) is ergodic if every \( G \)-invariant Borel set \( A \subset X \) satisfies \( \mu(A) = 0 \) or \( \mu(X \setminus A) = 0 \). A non-singular action \( G \acts (X, \mu) \) is called weakly mixing if for any ergodic pmp action \( G \acts (Y, \nu) \) the diagonal product action \( G \acts X \times Y \) is ergodic. If \( G \) is not compact and \( G \acts (X, \mu) \) is pmp, we say that \( G \acts X \) is mixing if

\[
\lim_{g \to \infty} \mu(g \cdot A \cap B) = \mu(A)\mu(B) \quad \text{for every pair of Borel subsets } A, B \subset X.
\]
Suppose that $G \actson (X, \mu)$ is a non-singular action and that $\mu$ is a probability measure. A sequence of Borel subsets $A_n \subset X$ is called almost invariant if

$$\sup_{g \in K} \mu(g \cdot A_n \triangle A_n) \to 0$$

for every compact subset $K \subset G$.

The action $G \actson (X, \mu)$ is called strongly ergodic if every almost invariant sequence $A_n \subset X$ is trivial, that is, $\mu(A_n)(1 - \mu(A_n)) \to 0$. The strong ergodicity of $G \actson (X, \mu)$ only depends on the measure class of $\mu$. When $(Y, \nu)$ is a standard measure space and $\nu$ is infinite, a non-singular action $G \actson (Y, \nu)$ is called strongly ergodic if $G \actson (Y, \nu')$ is strongly ergodic, where $\nu'$ is a probability measure that is equivalent to $\nu$.

Following [AIM19, Definition A.16], we say that a non-singular action $G \actson (X, \mu)$ is dissipative up to compact stabilizers if each ergodic component is of the form $G \actson G/K$, for a compact subgroup $K \subset G$. By [AIM19, Theorem A.29] a non-singular action $G \actson (X, \mu)$, with $\mu(X) = 1$, is dissipative up to compact stabilizers if and only if

$$\int_G \frac{dg \mu}{d\mu}(x) d\lambda(g) < +\infty$$

for a.e. $x \in X$,

where $\lambda$ denotes the left invariant Haar measure on $G$. We say that $G \actson (X, \mu)$ is infinitely recurrent if for every non-negligible subset $A \subset X$ and every compact subset $K \subset G$ there exists $g \in G \setminus K$ such that $\mu(g \cdot A \cap A) > 0$. By [AIM19, Proposition A.28] and Lemma 2.1 below, a non-singular action $G \actson (X, \mu)$, with $\mu(X) = 1$, is infinitely recurrent if and only if

$$\int_G \frac{dg \mu}{d\mu}(x) d\lambda(g) = +\infty$$

for a.e. $x \in X$.

A non-singular action $G \actson (X, \mu)$ is called dissipative if it is essentially free and dissipative up to compact stabilizers. In that case there exists a standard measure space $(X_0, \mu_0)$ such that $G \actson X$ is conjugate with the action $G \actson G \times X_0 : g \cdot (h, x) = (gh, x)$. A non-singular action $G \actson (X, \mu)$ decomposes, uniquely up to a null set, as $G \actson D \sqcup C$, where $G \actson D$ is dissipative up to compact stabilizers and $G \actson C$ is infinitely recurrent. When $G$ is a countable group and $G \actson (X, \mu)$ is essentially free, we say that $G \actson X$ is conservative if it is infinitely recurrent.

**Lemma 2.1.** Suppose that $G$ is an lcsc group with left invariant Haar measure $\lambda$ and that $(X, \mu)$ is a standard probability space. Assume that $G \actson (X, \mu)$ is a non-singular action that is infinitely recurrent. Then we have that

$$\int_G \frac{dg \mu}{d\mu}(x) d\lambda(g) = +\infty$$

for a.e. $x \in X$.

**Proof.** Note that the set

$$D = \left\{ x \in X : \int_G \frac{dg \mu}{d\mu}(x) d\lambda(g) < +\infty \right\}$$

is $G$-invariant. Therefore, it suffices to show that $G \actson X$ is not infinitely recurrent under the assumption that $D$ has full measure.
Let $\pi : (X, \mu) \to (Y, \nu)$ be the projection onto the space of ergodic components of $G \curvearrowright X$. Then there exist a conull Borel subset $Y_0 \subset Y$ and a Borel map $\theta : Y_0 \to X$ such that $(\pi \circ \theta)(y) = y$ for every $y \in Y_0$.

Write $X_y = \pi^{-1}(\{y\})$. By [AIM19, Theorem A.29], for a.e. $y \in Y$ there exists a compact subgroup $K_y \subset G$ such that $G \curvearrowright X_y$ is conjugate with $G \curvearrowright G/K_y$. Let $G_n \subset G$ be an increasing sequence of compact subsets of $G$ such that $\bigcup_{n \geq 1} \interior[G]{G_n} = G$. For every $x \in X$, write $G_x = \{g \in G : g \cdot x = x\}$ for the stabilizer subgroup of $x$. Using an argument as in [MRV11, Lemma 10], one shows that for each $n \geq 1$ the set $\{x \in X : G_x \subset G_n\}$ is Borel. Thus, for every $n \geq 1$ the set

$$U_n = \{y \in Y_0 : K_y \subset G_n\} = \{y \in Y_0 : G_{\theta(y)} \subset G_n\}$$

is a Borel subset of $Y$ and we have that $\nu(\bigcup_{n \geq 1} U_n) = 1$. Therefore, the sets

$$A_n = \{g \cdot \theta(y) : g \in G_n, y \in U_n\}$$

are analytic and exhaust $X$ up to a set of measure zero. So there exist an $n_0 \in \mathbb{N}$ and a non-negligible Borel set $B \subset A_{n_0}$. Suppose that $h \in G$ is such that $h \cdot B \cap B \neq \emptyset$. Then there exist $y \in U_{n_0}$ and $g_1, g_2 \in G_{n_0}$ such that $h g_1 \cdot \theta(y) = g_2 \cdot \theta(y)$, and we get that $h \in G_{n_0} K_y G_{n_0}^{-1} \subset G_{n_0} G_{n_0} G_{n_0}^{-1}$. In other words, for $h \in G$ outside the compact set $G_{n_0} G_{n_0} G_{n_0}^{-1}$ we have that $\mu(h \cdot B \cap B) = 0$, so that $G \curvearrowright X$ is not infinitely recurrent.

We will frequently use the following result of Schmidt and Walters. Suppose that $G \curvearrowright (X, \mu)$ is a non-singular action that is infinitely recurrent and suppose that $G \curvearrowright (Y, \nu)$ is pmp and mixing. Then by [SW81, Theorem 2.3] we have that

$$L^\infty(X \times Y)^G = L^\infty(X)^G \otimes 1,$$

where $G \curvearrowright X \times Y$ acts diagonally. Although [SW81, Theorem 2.3] demands proper ergodicity of the action $G \curvearrowright (X, \mu)$, the infinite recurrence assumption is sufficient as remarked in [AIM19, Remark 7.4].

### 2.2. The Maharam extension and crossed products.

Let $(X, \mu)$ be a standard measure space. For any non-singular automorphism $\varphi \in \text{Aut}(X, \mu)$, we define its Maharam extension by

$$\widetilde{\varphi} : X \times \mathbb{R} \to X \times \mathbb{R} : \widetilde{\varphi}(x, t) = (\varphi(x), t + \log(d\varphi^{-1}\mu/d\mu)(x)).$$

Then $\widetilde{\varphi}$ preserves the infinite measure $\mu \times \exp(-t)dt$. The assignment $\varphi \mapsto \widetilde{\varphi}$ is a continuous group homomorphism from $\text{Aut}(X)$ to $\text{Aut}(X \times \mathbb{R})$. Thus, for each non-singular group action $G \curvearrowright (X, \mu)$, by composing with this map, we obtain a non-singular group action $G \curvearrowright X \times \mathbb{R}$, which we call the Maharam extension of $G \curvearrowright X$. If $G \curvearrowright X$ is a non-singular group action, the translation action $\mathbb{R} \curvearrowright X \times \mathbb{R}$ in the second component commutes with the Maharam extension $G \curvearrowright X \times \mathbb{R}$. Therefore, we get a well-defined action $\mathbb{R} \curvearrowright L^\infty(X \times \mathbb{R})^G$, which is the Krieger flow associated to the action $G \curvearrowright X$. The Krieger flow is given by $\mathbb{R} \curvearrowright \mathbb{R}$ if and only if there exists a $G$-invariant $\sigma$-finite measure $\nu$ on $X$ that is equivalent to $\mu$. 

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Suppose that $M \subset B(H)$ is a von Neumann algebra represented on the Hilbert space $H$ and that $\alpha: G \curvearrowright M$ is a continuous action on $M$ of an lcsc group $G$. Then the crossed product von Neumann algebra $M \rtimes_\alpha G \subset B(L^2(G, H))$ is the von Neumann algebra generated by the operators $\{\pi(x)\}_{x \in M}$ and $\{u_h\}_{h \in G}$ acting on $\xi \in L^2(G, H)$ as

$$\pi(x)\xi(g) = \alpha_g^{-1}(x)\xi(g), \quad (u_h\xi)(g) = \xi(h^{-1}g).$$

In particular, if $G \curvearrowright (X, \mu)$ is a non-singular group action, the crossed product $L^\infty(X) \rtimes G \subset B(L^2(G \times X))$ is the von Neumann algebra generated by the operators

$$\pi(H)\xi(g, x) = H(g \cdot x)\xi(g, x), \quad (u_h\xi)(g, x) = \xi(h^{-1}g, x),$$

for $H \in L^\infty(X)$ and $h \in G$. If $G \curvearrowright X$ is non-singular essentially free and ergodic, then $L^\infty(X) \rtimes G$ is a factor. Moreover, when $G$ is a unimodular group, the Krieger flow of $G \curvearrowright X$ equals the flow of weights of the crossed product von Neumann algebra $L^\infty(X) \rtimes G$. For non-unimodular groups this is not necessarily true, motivating the following definition.

**Definition 2.2.** Let $G$ be an lcsc group with modular function $\Delta: G \to \mathbb{R}_{>0}$. Let $\lambda$ denote the Lebesgue measure on $\mathbb{R}$. Suppose that $\alpha: G \curvearrowright (X, \mu)$ is a non-singular action. We define the **modular Maharam extension** of $G \curvearrowright X$ as the non-singular action

$$\beta: G \curvearrowright (X \times \mathbb{R}, \mu \times \lambda): \quad g \cdot (x, t) = (g \cdot x, t + \log(\Delta(g)) + \log(dg^{-1}\mu/d\mu)(x)).$$

Let $L^\infty(X \times \mathbb{R})^\beta$ denote the subalgebra of $\beta$-invariant elements. We define the flow of weights associated to $G \curvearrowright X$ as the translation action $\mathbb{R} \curvearrowright L^\infty(X \times \mathbb{R})^\beta: (t \cdot H)(x, s) = H(x, s - t)$.

As we explain below, the flow of weights associated to an essentially free ergodic non-singular action $G \curvearrowright X$ equals the flow of weights of the crossed product factor $L^\infty(X) \rtimes G$, justifying the terminology. See also [Sa74, Proposition 4.1].

Let $\alpha: G \curvearrowright X$ be an essentially free ergodic non-singular group action with modular Maharam extension $\beta: G \curvearrowright X \times \mathbb{R}$. By [Sa74, Proposition 1.1] there is a canonical normal semifinite faithful weight $\varphi$ on $L^\infty(X) \rtimes_\alpha G$ such that the modular automorphism group $\sigma^\varphi$ is given by

$$\sigma^\varphi_t(\pi(H)) = \pi(H), \quad \sigma^\varphi_t(u_g) = \Delta(g)^{it}u_g\pi((dg^{-1}\mu/d\mu)^{it}),$$

where $\Delta: G \to \mathbb{R}_{>0}$ denotes the modular function of $G$.

For an element $\xi \in L^2(\mathbb{R}, L^2(G \times X))$ and $(g, x) \in G \times X$, write $\xi_{g,x}$ for the map given by $\xi_{g,x}(s) = \xi(s, g, x)$. Then by Fubini’s theorem $\xi_{g,x} \in L^2(\mathbb{R})$ for a.e. $(g, x) \in G \times X$. Let $U: L^2(\mathbb{R}, L^2(G \times X)) \to L^2(G, L^2(X \times \mathbb{R}))$ be the unitary given on $\xi \in L^2(\mathbb{R}, L^2(G \times X))$ by

$$(U\xi)(g, x, t) = F^{-1}(\xi_{g,x})(t + \log(\Delta(g)) + \log(dg^{-1}\mu/d\mu)(x)),$$

where $F^{-1}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denotes the inverse Fourier transform. One can check that conjugation by $U$ induces an isomorphism

$$\Psi: (L^\infty(X) \rtimes_\alpha G) \rtimes_{\sigma^\varphi} \mathbb{R} \to L^\infty(X \times \mathbb{R}) \rtimes_\beta G.$$
Let $\kappa : L^\infty(X \times \mathbb{R}) \to L^\infty(X \times \mathbb{R}) \rtimes_\beta G$ be the inclusion map and let $\gamma : \mathbb{R} \curvearrowright L^\infty(X \times \mathbb{R}) \rtimes_\beta G$ be the action given by
\[
\gamma_t(\kappa(H))(x, s) = \kappa(H)(x, s - t), \quad \gamma_t(u_g) = u_g.
\]
Then one can verify that $\Psi$ conjugates the dual action $\hat{\sigma} : \mathbb{R} \curvearrowright (L^\infty(X) \rtimes_\alpha G) \rtimes \sigma \mathbb{R}$ and $\gamma$. Therefore, we can identify the flow of weights $\mathbb{R} \curvearrowright \mathbb{Z}((L^\infty(X) \rtimes_\alpha G) \rtimes \sigma \mathbb{R})$ with $\mathbb{R} \curvearrowright \mathbb{Z}(L^\infty(X \times \mathbb{R}) \rtimes_\beta G) \cong L^\infty(X \times \mathbb{R})^\beta$, the flow of weights associated to $G \curvearrowright X$.

Remark 2.3. It will be useful to speak about the Krieger type of a non-singular ergodic action $G \curvearrowright X$. In light of the discussion above, we will only use this terminology for countable groups $G$, so that no confusion arises with the type of the crossed product von Neumann algebra $L^\infty(X) \rtimes G$. So assume that $G$ is countable and that $G \curvearrowright (X, \mu)$ is a non-singular ergodic action. Then the Krieger flow is ergodic and we distinguish several cases. If $\nu$ is atomic, we say that $G \curvearrowright X$ is of type I. If $\nu$ is non-atomic and finite, we say that $G \curvearrowright X$ is of type II$_1$. If $\nu$ is non-atomic and infinite, we say that $G \curvearrowright X$ is of type II$_\infty$. If the Krieger flow is given by $\mathbb{R} \curvearrowright \mathbb{R}/ \log(\lambda) \mathbb{Z}$ with $\lambda \in (0, 1)$, we say that $G \curvearrowright X$ is of type III. If the Krieger flow is properly ergodic (that is, every orbit has measure zero), we say that $G \curvearrowright X$ is of type III$_0$.

2.3. Non-singular Bernoulli actions. Suppose that $G$ is a countable infinite group and that $(\mu_g)_{g \in G}$ is a family of equivalent probability measures on a standard Borel space $X_0$. The action
\[
G \curvearrowright (X, \mu) = \prod_{h \in G} (X_0, \mu_h) : \quad (g \cdot x)_h = x_{g^{-1}h}
\]  
(2.1)
is called the Bernoulli action. For two probability measures $\nu, \eta$ on a standard Borel space $Y$, the Hellinger distance $H^2(\nu, \eta)$ is defined by
\[
H^2(\nu, \eta) = \frac{1}{2} \int_Y \left( \sqrt{d\nu/d\zeta} - \sqrt{d\eta/d\zeta} \right)^2 d\zeta,
\]
(2.2)
where $\zeta$ is any probability measure on $Y$ such that $\nu, \eta \prec \zeta$. By Kakutani’s criterion for equivalence of infinite product measures [Kak48] the Bernoulli action (2.1) is non-singular if and only if
\[
\sum_{h \in G} H^2(\mu_h, \mu_{gh}) < +\infty \quad \text{for every } g \in G.
\]
If $(X, \mu)$ is non-atomic and the Bernoulli action (2.1) is non-singular, then it is essentially free by [BKV19, Lemma 2.2].

Suppose that $I$ is a countable infinite set and that $(\mu_i)_{i \in I}$ is a family of equivalent probability measures on a standard Borel space $X_0$. If $G$ is an lcsc group that acts on $I$, the action
\[
G \curvearrowright (X, \mu) = \prod_{i \in I} (X_0, \mu_i) : \quad (g \cdot x)_i = x_{g^{-1}i}
\]  
(2.3)
is called the \textit{generalized Bernoulli action} and it is non-singular if and only if \( \sum_{i \in I} H^2(\mu_i, \mu_{g_i}) < +\infty \) for every \( g \in G \). When \( \nu \) is a probability measure on \( X_0 \) such that \( \mu_i = \nu \) for every \( i \in I \), the generalized Bernoulli action \((2.3)\) is pmp and it is mixing if and only if the stabilizer subgroup \( G_i = \{ g \in G : g \cdot i = i \} \) is compact for every \( i \in I \). In particular, if \( G \) is countable infinite, the pmp Bernoulli action \( G \acts (X_0, \mu_0)^G \) is mixing.

2.4. \textit{Groups acting on trees}. Let \( T = (V(T), E(T)) \) be a locally finite tree, so that the edge set \( E(T) \) is a symmetric subset of \( V(T) \times V(T) \) with the property that vertices \( v, w \in V(T) \) are adjacent if and only if \( (v, w), (w, v) \in E(T) \). When \( T \) is clear from the context, we will write \( E \) instead of \( E(T) \). Also we will often write \( T \) instead of \( V(T) \) for the vertex set. For any two vertices \( v, w \in T \) let \([v, w]\) denote the smallest subtree of \( T \) that contains \( v \) and \( w \). The distance between vertices \( v, w \in T \) is defined as \( d(v, w) = |V([v, w])| - 1 \). Fixing a root \( \rho \in T \), we define the boundary \( \partial T \) of \( T \) as the collection of all infinite line segments starting at \( \rho \). We equip \( \partial T \) with a metric \( d_\rho \) as follows. If \( \omega, \omega' \in \partial T \), let \( v \in T \) be the unique vertex such that \( d(\rho, v) = \sup_{v \in \omega' \cap \omega} d(\rho, v) \) and define
\[
d_\rho(\omega, \omega') = \exp(-d(\rho, v)).
\]
Then, up to homeomorphism, the space \((\partial T, d_\rho)\) does not depend on the chosen root \( \rho \in T \). Furthermore, the Hausdorff dimension \( \dim_H \partial T \) of \((\partial T, d_\rho)\) is also independent of the choice of \( \rho \in T \).

Let \( \text{Aut}(T) \) denote the group of automorphisms of \( T \). By [Tit70, Proposition 3.2], if \( g \in \text{Aut}(T) \), then either:
- \( g \) fixes a vertex or interchanges a pair of vertices (in this case we say that \( g \) is \textit{elliptic});
- or there exists a bi-infinite line segment \( L \subset T \), called the \textit{axis} of \( g \), such that \( g \) acts on \( L \) by non-trivial translation (in this case we say that \( g \) is \textit{hyperbolic}).

We equip \( \text{Aut}(T) \) with the topology of pointwise convergence. A subgroup \( G \subset \text{Aut}(T) \) is closed with respect to this topology if and only if for every \( v \in T \) the stabilizer subgroup \( G_v = \{ g \in G : g \cdot v = v \} \) is compact. An action of an lcsc group \( G \) on \( T \) is a continuous homomorphism \( G \rightarrow \text{Aut}(T) \). We say that the action \( G \acts T \) is \textit{cocompact} if there is a finite set \( F \subset E(T) \) such that \( G \cdot F = E(T) \). A subgroup \( G \subset \text{Aut}(T) \) is called \textit{non-elementary} if it does not fix any point in \( T \cup \partial T \) and does not interchange any pair of points in \( T \cup \partial T \). Equivalently, \( G \subset \text{Aut}(T) \) is non-elementary if there exist hyperbolic elements \( h, g \in G \) with axes \( L_h \) and \( L_g \) such that \( L_h \cap L_g \) is finite. If \( G \subset \text{Aut}(T) \) is a non-elementary closed subgroup, there exists a unique minimal \( G \)-invariant subtree \( S \subset T \) and \( G \) is compactly generated if and only if \( G \acts S \) is cocompact (see [CM11, §2]). Recall from (1.5) the definition of the Poincaré exponent \( \delta(G \acts T) \) of a subgroup \( G \subset \text{Aut}(T) \). If \( G \subset \text{Aut}(T) \) is a closed subgroup such that \( G \acts T \) is cocompact, then we have that \( \delta(G \acts T) = \dim_H \partial T \).

3. \textit{Phase transitions for non-singular Bernoulli actions: proof of Theorems A and B}
Let \( G \) be a countable infinite group and let \( (\mu_g)_{g \in G} \) be a family of equivalent probability measures on a standard Borel space \( X_0 \). Let \( \nu \) also be a probability measure on \( X_0 \). For \( t \in [0, 1] \) we define the family of probability measures

\[ 
\mu_t \]
\[ \mu_g^t = (1 - t) \nu + t \mu_g, \quad g \in G. \]  

We write \( \mu_t \) for the infinite product measure \( \mu_t = \prod_{g \in G} \mu_g^t \) on \( X = \prod_{g \in G} X_0 \). We prove Theorem 3.1 below, which is slightly more general than Theorem A.

**Theorem 3.1.** Let \( G \) be a countable infinite group and let \( (\mu_g)_{g \in G} \) be a family of equivalent probability measures on a standard probability space \( X_0 \), which is not supported on a single atom. Assume that the Bernoulli action \( G \curvearrowright \prod_{g \in G} (X_0, \mu_g) \) is non-singular. Let \( \nu \) also be a probability measure on \( X_0 \). Then for every \( t \in [0, 1] \) the Bernoulli action

\[ G \curvearrowright (X, \mu_t) = \prod_{g \in G} (X_0, (1 - t) \nu + t \mu_g) \]  

is non-singular. Assume, in addition, that one of the following conditions holds.

1. \( \nu \sim \mu_e \).
2. \( \nu \prec \mu_e \) and \( \sup_{g \in G} |\log d\mu_g/d\mu_e(x)| < +\infty \) for a.e. \( x \in X_0 \).

Then there exists a \( t_1 \in [0, 1] \) such that \( G \curvearrowright (X, \mu_t) \) is dissipative for every \( t > t_1 \) and weakly mixing for every \( t < t_1 \).

**Remark 3.2.** One might hope to prove a completely general phase transition result that only requires \( \nu \prec \mu_e \), and not the additional assumption that \( \sup_{g \in G} |\log d\mu_g/d\mu_e(x)| < +\infty \) for a.e. \( x \in X_0 \). However, the following example shows that this is not possible.

Let \( G \) be any countable infinite group and let \( G \curvearrowright \prod_{g \in G} (C_0, \eta_g) \) be a conservative non-singular Bernoulli action. Note that Theorem 3.1 implies that \( G \curvearrowright \prod_{g \in G} (X_0, (1 - t) \eta_e + t \eta_g) \) is conservative for every \( t < 1 \). Let \( C_1 \) be a standard Borel space and let \( (\mu_g)_{g \in G} \) be a family of equivalent probability measures on \( X_0 = C_0 \sqcup C_1 \) such that \( 0 < \sum_{g \in G} \mu_g(C_1) < +\infty \) and such that \( \mu_g|_{C_0} = \mu_g(C_0) \eta_g \). Then the Bernoulli action \( G \curvearrowright (X, \mu) = \prod_{g \in G} (X_0, \mu_g) \) is non-singular with non-negligible conservative part \( C_0^G \subset G \) and dissipative part \( X \setminus C_0^G \). Taking \( \nu = \eta_e \prec \mu_e \), for each \( t < 1 \) the Bernoulli action \( G \curvearrowright (X, \mu_t) = \prod_{g \in G} (X_0, (1 - t) \eta_e + t \eta_g) \) is constructed in the same way, by starting with the conservative Bernoulli action \( G \curvearrowright \prod_{g \in G} (C_0, (1 - t) \eta_e + t \eta_g) \). So for every \( t \in (0, 1) \) the Bernoulli action \( G \curvearrowright (X, \mu_t) \) has non-negligible conservative part and non-negligible dissipative part.

We can also prove a version of Theorem B in the more general setting of Theorem 3.1.

**Theorem 3.3.** Let \( G \) be a countable infinite non-amenable group. Make the same assumptions as in Theorem 3.1 and consider the non-singular Bernoulli actions \( G \curvearrowright (X, \mu_t) \) given by (3.2). Assume, moreover, that:

1. \( \nu \sim \mu_e \), or
2. \( \nu \prec \mu_e \) and \( \sup_{g \in G} |\log d\mu_g/d\mu_e(x)| < +\infty \) for a.e. \( x \in X_0 \).

Then there exists a \( t_0 > 0 \) such that \( G \curvearrowright (X, \mu_t) \) is strongly ergodic for every \( t < t_0 \).
Proof of Theorem 3.1. Assume that $G \curvearrowright (X, \mu_1) = \prod_{g \in G} (X_0, \mu_g)$ is non-singular. For every $t \in [0, 1]$ we have that

$$\sum_{h \in G} H^2(\mu^h_1, \mu^h_{gh}) \leq t \sum_{h \in G} H^2(\mu_h, \mu_{gh}) \quad \text{for every } g \in G,$$

so that $G \curvearrowright (X, \mu_t)$ is non-singular for every $t \in [0, 1]$. The rest of the proof we divide into two steps.

CLAIM 1. If $G \curvearrowright (X, \mu_t)$ is conservative, then $G \curvearrowright (X, \mu_s)$ is weakly mixing for every $s < t$.

Proof of Claim 1. Note that for every $g \in G$ we have that

$$(\mu^s_g)^r = (1 - r)v + r\mu^s_g = (1 - r)v + r(1 - s)v + rs\mu_g = \mu^s_{gr},$$

so that $(\mu^s_g)^r = \mu^s_{sr}$. Therefore, it suffices to prove that $G \curvearrowright (X, \mu_s)$ is weakly mixing for every $s < 1$, assuming that $G \curvearrowright (X, \mu_1)$ is conservative.

The claim is trivially true for $s = 0$. So assume that $G \curvearrowright (X, \mu_1)$ is conservative and fix $s \in (0, 1)$. Let $G \curvearrowright (Y, \eta)$ be an ergodic pmp action. Define $Y_0 = X_0 \times X_0 \times \{0, 1\}$ and define the probability measures $\lambda$ on $[0, 1]$ by $\lambda(0) = s$. Define the map $\theta : Y_0 \to X_0$ by

$$\theta(x, x', j) = \begin{cases} x & \text{if } j = 0, \\ x' & \text{if } j = 1. \end{cases}$$

(3.3)

Then for every $g \in G$ we have that $\theta_g(\mu^s_g \times v \times \lambda) = \mu^s_g$. Write $Z = \{0, 1\}^G$ and equip $Z$ with the probability measure $\lambda^G$. We identify the Bernoulli action $G \curvearrowright Y_0^G$ with the diagonal action $G \curvearrowright X \times X \times Z$. By applying $\theta$ in each coordinate we obtain a $G$-equivariant factor map

$$\Psi : X \times X \times Z \to X : \Psi(x, x', z)_h = \theta(x_h, x'_h, z_h).$$

(3.4)

Then the map $\text{id}_Y \times \Psi : Y \times X \times X \times Z \to Y \times X$ is $G$-equivariant and we have that $(\text{id}_Y \times \Psi)_* (\eta \times \mu_1 \times \mu^G_0) = \eta \times \mu_s$. The construction above is similar to [KS20, §4].

Take $F \in L^\infty(Y \times X, \eta \times \mu_s)^G$. Note that the diagonal action $G \curvearrowright (Y \times X, \eta \times \mu_1)$ is conservative, since $G \curvearrowright (Y, \eta)$ is pmp. The action $G \curvearrowright (X \times Z, \mu_0 \times \lambda^G)$ can be identified with a pmp Bernoulli action with base space $(X_0 \times \{0, 1\}, \nu \times \lambda)$, so that it is mixing. By [SW81, Theorem 2.3] we have that

$$L^\infty(Y \times X \times X \times Z, \eta \times \mu_1 \times \mu^G_0) = L^\infty(Y \times X, \eta \times \mu_1)^G \otimes 1 \otimes 1,$$

which implies that the assignment $(y, x, x', z) \mapsto F(y, \Psi(x, x', z))$ is essentially independent of $x'$ and $z$. Choosing a finite set of coordinates $\mathcal{F} \subset G$ and changing, for $g \in \mathcal{F}$, the value $z_g$ between 0 and 1, we see that $F$ is essentially independent of the $x_g$-coordinates for $g \in \mathcal{F}$. As this is true for any finite set $\mathcal{F} \subset G$, we have that $F \in L^\infty(Y)^G \otimes 1$. The action $G \curvearrowright (Y, \eta)$ is ergodic and therefore $F$ is essentially constant. We conclude that $G \curvearrowright (X, \mu_s)$ is weakly mixing. 

\hfill \Box
CLAIM 2. If \( v \sim \mu_s \) and if \( G \rtimes (X, \mu_1) \) is not dissipative, then \( G \rtimes (X, \mu_s) \) is conservative for every \( s < t \).

Proof of Claim 2. Again it suffices to assume that \( G \rtimes (X, \mu_1) \) is not dissipative and to show that \( G \rtimes (X, \mu_s) \) is conservative for every \( s < 1 \).

When \( s = 0 \), the statement is trivial, so assume that \( G \rtimes (X, \mu_1) \) is not dissipative and fix \( s \in (0, 1) \). Let \( C \subset X \) denote the non-negligible conservative part of \( G \rtimes (X, \mu_1) \).

As in the proof of Claim 1, write \( Z = [0, 1]^G \) and let \( \lambda \) be the probability measure on \([0, 1]\) given by \( \lambda(0) = s \). Writing \( \Psi : X \times X \times Z \to X \) for the \( G \)-equivariant map (3.4), we claim that \( \Psi_*((\mu_1 \times \mu_0 \times \lambda^G)|_{C \times X \times Z}) \sim \mu_s \), so that \( G \rtimes (X, \mu_s) \) is a factor of a conservative non-singular action, and therefore must be conservative itself.

As \( \Psi_*((\mu_1 \times \mu_0 \times \lambda^G)|_{C \times X \times Z}) \sim \mu_s \), we have that \( \Psi_*((\mu_1 \times \mu_0 \times \lambda^G)|_{C \times X \times Z}) \sim \mu_s \). Let \( \mathcal{U} \subset X \) be the Borel set, uniquely determined up to a set of measure zero, such that \( \Psi_*((\mu_1 \times \mu_0 \times \lambda^G)|_{C \times X \times Z}) \sim \mu_s|\mathcal{U} \). We have to show that \( \mu_s(X \setminus \mathcal{U}) = 0 \). Fix a finite subset \( \mathcal{F} \subset G \). For every \( t \in [0, 1] \) define

\[
(X_1, y_1^t) = \prod_{g \in \mathcal{F}} (X_0, (1-t)v + t\mu_g),
\]

\[
(X_2, y_2^t) = \prod_{g \in G \setminus \mathcal{F}} (X_0, (1-t)v + t\mu_g).
\]

We shall write \( y_1 = y_1^1, y_2 = y_2^1 \). Also define

\[
(Y_1, \xi_1) = \prod_{g \in \mathcal{F}} (X_0 \times X_0 \times [0, 1], \mu_g \times v \times \lambda),
\]

\[
(Y_2, \xi_2) = \prod_{g \in G \setminus \mathcal{F}} (X_0 \times X_0 \times [0, 1], \mu_g \times v \times \lambda).
\]

By applying the map (3.3) in every coordinate, we get factor maps \( \Psi_j : Y_j \to X_j \) that satisfy \( (\Psi_j)_*(\xi_j) = y_j^t \) for \( j = 1, 2 \). Identify \( X_1 \times Y_2 \equiv X \times (X_0 \times [0, 1])^{G \setminus \mathcal{F}} \) and define the subset \( C' \subset X_1 \times Y_2 \) by \( C' = C \times (X_0 \times [0, 1])^{G \setminus \mathcal{F}} \). Let \( \mathcal{U}' \subset X \) be Borel such that

\[
(id_{X_1} \times \Psi_2)_*((y_1 \times \xi_2)|_{C'}) \sim (y_1 \times y_2^t)|_{\mathcal{U}'}.
\]

Identify \( Y_1 \times X_2 \equiv X \times (X_0 \times [0, 1])^{G \setminus \mathcal{F}} \) and define \( V \subset Y_1 \times X_2 \) by \( V = \mathcal{U}' \times (X_0 \times [0, 1])^{G \setminus \mathcal{F}} \). Then we have that

\[
(\Psi_1 \times id_{X_2})_*(((\xi_1 \times y_2^t)|_{V}) \sim (\Psi_1 \times id_{X_2})_*((y_1 \times \xi_1)|_{C'} \times v^G \times \lambda^\mathcal{F})
\]

\[
= \Psi_*(((\xi_1 \times \xi_2)|_{C' \times X \times Z}) \sim \mu_s|\mathcal{U}.
\]

Let \( \pi : X_1 \times X_2 \to X_2 \) and \( \pi' : Y_1 \times X_2 \to X_2 \) denote the coordinate projections. Note that by construction we have that

\[
\pi'_*(y_1 \times y_2^t)|_V \sim \pi_*((y_1 \times y_2^t)|_{\mathcal{U}'}) \sim \pi_*(\mu_s|\mathcal{U}). \tag{3.5}
\]

Let \( W \subset X_2 \) be Borel such that \( \pi_*(\mu_s|\mathcal{U}) \sim y_2^t|_W \). For every \( y \in X_2 \) define the Borel sets

\[
\mathcal{U}_y = \{ x \in X_1 : (x, y) \in \mathcal{U} \} \quad \text{and} \quad \mathcal{U}'_y = \{ x \in X_1 : (x, y) \in \mathcal{U}' \}.
\]
As $\pi_*((\gamma_1 \times \gamma_2^s)|_{U'}) \sim \gamma_2^s|_W$, we have that
\[ \gamma_1(U'_y) > 0 \quad \text{for } \gamma_2^s\text{-a.e. } y \in W. \]

The disintegration of $(\gamma_1 \times \gamma_2^s)|_{U'}$ along $\pi$ is given by $(\gamma_1|_{U'_y})_{y \in W}$. Therefore, the disintegration of $(\zeta_1 \times \gamma_2^s)|_{\nu}$ along $\pi'$ is given by $(\gamma_1|_{U'_y} \times v^F \times \lambda^F)_{y \in W}$. We conclude that the disintegration of $(\Psi_1 \times \text{id}_{X_2})_\ast((\zeta_1 \times \gamma_2^s)|_{\nu})$ along $\pi$ is given by $((\Psi_1)_\ast(\gamma_1|_{U'_y} \times v^F \times \lambda^F))_{y \in W}$. The disintegration of $\mu_s|_{U}$ along $\pi$ is given by $(\gamma_2^s|_{U'})_{y \in W}$. Since $\mu_s|_{U} \sim (\Psi_1 \times \text{id}_{X_2})_\ast((\zeta_1 \times \gamma_2^s)|_{\nu})$, we conclude that
\[ (\Psi_1)_\ast(\gamma_1|_{U'_y} \times v^F \times \lambda^F) \sim \gamma_1^s|_{U_y} \quad \text{for } \gamma_2^s\text{-a.e. } y \in W. \]

As $\gamma_1(U'_y) > 0$ for $\gamma_2^s$-a.e. $y \in W$, and using that $v \sim \mu_e$, we see that
\[
\gamma_1^s \sim v^F \sim (\Psi_1)_\ast((\gamma_1 \times v^F \times \lambda^F)|_{U'_y} \times X_0^F \times \{1\}^F)
\leq (\Psi_1)_\ast(\gamma_1|_{U'_y} \times v^F \times \lambda^F).
\]

for $\gamma_2^s$-a.e. $y \in W$. It is clear that also $(\Psi_1)_\ast(\gamma_1|_{U'_y} \times v^F \times \lambda^F) \sim \gamma_1^s$, so that $\gamma_1^s|_{U_y} \sim \gamma_1^s$ for $\gamma_2^s$-a.e. $y \in W$. Therefore, we have that $\gamma_1^s(X_1 \setminus U_y) = 0$ for $\gamma_2^s$-a.e. $y \in W$, so that
\[ \mu_s(U \Delta (X_0^F \times W)) = 0. \]

Since this is true for every finite subset $F \subset G$, we conclude that $\mu_s(X \setminus U) = 0$. \hfill $\Box$

The conclusion of the proof now follows by combining both claims. Assume that $G \lhd (X, \mu_t)$ is not dissipative and fix $s < t$. Choose $r$ such that $s < r < t$.

\[ v \sim \mu_e. \] By Claim 2 we have that $G \lhd (X, \mu_r)$ is conservative. Then by Claim 1 we see that $G \lhd (X, \mu_s)$ is weakly mixing.

\[ v \prec \mu_e. \] As $v \prec \mu_e$, the measures $\mu_r^t$ and $\mu_e$ are equivalent. We have that
\[
\frac{d\mu_r^t}{d\mu_e} = \left(1 - t \right) \frac{dv}{d\mu_e} + t \frac{d\mu_g}{d\mu_e} \frac{d\mu_e}{d\mu_r^t}.
\]

So if $\sup_{g \in G} |\log d\mu_g/d\mu_e(x)| < +\infty$ for a.e $x \in X_0$, we also have that
\[
\sup_{g \in G} |\log d\mu_r^t/d\mu_e(x)| < +\infty \quad \text{for a.e. } x \in X_0.
\]

It follows from [BV20, Proposition 4.3] that $G \lhd (X, \mu_t)$ is conservative. Then by Claim 1 we have that $G \lhd (X, \mu_s)$ is weakly mixing. \hfill $\Box$

Remark 3.4. Let $I$ be a countably infinite set and suppose that we are given a family of equivalent probability measures $(\mu_i)_{i \in I}$ on a standard Borel space $X_0$. Let $\nu$ be a probability measure on $X_0$ that is equivalent to all the $\mu_i$. If $G$ is an lcsc group that acts
on $I$ such that for each $i \in I$ the stabilizer subgroup $G_i = \{ g \in G : g \cdot i = i \}$ is compact, then the pmp generalized Bernoulli action

$$G \acts \prod_{i \in I} (X_0, \nu), \quad (g \cdot x)_i = x_{g^{-1} \cdot i}$$

is mixing. For $t \in [0, 1]$ write

$$(X, \mu_t) = \prod_{i \in I} (X_0, (1 - t)\nu + t\mu_i)$$

and assume that the generalized Bernoulli action $G \acts (X, \mu_1)$ is non-singular.

Since [SW81, Theorem 2.3] still applies to infinitely recurrent actions of lcsc groups (see [AIM19, Remark 7.4]), it is straightforward to adapt the proof of Claim 1 in the proof of Theorem 3.1 to prove that if $G \acts (X, \mu_s)$ is infinitely recurrent, then $G \acts (X, \mu_t)$ is weakly mixing for every $s < t$. Similarly, we can adapt the proof of Claim 2, using that a factor of an infinitely recurrent action is again infinitely recurrent. Together, this leads to the following phase transition result in the lcsc setting.

Assume that $G_i = \{ g \in G : g \cdot i = i \}$ is compact for every $i \in I$ and that $\nu \sim \mu_e$. Then there exists a $t_1 \in [0, 1]$ such that $G \acts (X, \mu_t)$ is dissipative up to compact stabilizers for every $t > t_1$ and weakly mixing for every $t < t_1$.

Recall the following definition from [BKV19, Definition 4.2]. When $G$ is a countable infinite group and $G \acts (X, \mu)$ is a non-singular action on a standard probability space, a sequence $(\eta_n)$ of probability measures on $G$ is called strongly recurrent for the action $G \acts (X, \mu)$ if

$$\sum_{h \in G} \eta_n^2(h) \int_X \frac{d\mu(x)}{\sum_{k \in G} \eta_n(hk^{-1})dk^{-1}\mu/d\mu(x)} \to 0.$$ 

We say that $G \acts (X, \mu)$ is strongly conservative if there exists a sequence $(\eta_n)$ of probability measures on $G$ that is strongly recurrent for $G \acts (X, \mu)$.

**Lemma 3.5.** Let $G \acts (X, \mu)$ and $G \acts (Y, \nu)$ be non-singular actions of a countable infinite group $G$ on standard probability spaces $(X, \mu)$ and $(Y, \nu)$. Suppose that $\psi : (X, \mu) \to (Y, \nu)$ is a measure-preserving $G$-equivariant factor map and that $(\eta_n)$ is a sequence of probability measures on $G$ that is strongly recurrent for the action $G \acts (X, \mu)$. Then $(\eta_n)$ is strongly recurrent for the action $G \acts (Y, \nu)$.

**Proof.** Let $E : L^0(X, [0, +\infty)) \to L^0(Y, [0, +\infty))$ denote the conditional expectation map that is uniquely determined by

$$\int_Y E(F)H d\nu = \int_X F(H \circ \psi) d\mu$$

for all positive measurable functions $F : X \to [0, +\infty)$ and $H : Y \to [0, +\infty)$. Since

$$\frac{dk^{-1}v}{dv} = \frac{d\psi_o(k^{-1}\mu)}{d\psi_o\mu} = E\left(\frac{dk^{-1}\mu}{d\mu}\right)$$

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for every \( k \in G \), we have that
\[
\sum_{k \in G} \eta_n(hk^{-1}) \frac{dk^{-1}v}{dv}(y) = E\left( \sum_{k \in G} \eta_n(hk^{-1}) \frac{dk^{-1}\mu}{d\mu} \right)(y) \quad \text{for a.e. } y \in Y. \tag{3.6}
\]

By Jensen’s inequality for conditional expectations, applied to the convex function \( t \mapsto \frac{1}{t} \), we also have that
\[
\frac{1}{E(\sum_{k \in G} \eta_n(hk^{-1})dk^{-1}\mu/d\mu)(y)} \leq E\left( \frac{1}{\sum_{k \in G} \eta_n(hk^{-1})dk^{-1}\mu/d\mu} \right)(y) \quad \text{for a.e. } y \in Y. \tag{3.7}
\]

Combining (3.6) and (3.7), we see that
\[
\sum_{h \in G} \eta_n^2(h) \int_Y \sum_{k \in G} \eta_n(hk^{-1})dk^{-1}v/dv(y) \frac{dv(y)}{\sum_{k \in G} \eta_n(hk^{-1})dk^{-1}v/dv(y)} \\
\leq \sum_{h \in G} \eta_n^2(h) \int_Y E\left( \frac{1}{\sum_{k \in G} \eta_n(hk^{-1})dk^{-1}\mu/d\mu} \right)(y) dv(y) \\
= \sum_{h \in G} \eta_n^2(h) \int_X \frac{d\mu(x)}{\sum_{k \in G} \eta_n(hk^{-1})dk^{-1}\mu/d\mu(x)},
\]
which converges to 0 as \( \eta_n \) is strongly recurrent for \( G \acts (X, \mu) \).

We say that a non-singular group action \( G \acts (X, \mu) \) has an \textit{invariant mean} if there exists a \( G \)-invariant linear functional \( \varphi \in L^\infty(X)^* \). We say that \( G \acts (X, \mu) \) is \textit{amenable (in the sense of Zimmer)} if there exists a \( G \)-equivariant conditional expectation \( E : L^\infty(G \times X) \rightarrow L^\infty(X) \), where the action \( G \acts G \times X \) is given by \( g \cdot (h, x) = (gh, g \cdot x) \).

**Proposition 3.6.** Let \( G \) be a countable infinite group and let \((\mu_g)_{g \in G}\) be a family of equivalent probability measures on a standard Borel space \( X_0 \) that is not supported on a single atom. Let \( v \) be a probability measure on \( X_0 \) and for each \( t \in [0, 1] \) consider the Bernoulli action (3.2). Assume that \( G \acts (X, \mu_t) \) is non-singular.

1. If \( G \acts (X, \mu_t) \) has an invariant mean, then \( G \acts (X, \mu_s) \) has an invariant mean for every \( s < t \).
2. If \( G \acts (X, \mu_t) \) is amenable, then \( G \acts (X, \mu_s) \) is amenable for every \( s > t \).
3. If \( G \acts (X, \mu_t) \) is strongly conservative, then \( G \acts (X, \mu_s) \) is strongly conservative for every \( s < t \).

**Proof.** (1) We may assume that \( t = 1 \). So suppose that \( G \acts (X, \mu_1) \) has an invariant mean and fix \( s < 1 \). Let \( \lambda \) be the probability measure on \( \{0, 1\} \) that is given by \( \lambda(0) = s \). Then by [AIM19, Proposition A.9] the diagonal action \( G \acts (X \times X \times \{0, 1\}^G, \mu_1 \times \mu_0 \times \lambda^G) \) has an invariant mean. Since \( G \acts (X, \mu_s) \) is a factor of this diagonal action, it admits a \( G \)-invariant mean as well.

(2) It suffices to show that \( G \acts (X, \mu_t) \) is amenable whenever there exists a \( t \in (0, 1) \) such that \( G \acts (X, \mu_t) \) is amenable. Write \( \lambda \) for the probability measure on \( \{0, 1\} \) given by \( \lambda(0) = t \). Then \( G \acts (X, \mu_t) \) is a factor of the diagonal action \( G \acts (X \times X \times X_0) \)
[0, 1]_G,\ \mu_1 \times \mu_0 \times \lambda^G),\ so\ by\ [Zim78,\ Theorem\ 2.4]\ also\ the\ latter\ action\ is\ amenable.\ Since\ G \actson (X \times [0, 1]^G, \mu_0 \times \lambda^G)\ is\ pmp,\ we\ have\ that\ G \actson (X, \mu_1)\ is\ amenable.

(3) We may again assume that t = 1. Suppose that (\eta_n) is a strongly recurrent sequence of probability measures on G for the action G \actson (X, \mu_1). Fix s < 1 and let \lambda be the probability measure on \{0, 1\} defined by \lambda(0) = s. As the diagonal action G \actson (X \times X \times [0, 1], \mu_1 \times \mu_0 \times \lambda^G). Since G \actson (X, \mu_t) is a factor of G \actson (X \times X \times [0, 1], \mu_1 \times \mu_0 \times \lambda^G), it follows from Lemma 3.5 that the sequence \eta_n is strongly recurrent for G \actson (X, \mu_t).

We finally prove Theorem 3.3. The proof relies heavily upon the techniques developed in [MV20, §5].

Proof of Theorem 3.3. For every t \in (0, 1] write \rho^t for the Koopman representation

$$\rho^t : G \actson L^2(X, \mu_t) : \ (\rho^t_g(\xi))(x) = \left(\frac{d\mu_t}{d\mu_t^g}(x)\right)^{1/2} \xi(g^{-1} \cdot x).$$

Fix s \in (0, 1) and let C > 0 be such that \log(1 - x) \geq -Cx for every x \in [0, s). Then for every t < s and every g \in G we have that

$$\log(\langle \rho^t_g(1), 1 \rangle) = \sum_{h \in G} \log(1 - H^2(\mu^t_{gh}, \mu_h)) \geq \sum_{h \in G} \log(1 - tH^2(\mu^t_g, \mu_h)) \geq -Ct \sum_{h \in G} H^2(\mu^t_{gh}, \mu_h).$$

Because G \actson (X, \mu_1) is non-singular we get that

$$\langle \rho^t_g(1), 1 \rangle \to 1 \text{ as } t \to 0, \text{ for every } g \in G. \quad (3.8)$$

We claim that there exists a t' > 0 such that G \actson (X, \mu_t) is non-amenable for every t < t'. Suppose, to the contrary, that t_n is a sequence that converges to zero such that G \actson (X, \mu_{t_n}) is amenable for every n \in \mathbb{N}. Then it follows from [Nev03, Theorem 3.7] that \rho^{t_n} is weakly contained in the left regular representation \lambda_G for every n \in \mathbb{N}. Write 1_G for the trivial representation of G. It follows from (3.8) that \bigoplus_{n \in \mathbb{N}} \rho^{t_n} has almost invariant vectors, so that

$$1_G \prec \bigoplus_{n \in \mathbb{N}} \rho^{t_n} \prec \infty \lambda_G < \lambda_G,$$

which is in contradiction to the non-amenability of G. By Theorem 3.1 there exists a t_1 \in [0, 1] such that G \actson (X, \mu_{t_1}) is weakly mixing for every t < t_1. Since every dissipative action is amenable (see, for example, [AIM19, Theorem A.29]) it follows that t_1 \geq t' > 0.
Write $Z_0 = [0, 1)$ and let $\lambda$ denote the Lebesgue probability measure on $Z_0$. Let $\rho^0$ denote the reduced Koopman representation

$$\rho^0 : G \curvearrowright L^2(X \times Z^G_0, \mu_0 \times \lambda^G) \cong C1 : (\rho^0_0(\xi))(x) = \xi(g^{-1} \cdot x).$$

As $G$ is non-amenable, $\rho^0$ has stable spectral gap. Suppose that for every $s > 0$ we can find $0 < s' < s$ such that $\rho^{s'}$ is weakly contained in $\rho^s \otimes \rho^0$. Then there exists a sequence $s_n$ that converges to zero, such that $\rho^{s_n}$ is weakly contained in $\rho^{s_n} \otimes \rho^0$ for every $n \in \mathbb{N}$. This implies that $\bigoplus_{n \in \mathbb{N}} \rho^{s_n}$ is weakly contained in $\bigoplus_{n \in \mathbb{N}} (\rho^{s_n} \otimes \rho^0)$. But by (3.8), the representation $\bigoplus_{n \in \mathbb{N}} \rho^{s_n}$ has almost invariant vectors, so that $\bigoplus_{n \in \mathbb{N}} (\rho^{s_n} \otimes \rho^0)$ weakly contains the trivial representation. This is in contradiction to $\rho^0$ having stable spectral gap. We conclude that there exists an $s > 0$ such that $\rho^t$ is not weakly contained in $\rho^t \otimes \rho^0$ for every $t < s$.

We prove that $G \curvearrowright (X, \mu_t)$ is strongly ergodic for every $t < \min\{t', s\}$, in which case we can apply [MV20, Lemma 5.2] to the non-singular action $G \curvearrowright (X, \mu_t)$ and the pmp action $G \curvearrowright (X \times Z^G_0, \mu_0 \times \lambda^G)$ by our choice of $t'$ and $s$. After rescaling, we may assume that $G \curvearrowright (X, \mu_1)$ is ergodic and that $\rho^t$ is not weakly contained in $\rho^t \otimes \rho^0$ for every $t \in (0, 1)$.

Let $t \in (0, 1)$ be arbitrary and define the map

$$\Psi : X \times X \times Z^G_0 \to X : \Psi(x, y, z)_h = \begin{cases} x_h & \text{if } z_h \leq t, \\ y_h & \text{if } z_h > t. \end{cases}$$

Then $\Psi$ is $G$-equivariant and we have that $\Psi((\mu_1 \times \mu_0) \times \lambda^G) = \mu_t$. Suppose that $G \curvearrowright (X, \mu_t)$ is not strongly ergodic. Then we can find a bounded almost invariant sequence $f_n \in L^\infty(X, \mu_t)$ such that $\|f_n\|_2 = 1$ and $\mu_t(f_n) = 0$ for every $n \in \mathbb{N}$. Therefore, $\Psi_*(f_n)$ is a bounded almost invariant sequence for $G \curvearrowright (X \times X \times Z^G_0, \mu_1 \times \mu_0 \times \lambda^G)$. Let $E : L^\infty(X \times X \times Z^G_0) \to L^\infty(X)$ be the conditional expectation that is uniquely determined by $\mu_1 \circ E = \mu_1 \times \mu_0 \times \lambda^G$. By [MV20, Lemma 5.2] we have that $\lim_{n \to \infty} \|(E \circ \Psi_*)(f_n) - \Psi_*(f_n)\|_2 = 0$. As $\Psi$ is measure-preserving we get, in particular, that

$$\lim_{n \to \infty} \|(E \circ \Psi_*)(f_n)\|_2 = 1. \quad (3.9)$$

Note that if $\mu_t(f) = 0$ for some $f \in L^2(X, \mu_t)$, we have that $\mu_1((E \circ \Psi_*)(f)) = 0$. So we can view $E \circ \Psi_*$ as a bounded operator

$$E \circ \Psi_* : L^2(X, \mu_t) \otimes C1 \to L^2(X, \mu_t) \otimes C1.$$

**Claim.** The bounded operator $E \circ \Psi_* : L^2(X, \mu_t) \otimes C1 \to L^2(X, \mu_1) \otimes C1$ has norm strictly less than 1.

The claim is in direct contradiction to (3.9), so we conclude that $G \curvearrowright (X, \mu_t)$ is strongly ergodic.

**Proof of claim.** For every $g \in G$, let $\varphi_g$ be the map

$$\varphi_g : L^2(X_0, \mu'_g) \to L^2(X_0, \mu_g) : \varphi_g(F) = tF + (1-t)\nu(F) \cdot 1.$$
Then \( E \circ \Psi_\rho : L^2(X_0, \mu_t) \to L^2(X, \mu_1) \) is given by the infinite product \( \bigotimes_{g \in G} \varphi_g \). For every \( g \in G \) we have that
\[
\| F \|_{2, \mu_g} = \| (d \mu_g / d \mu_t)^{-1/2} F \|_{2, \mu_t} \leq t^{-1/2} \| F \|_{2, \mu_g},
\]
so that the inclusion map \( t_g : L^2(X_0, \mu_t) \hookrightarrow L^2(X_0, \mu_g) \) satisfies \( \| t_g \| \leq t^{-1/2} \) for every \( g \in G \). We have that
\[
\varphi_g(F) = t(F - \mu_g(F) \cdot 1) + \mu_t(F) \cdot 1 \quad \text{for every } F \in L^2(X_0, \mu_g).
\]
So if we write \( P^t_g \) for the projection map onto \( L^2(X_0, \mu_t) \ominus C_1 \) and \( P_g \) for the projection map onto \( L^2(X_0, \mu_g) \ominus C_1 \), we have that
\[
\varphi_g \circ P^t_g = t(P_g \circ t_g) \quad \text{for every } g \in G. \tag{3.10}
\]
For a non-empty finite subset \( F \subset G \) let \( V(F) \) be the linear subspace of \( L^2(X_0, \mu_t) \ominus C_1 \) spanned by \( \bigotimes_{g \in F} L^2(X_0, \mu_g) \ominus C_1 \otimes \bigotimes_{g \in G \setminus F} 1. \)

Then, using (3.10), we see that
\[
\|(E \circ \Psi_\rho)(f)\|_2 \leq |F|^{1/2} \| f \|_2 \quad \text{for every } f \in V(F).
\]
Since \( \bigoplus_{F \neq \emptyset} V(F) \) is dense inside \( L^2(X, \mu_t) \ominus C_1 \), we have that
\[
\|(E \circ \Psi_\rho)|_{L^2(X, \mu_t) \ominus C_1}\| \leq t^{1/2} < 1. \tag*{\square}
\]
This also concludes the proof of Theorem 3.3. \( \tag*{\square} \)

4. Non-singular Bernoulli actions arising from groups acting on trees: proof of Theorem C

Let \( T \) be a locally finite tree and choose a root \( \rho \in T \). Let \( \mu_0 \) and \( \mu_1 \) be equivalent probability measures on a standard Borel space \( X_0 \). Following [AIM19, §10], we define a family of equivalent probability measures \( (\mu_e)_{e \in E} \) by
\[
\mu_e = \begin{cases} 
\mu_0 & \text{if } e \text{ is oriented towards } \rho, \\
\mu_1 & \text{if } e \text{ is oriented away from } \rho.
\end{cases} \tag{4.1}
\]
Let \( G \subset \text{Aut}(T) \) be a subgroup. When \( g \in G \) and \( e \in E \), the edges \( e \) and \( g \cdot e \) are simultaneously oriented towards, or away from, \( \rho \), unless \( e \in E([\rho, g \cdot \rho]) \). As \( E([\rho, g \cdot \rho]) \) is finite for every \( g \in G \), the generalized Bernoulli action
\[
G \actson (X, \mu) = \prod_{e \in E} (X_0, \mu_e) : (g \cdot x)_e = x_{g^{-1} \cdot e} \tag{4.2}
\]
is non-singular. If we start with a different root \( \rho' \in T \), let \( (\mu'_e)_{e \in E} \) denote the corresponding family of probability measures on \( X_0 \). Then we have that \( \mu_e = \mu'_e \) for all but finitely many \( e \in E \), so that the measures \( \prod_{e \in E} \mu_e \) and \( \prod_{e \in E} \mu'_e \) are equivalent. Therefore, up to conjugacy, the action (4.2) is independent of the choice of root \( \rho \in T \).
**Lemma 4.1.** Let $T$ be a locally finite tree such that each vertex $v \in V(T)$ has degree at least 2. Suppose that $G \subset \text{Aut}(T)$ is a countable subgroup. Let $\mu_0$ and $\mu_1$ be equivalent probability measures on a standard Borel space $X_0$ and fix a root $\rho \in T$. Then the action $\alpha : G \curvearrowright (X, \mu)$ given by (4.2) is essentially free.

**Proof.** Take $g \in G \setminus \{e\}$. It suffices to show that $\mu(\{x \in X : g \cdot x = x\}) = 0$. If $g$ is elliptic, there exist disjoint infinite subtrees $T_1, T_2 \subset T$ such that $g \cdot T_1 = T_2$. Note that

$$
(X_1, \mu_1) = \prod_{e \in E(T_1)} (X_0, \mu_e) \quad \text{and} \quad (X_2, \mu_2) = \prod_{e \in E(T_2)} (X_0, \mu_e)
$$

are non-atomic and that $g$ induces a non-singular isomorphism $\varphi : (X_1, \mu_1) \to (X_2, \mu_2)$: $\varphi(x)_e = x_{g^{-1} \cdot e}$. We get that

$$
\mu_1 \times \mu_2(\{(x, \varphi(x)) : x \in X_1\}) = 0.
$$

A fortiori $\mu(\{x \in X : g \cdot x = x\}) = 0$. If $g$ is hyperbolic, let $L_g \subset T$ denote its axis on which it acts by non-trivial translation. Then $\prod_{e \in E(L_g)} (X_0, \mu_e)$ is non-atomic and by [BKV19, Lemma 2.2] the action $g^\mathbb{Z} \curvearrowright \prod_{e \in E(L_g)} (X_0, \mu_e)$ is essentially free. This implies that also $\mu(\{x \in X : g \cdot x = x\}) = 0$. \qed

We prove Theorem 4.2 below, which implies Theorem C and also describes the stable type when the action is weakly mixing.

**Theorem 4.2.** Let $T$ be a locally finite tree with root $\rho \in T$. Let $G \subset \text{Aut}(T)$ be a closed non-elementary subgroup with Poincaré exponent $\delta = \delta(G \curvearrowright T)$ given by (1.5). Let $\mu_0$ and $\mu_1$ be non-trivial equivalent probability measures on a standard Borel space $X_0$. Consider the generalized non-singular Bernoulli action $\alpha : G \curvearrowright (X, \mu)$ given by (4.2). Then $\alpha$ is:

- weakly mixing if $1 - H^2(\mu_0, \mu_1) > \exp(-\delta/2)$;
- dissipative up to compact stabilizers if $1 - H^2(\mu_0, \mu_1) < \exp(-\delta/2)$.

Let $G \curvearrowright (Y, \nu)$ be an ergodic pmp action and let $\Lambda \subset \mathbb{R}$ be the smallest closed subgroup that contains the essential range of the map

$$
X_0 \times X_0 \to \mathbb{R} : (x, x') \mapsto \log(d\mu_0/d\mu_1)(x) - \log(d\mu_0/d\mu_1)(x').
$$

Let $\Delta : G \to \mathbb{R}_{>0}$ denote the modular function and let $\Sigma$ be the smallest subgroup generated by $\Lambda$ and $\log(\Delta(G))$.

Suppose that $1 - H^2(\mu_0, \mu_1) > \exp(-\delta/2)$. Then the Krieger flow and the flow of weights of $\beta : G \curvearrowright X \times Y$ are determined by $\Lambda$ and $\Sigma$ as follows.

1. If $\Lambda$ (respectively, $\Sigma$) is trivial, then the Krieger flow (respectively, flow of weights) is given by $\mathbb{R} \curvearrowright \mathbb{R}$.
2. If $\Lambda$ (respectively, $\Sigma$) is dense, then the Krieger flow (respectively, flow of weights) is trivial.
3. If $\Lambda$ (respectively, $\Sigma$) equals $a\mathbb{Z}$, with $a > 0$, then the Krieger flow (respectively, flow of weights) is given by $\mathbb{R} \curvearrowright \mathbb{R}/a\mathbb{Z}$.  

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In general, we do not know the behaviour of the action \( (4.2) \) in the critical situation \( 1 - H^2(\mu_0, \mu_1) = \exp(-\delta/2) \). However, if \( T \) is a regular tree and \( G \rhd T \) has full Poincaré exponent, we prove in Proposition 4.3 below that the action is dissipative up to compact stabilizers. This is similar to [AIM19, Theorems 8.4 and 9.10].

**Proposition 4.3.** Let \( T \) be a \( q \)-regular tree with root \( \rho \in T \) and let \( G \subset \mathrm{Aut}(T) \) be a closed subgroup with Poincaré exponent \( \delta = \delta(G \rhd T) = \log(q-1) \). Let \( \mu_0 \) and \( \mu_1 \) be equivalent probability measures on a standard Borel space \( X_0 \).

If \( 1 - H^2(\mu_0, \mu_1) = (q-1)^{-1/2} \), then the action \( (4.2) \) is dissipative up to compact stabilizers.

Interesting examples of actions of the form \( (4.2) \) arise when \( G \subset \mathrm{Aut}(T) \) is the free group on a finite set of generators acting on its Cayley tree. In that case, following [AIM19, §6] and [MV20, Remark 5.3], we can also give a sufficient criterion for strong ergodicity.

**Proposition 4.4.** Let the free group \( \mathbb{F}_d \) on \( d \geq 2 \) generators act on its Cayley tree \( T \). Let \( \mu_0 \) and \( \mu_1 \) be equivalent probability measures on a standard Borel space \( X_0 \).

Then the action \( (4.2) \) dissipative if \( 1 - H^2(\mu_0, \mu_1) \leq (2d-1)^{-1/2} \) and weakly mixing and non-amenable if \( 1 - H^2(\mu_0, \mu_1) > (2d-1)^{-1/2} \). Furthermore, the action \( (4.2) \) is strongly ergodic when \( 1 - H^2(\mu_0, \mu_1) > (2d-1)^{-1/4} \).

The proof of Theorem 4.2 below is similar to that of [LP92, Theorem 4] and [AIM19, Theorems 10.3 and 10.4].

**Proof of Theorem 4.2.** Define a family \( (X_e)_{e \in E} \) of independent random variables on \( (X, \mu) = \prod_{e \in E}(X_0, \mu_e) \) by

\[
X_e(x) = \begin{cases} 
\log(d\mu_1/d\mu_0)(x_e) & \text{if } e \text{ is oriented towards } \rho, \\
\log(d\mu_0/d\mu_1)(x_e) & \text{if } e \text{ is oriented away from } \rho.
\end{cases}
\]

(4.3)

For \( v \in T \) we write

\[
S_v = \sum_{e \in E([\rho, v])} X_e.
\]

Then we have that

\[
\frac{dg\mu}{d\mu} = \exp(S_{g\rho}) \quad \text{for every } g \in G.
\]

Since \( G \subset \mathrm{Aut}(T) \) is a closed subgroup, for each \( v \in T \) the stabilizer subgroup \( G_v = \{ g \in G : g \cdot v = v \} \) is a compact open subgroup of \( G \).

Suppose that \( 1 - H^2(\mu_0, \mu_1) < \exp(-\delta/2) \). Then we have that

\[
\int_X \sum_{v \in G_\rho} \exp(S_v(x)/2) \, d\mu(x) = \sum_{v \in G_\rho} (1 - H^2(\mu_0, \mu_1))^{2d(\rho, v)} < +\infty,
\]

by definition of the Poincaré exponent. Therefore, we have that \( \sum_{v \in G_\rho} \exp(S_v(x)/2) < +\infty \) for a.e. \( x \in X \). Let \( \lambda \) denote the left invariant Haar measure on \( G \) and define \( L = \lambda(G_\rho) \), where \( G_\rho = \{ g \in G : g \cdot \rho = \rho \} \). Then we have that

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Phase transitions for non-singular Bernoulli actions

\[ \int_G \frac{d\mu}{d\lambda}(x) \, d\lambda(g) = L \sum_{v \in G \cdot \rho} \exp(S_v(x)) < +\infty \quad \text{for a.e. } x \in X. \]

We conclude that \( G \curvearrowright (X, \mu) \) is dissipative up to compact stabilizers.

Now assume that \( 1 - H^2(\mu_0, \mu_1) > \exp(-\delta/2) \). We start by proving that \( G \curvearrowright (X, \mu) \) is infinitely recurrent. By [AIM19, Theorem 8.17] we can find a non-elementary closed compactly generated subgroup \( G' \subset G \) such that \( 1 - H^2(\mu_0, \mu_1) > \exp(-\delta(G')/2) \). Let \( T' \subset T \) be the unique minimal \( G' \)-invariant subtree. Then \( G' \) acts cocompactly on \( T' \) and we have that \( \delta(G') = \dim_H \partial T' \). Let \( X \) and \( Y \) be independent random variables with distributions \( \log d\mu_1/d\mu_0 + \mu_0 \) and \( \log d\mu_0/d\mu_1 + \mu_1 \), respectively. Set \( Z = X + Y \) and write

\[ \varphi(t) = \mathbb{E}(\exp(tZ)). \]

The assignment \( t \mapsto \varphi(t) \) is convex, \( \varphi(t) = \varphi(1 - t) \) for every \( t \) and \( \varphi(1/2) = (1 - H^2(\mu_0, \mu_1))^2 \). We conclude that

\[ \inf_{t \geq 0} \varphi(t) = (1 - H^2(\mu_0, \mu_1))^2. \]

Write \( R_k \) for the sum of \( k \) independent copies of \( Z \). By the Chernoff–Cramér theorem, as stated in [LP92], there exists an \( M \in \mathbb{N} \) such that

\[ \mathbb{P}(R_M \geq 0) > \exp(-M\delta(G')). \quad (4.4) \]

Below we define a new unoriented tree \( S \). This means that the edge set of \( S \) consists of subsets \( \{v, w\} \subset V(S) \). Fix a vertex \( \rho' \in T' \) and define the unoriented tree \( S \) as follows.

- \( S \) has vertices \( v \in T' \) so that \( d_{T'}(\rho', v) \) is divisible by \( M \).
- There is an edge \( \{v, w\} \in E(S) \) between two vertices \( v, w \in S \) if \( d_{T'}(v, w) = M \) and \( [\rho', v]_{T'} \subset [\rho', w]_{T'} \).

Here the notation \( [\rho', v]_{T'} \) means that we consider the line segment \( [\rho', v] \) as a subtree of \( T' \). We have that \( \dim_H \partial S = M \dim_H \partial T' = M\delta(G') \). Form a random subgraph \( S(x) \) of \( S \) by deleting those edges \( \{v, w\} \in E(S) \) where

\[ \sum_{e \in E([v, w]_{T'})} X_e(x) < 0. \]

This is an edge percolation on \( S \), where each edge remains with probability \( p = \mathbb{P}(R_M \geq 0) \). So by (4.4) we have that \( p \exp(\dim_H S) > 1 \). Furthermore, if \( \{v, w\} \) and \( \{v', w'\} \) are edges of \( S \) so that \( E([v, w]_{T'}) \cap E([v', w']_{T'}) = \emptyset \), their presence in \( S(x) \) constitutes independent events. So the percolation process is a quasi-Bernoulli percolation as introduced in [Lyo89]. Taking \( w \in (1, p \exp(\dim_H S)) \) and setting \( u_n = w^{-n} \), it follows from [Lyo89, Theorem 3.1] that percolation occurs almost surely, that is, \( S(x) \) contains an infinite connected component for a.e. \( x \in X \). Writing

\[ S'_v(x) = \sum_{e \in E([\rho', v]_{T'})} X_e(x), \]
this means that for a.e. \( x \in (X, \mu) \) we can find a constant \( a_x > -\infty \) such that \( S'_v(x) > a_x \) for infinitely many \( v \in T' \). As \( T'/G' \) is finite, there exists a vertex \( w \in T' \) such that

\[
\sum_{v \in G'/w} \exp(S'_v(x)) = +\infty \text{ with positive probability.} \tag{4.5}
\]

Therefore, by Kolmogorov's zero–one law, we have that \( \sum_{v \in G'/w} \exp(S'_v(x)) = +\infty \) almost surely. Since a change of root results in a conjugate action, we may assume that \( \rho = w \). Then (4.5) implies that \( \sum_{v \in G'} \exp(S'_v(x)) = +\infty \) for a.e. \( x \in X \). Writing again \( L \) for the Haar measure of the stabilizer subgroup \( G_{\rho} = \{ g \in G : g \cdot \rho = \rho \} \), we see that

\[
\int_G \frac{dg}{d\mu} \, d\lambda(g) = L \sum_{v \in G} \exp(S_v) = +\infty \text{ almost surely.}
\]

We conclude that \( G \curvearrowright (X, \mu) \) is infinitely recurrent. We prove that \( G \curvearrowright (X, \mu) \) is weakly mixing using a phase transition result from the previous section. Define the measurable map

\[
\psi : X_0 \to (0, 1] : \quad \psi(x) = \min\{d\mu_1/d\mu_0(x), 1\}.
\]

Let \( \nu \) be the probability measure on \( X_0 \) determined by

\[
d\nu/d\mu_0(x) = \rho^{-1} \psi(x) \quad \text{where} \quad \rho = \int_{X_0} \psi(x) \, d\mu_0(x).
\]

Then we have that \( \nu \sim \mu_0 \) and for every \( s > 1 - \rho \) the probability measures

\[
\eta^s_0 = s^{-1} (\mu_0 - (1 - s)\nu),
\]

\[
\eta^s_1 = s^{-1} (\mu_1 - (1 - s)\nu)
\]

are well defined. We consider the non-singular actions \( G \curvearrowright (X, \eta_s) = \prod_{e \in E} (X_0, \eta^s_e) \), where

\[
\eta^s_e = \begin{cases} 
\eta^s_0 & \text{if } e \text{ is oriented towards } \rho, \\
\eta^s_1 & \text{if } e \text{ is oriented away from } \rho.
\end{cases}
\]

By the dominated convergence theorem we have that \( H^2(\eta^s_0, \eta^s_1) \to H^2(\mu_0, \mu_1) \) as \( s \to 1 \). So we can choose \( s \) close enough to 1, but not equal to 1, such that \( 1 - H^2(\eta^s_0, \eta^s_1) > \exp(-\delta/2) \). By the first part of the proof we have that \( G \curvearrowright (X, \eta_s) \) is infinitely recurrent. Note that

\[
\mu_j = (1 - s)v + s\eta^s_j \quad \text{for } j = 0, 1.
\]

Since we assumed that \( G \subset \text{Aut}(T) \) is closed, all the stabilizer subgroups \( G_v = \{ g \in G : g \cdot v = v \} \) are compact. By Remark 3.4 we conclude that \( G \curvearrowright (X, \mu) \) is weakly mixing.

Let \( G \curvearrowright (Y, v) \) be an ergodic pmp action. To determine the Krieger flow and the flow of weights of \( \beta : G \curvearrowright X \times Y \) we use a similar approach to [AIM19, Theorem 10.4] and [VW17, Proposition 7.3]. First we determine the Krieger flow and then we deal with the flow of weights.
As before, let $G' \subset G$ be a non-elementary compactly generated subgroup such that $1 - H^2(\mu_0, \mu_1) > \exp(-\delta(G')/2)$. By [AIM19, Theorem 8.7] we may assume that $G/G'$ is not compact. Let $T' \subset T$ be the minimal $G'$-invariant subtree. Let $v \in T'$ be as in Lemma 4.5 below so that

$$\bigcap_{g \in G} \left( E(gT') \cup E([v, g^{-1} \cdot v]) \right) = \emptyset. \quad (4.6)$$

Since changing the root yields a conjugate action, we may assume that $\rho = v$. Let $(Z_0, \zeta_0)$ be a standard probability space such that there exist measurable maps $\theta_0, \theta_1 : Z_0 \to X_0$ that satisfy $(\theta_0)_* \zeta_0 = \mu_0$ and $(\theta_1)_* \zeta_0 = \mu_1$. Write

$$\begin{align*}
(Z, \zeta) &= \prod_{e \in E(T) \setminus E(T')} (Z_0, \zeta_0), \\
(X_1, \rho_1) &= \prod_{e \in E(T) \setminus E(T')} (X_0, \mu_e), \\
(X_2, \rho_2) &= \prod_{e \in E(T')} (X_0, \mu_e).
\end{align*}$$

By the first part of the proof we have that $G' \curvearrowright (X_2, \rho_2)$ is infinitely recurrent. Define the pmp map

$$\Psi : (Z, \zeta) \to (X_1, \rho_1) : \quad (\Psi(z))_e = \begin{cases} 
\theta_0(ze) & \text{if } e \text{ is oriented towards } \rho, \\
\theta_1(ze) & \text{if } e \text{ is oriented away from } \rho.
\end{cases}$$

Consider

$$U = \{ e \in E(T) : e \text{ is oriented towards } \rho \}.$$

Since $gU \triangle U = E(T)([\rho, g \cdot \rho]) \subset E(T')$ for any $g \in G'$, the set $(E(T) \setminus E(T')) \cap U$ is $G'$-invariant. Therefore, $\Psi$ is a $G'$-equivariant factor map. Consider the Maharam extensions

$$G' \curvearrowright Z \times X_2 \times Y \times \mathbb{R} \quad \text{and} \quad G \curvearrowright X \times Y \times \mathbb{R}$$

of the diagonal actions $G' \curvearrowright Z \times X_2 \times Y$ and $G' \curvearrowright X \times Y \times \mathbb{R}$, respectively. Identifying $(X, \mu) = (X_1, \rho_1) \times (X_2, \rho_2)$, we obtain a $G'$-equivariant factor map

$$\Phi : Z \times X_2 \times Y \times \mathbb{R} \to X_1 \times X_2 \times Y \times \mathbb{R} : \quad \Phi(z, x, y, t) = (\Psi(z), x, y, t).$$

Take $F \in L^\infty(X \times Y \times \mathbb{R})^G$. By [AIM19, Proposition A.33] the Maharam extension $G' \curvearrowright Z \times X_2 \times Y$ is infinitely recurrent. Since $G' \curvearrowright Z$ is a mixing pmp generalized Bernoulli action we have that $F \circ \Phi \in L^\infty(Z \times X_2 \times Y \times \mathbb{R})^G \subset 1 \overline{\mathbb{S}} L^\infty(X \times Y \times \mathbb{R})^G$ by [SW81, Theorem 2.3]. Therefore, $F$ is essentially independent of the $E(T) \setminus E(T')$-coordinates. Thus, for any $g \in G$ the assignment

$$(x, y, t) \mapsto F(g \cdot x, y, t) = F(x, y, t - \log(dg^{-1}\mu/d\mu)(x))$$

is essentially independent of the $E(T) \setminus E(gT')$-coordinates. Since $\log(dg^{-1}\mu/d\mu)$ only depends on the $E([\rho, g^{-1} \cdot \rho])$-coordinates, we deduce that $F$ is essentially independent of
the $E(T) \setminus (E(g T') \cup E([\rho, g^{-1} \cdot \rho]))$-coordinates, for every $g \in G$. Therefore, by (4.6), we have that $F \in 1 \mathbb{R} L^\infty(Y \times \mathbb{R})$.

So we have proven that any $G$-invariant function $F \in L^\infty(X \times Y \times \mathbb{R})$ is of the form $F(x, y, t) = H(y, t)$, for some $H \in L^\infty(Y \times \mathbb{R})$ that satisfies

$$H(y, t) = H(g \cdot y, t + \log(dg^{-1} \mu/d\mu)(x)) \quad \text{for a.e. } (x, y, t) \in X \times Y \times \mathbb{R}. $$

Since $0$ is in the essential range of the maps $\log(dg\mu/d\mu)$, for every $g \in G$, we see that $H(g \cdot y, t) = H(y, t)$ for a.e. $(y, t) \in Y \times \mathbb{R}$. By ergodicity of $G \curvearrowleft Y$, we conclude that $H$ is of the form $H(y, t) = P(t)$, for some $P \in L^\infty(\mathbb{R})$ that satisfies

$$P(t) = P(t + \log(dg^{-1} \mu/d\mu)(x)) \quad \text{for a.e. } (x, t) \in X \times \mathbb{R}, \text{ for every } g \in G. \quad (4.7)$$

Let $\Gamma \subseteq \mathbb{R}$ be the subgroup generated by the essential ranges of the maps $\log(dg\mu/d\mu)$, for $g \in G$. If $\Gamma = \{0\}$ we can identify $L^\infty(X \times Y \times \mathbb{R})^G \cong L^\infty(\mathbb{R})$. If $\Gamma \subseteq \mathbb{R}$ is dense, then it follows that $P$ is essentially constant so that the Maharam extension $G \curvearrowleft X \times Y \times \mathbb{R}$ is ergodic, that is, the Krieger flow of $G \curvearrowleft X \times Y$ is trivial. If $\Gamma = a\mathbb{Z}$, with $a > 0$, we conclude by (4.7) that we can identify $L^\infty(X \times Y \times \mathbb{R})^G \cong L^\infty(\mathbb{R}/a\mathbb{Z})$, so that the Krieger flow of $G \curvearrowleft X \times Y$ is given by $\mathbb{R} \curvearrowleft \mathbb{R}/a\mathbb{Z}$. Finally, note that the closure of $\Gamma$ equals the closure of the subgroup generated by the essential range of the map

$$X_0 \times X_0 \rightarrow \mathbb{R}: \quad (x, x') \mapsto \log(d\mu_0/d\mu_1)(x) - \log(d\mu_0/d\mu_1)(x').$$

So we have calculated the Krieger flow in every case, concluding the proof of the theorem in the case where $G$ is unimodular.

When $G$ is not unimodular, let $G_0 = \ker \Delta$ be the kernel of the modular function. Let $G \curvearrowleft X \times Y \times \mathbb{R}$ be the modular Maharam extension and let $\alpha: G_0 \curvearrowleft X \times Y \times \mathbb{R}$ be its restriction to the subgroup $G_0$. Then we have that

$$L^\infty(X \times Y \times \mathbb{R})^G \subset L^\infty(X \times Y \times \mathbb{R})^\alpha.$$

By [AIM19, Theorem 8.16] we have that $\delta(G_0) = \delta$, and we can apply the argument above to conclude that $L^\infty(X \times Y \times \mathbb{R})^\alpha \subset 1 \mathbb{R} 1 \mathbb{R} L^\infty(\mathbb{R})$. So for every $F \in L^\infty(X \times Y \times \mathbb{R})^G$ there exists a $P \in L^\infty(\mathbb{R})$ such that

$$P(t) = P(t + \log(dg^{-1} \mu/d\mu)(x) + \log(\Delta(g))) \quad \text{for a.e. } (x, t) \in X \times \mathbb{R}, \text{ for every } g \in G. \quad (4.8)$$

Let $\Pi$ be the subgroup of $\mathbb{R}$ generated by the essential range of the maps

$$x \mapsto \log(dg^{-1} \mu/d\mu)(x) + \log(\Delta(g)) \quad \text{with } g \in G.$$

As $0$ is contained in the essential range of $\log(dg^{-1} \mu/d\mu)$, for every $g \in G$, we get that $\log(\Delta(G)) \subseteq \Pi$. Therefore, $\Pi$ also contains the subgroup $\Gamma \subseteq \mathbb{R}$ defined above. Thus, the closure of $\Pi$ equals the closure of $\Sigma$, where $\Sigma \subseteq \mathbb{R}$ is the subgroup as in the statement of the theorem. From (4.8) we conclude that we may identify $L^\infty(X \times Y \times \mathbb{R})^G \cong L^\infty(\mathbb{R})^\Sigma$, so that the flow of weights of $G \curvearrowleft X \times Y$ is as stated in the theorem. 

$$\square$$
LEMMA 4.5. Let $T$ be a locally finite tree and let $G \subset \text{Aut}(T)$ be a closed subgroup. Suppose that $H \subset G$ is a closed compactly generated subgroup that contains a hyperbolic element and assume that $G/H$ is not compact. Let $S \subset T$ be the unique minimal $H$-invariant subtree. Then there exists a vertex $v \in S$ such that
\begin{equation}
\bigcap_{g \in G} (gS \cup [v, g^{-1} \cdot v]) = \{v\}.
\end{equation}

Proof. Let $k \in H$ be a hyperbolic element and let $L \subset T$ be its axis, on which $k$ acts by a non-trivial translation. Then $L \subset S$, as one can show for instance as in the proof of [CM11, Proposition 3.8]. Pick any vertex $v \in L$. We claim that this vertex will satisfy (4.9). Take any $w \in V(T) \setminus \{v\}$. As $G/H$ is not compact, one can show as in [AIM19, Theorem 9.7] that there exists a $g \in G$ such that $g \cdot w \notin S$. Since $k$ acts by translation on $L$, there exists an $n \in \mathbb{N}$ large enough such that

$[v, k \cdot v] \subset [v, k^n g \cdot v]$ and $[v, k^{-1} \cdot v] \subset [v, k^{-n} g \cdot v],$

so that in particular we have that $w \notin [v, k^n g \cdot v] \cap [v, k^{-n} g \cdot v] = \{v\}$. Since $S$ is $H$-invariant, we also have that $k^n g \cdot w \notin S$ and $k^{-n} g \cdot w \notin S$ and we conclude that

$w \notin ((k^n g)^{-1}S \cup [v, k^n g \cdot v]) \cap ((k^{-n} g)^{-1}S \cup [v, k^{-n} g \cdot v]).$

Proof of Proposition 4.3. Define the family $(X_e)_{e \in E}$ of independent random variables on $(X, \mu)$ by (4.3) and write

$S_v = \sum_{e \in E(\rho, v)} X_e.$

CLAIM. There exists a $\delta > 0$ such that

$\mu(\{x \in X : S_v(x) \leq -\delta \text{ for every } v \in T \setminus \{\rho\}\}) > 0.$

Proof of claim. Note that $\mathbb{E}(\exp(X_e/2)) = 1 - H^2(\mu_0, \mu_1)$ for every $e \in E$. Define a family of random variables $(W_n)_{n \geq 0}$ on $(X, \mu)$ by

$W_n = \sum_{\substack{v \in T \\mid d(v, \rho) = n}} \exp(S_v/2).$

Using that $1 - H^2(\mu_0, \mu_1) = (q - 1)^{-1/2}$, one computes that

$\mathbb{E}(W_{n+1} \mid S_v, d(v, \rho) \leq n) = W_n$ for every $n \geq 1.$

So the sequence $(W_n)_{n \geq 0}$ is a martingale, and since it is positive it converges almost surely to a finite limit when $n \to +\infty$. Write $\Sigma_n = \{v \in T : d(v, \rho) = n\}$. As $W_n \geq \max_{v \in \Sigma_n} \exp(S_v/2)$ we conclude that there exists a positive constant $C < +\infty$ such that

$\mathbb{P}(S_v \leq C \text{ for every } v \in T) > 0.$

For any vertex $w \in T$, write $T_w = \{v \in T : [\rho, w] \subset [\rho, v]\}$: the set of children of $w$, including $w$ itself. Using the symmetry of the tree and changing the root from $\rho$ to $w \in T$, we also have that
\(\mathbb{P}(S_v - S_w \leq C \text{ for every } v \in T_w) > 0 \text{ for every } w \in T.\) \hspace{1cm} (4.10)

Set \(v_0 = (\log d\mu_1/d\mu_0)_*\mu_0\) and \(v_1 = (\log d\mu_0/d\mu_1)_*\mu_1.\) Because \(1 - H^2(\mu_0, \mu_1) \neq 0\) we have that \(\mu_0 \neq \mu_1,\) so that there exists a \(\delta > 0\) such that

\[v_0 * v_1((-\infty, -\delta)) > 0.\]

Here \(v_0 * v_1\) denotes the convolution product of \(v_0\) with \(v_1.\) Therefore, there exists \(N \in \mathbb{N}\) large enough such that

\[\mathbb{P}(S_w \leq -C - \delta \text{ for every } w \in \Sigma_N \text{ and } S_{w'} \leq -\delta \text{ for every } w' \in \Sigma_n \text{ with } n \leq N) > 0.\] \hspace{1cm} (4.11)

Since for any \(w \in \Sigma_N\) and \(w' \in \Sigma_n\) with \(n \leq N,\) we have that \(S_v - S_w\) is independent of \(S_{w'}\) for every \(v \in T_w,\) and since \(\Sigma_N\) is a finite set, it follows from (4.10) and (4.11) that

\[\mathbb{P}(S_v \leq -\delta \text{ for every } v \in T \setminus \{\rho\}) > 0.\]

This concludes the proof of the claim.

Let \(\delta > 0\) be as in the claim and define

\[\mathcal{U} = \{x \in X : S_v(x) \leq -\delta \text{ for every } v \in T \setminus \{\rho\}\},\]

so that \(\mu(\mathcal{U}) > 0.\) Let \(G_\rho\) be the stabilizer subgroup of \(\rho.\) Note that for every \(g, h \in G\) we have that \(S_{hg\cdot \rho}(x) = S_{g\cdot \rho}(h^{-1} \cdot x) + S_{h\cdot \rho}(x)\) for a.e. \(x \in X,\) so that for \(h \in G\) we have that

\[h \cdot \mathcal{U} \subset \{x \in X : S_{hg\cdot \rho}(x) \leq -\delta + S_{h\cdot \rho}(x) \text{ for every } g \neq G_\rho\}.\]

It follows that if \(h \notin G_\rho,\) we have that

\[\mathcal{U} \cap h \cdot \mathcal{U} \subset \{x \in X : S_{h\cdot \rho}(x) \leq -\delta \text{ and } S_{h\cdot \rho}(x) \geq \delta\} = \emptyset.\]

Since \(G \subset \text{Aut}(T)\) is closed, we have that \(G_\rho\) is compact. So the action \(G \acts (X, \mu)\) is not infinitely recurrent. Let \(\lambda\) denote the left invariant Haar measure on \(G.\) By an adaptation of the proof of [BV20, Proposition 4.3], the set

\[D = \left\{x \in X : \int_G \frac{dg \mu}{d\mu}(x) d\lambda(g) < +\infty\right\} = \left\{x \in X : \int_G \exp(S_{g\cdot \rho}(x)) d\lambda(g) < +\infty\right\}\]

satisfies \(\mu(D) \in \{0, 1\}.\) Since \(G \acts (X, \mu)\) is not infinitely recurrent, it follows from [AIM19, Proposition A.28] that \(\mu(D) > 0,\) so that we must have that \(\mu(D) = 1.\) By [AIM19, Theorem A.29] the action \(G \acts (X, \mu)\) is dissipative up to compact stabilizers.

We use a similar approach to [MV20, §6] in the proof of Proposition 4.4.

**Proof of Proposition 4.4.** It follows from Theorem 4.2 and Proposition 4.3 that the action \(G \acts (X, \mu),\) given by (4.2), is dissipative when \(1 - H^2(\mu_0, \mu_1) \leq (2d - 1)^{-1/2}\) and weakly mixing when \(1 - H^2(\mu_0, \mu_1) > (2d - 1)^{-1/2}\). So it remains to show that \(G \acts (X, \mu)\) is non-amenable when \(1 - H^2(\mu_0, \mu_1) > (2d - 1)^{-1/2}\) and strongly ergodic when \(1 - H^2(\mu_0, \mu_1) > (2d - 1)^{-1/4}\).
Assume first that $1 - H^2(\mu_0, \mu_1) > (2d - 1)^{-1/2}$. By taking the kernel of a surjective homomorphism $\mathbb{F}_d \to \mathbb{Z}$ we find a normal subgroup $H_1 \subset \mathbb{F}_d$ that is free on infinitely many generators. By [RT13, Théorème 0.1] we have that $\delta(H_1) = (2d - 1)^{-1/2}$. Then, using [Sul79, Corollary 6], we can find a finitely generated free subgroup $H_2 \subset H_1$ such that $H_1 = H_2 * H_3$ for some free subgroup $H_3 \subset H_1$ and such that $1 - H^2(\mu_0, \mu_1) > \exp(-\delta(H_2)/2)$. Let $\psi : H_1 \to H_3$ be the surjective group homomorphism uniquely determined by

$$\psi(h) = \begin{cases} e & \text{if } h \in H_2, \\ h & \text{if } h \in H_3. \end{cases}$$

We set $N = \ker \psi$, so that $H_2 \subset N$ and we get that $1 - H^2(\mu_0, \mu_1) > \exp(-\delta(N)/2)$. Therefore, $N \lhd (X, \mu)$ is ergodic by Theorem 4.2. Also we have that $H_1/N \cong H_3$, which is a free group on infinitely many generators. Therefore, $H_1 \lhd (X, \mu)$ is non-amenable by [MV20, Lemma 6.4]. A posteriori also $\mathbb{F}_d \lhd (X, \mu)$ is non-amenable.

Let $\pi$ be the Koopman representation of the action $\mathbb{F}_d \lhd (X, \mu)$:

$$\pi : G \lhd L^2(X, \mu) : (\pi_g(\xi))(x) = \left( \frac{dg \mu}{d\mu} (x) \right)^{1/2} \xi(g^{-1} \cdot x).$$

**Claim.** If $1 - H^2(\mu_0, \mu_1) > (2d - 1)^{-1/4}$, then $\pi$ is not weakly contained in the left regular representation.

**Proof of claim.** Let $\eta$ denote the canonical symmetric measure on the generator set of $\mathbb{F}_d$ and define

$$P = \sum_{g \in \mathbb{F}_d} \eta(g) \pi_g.$$ 

The $\eta$-spectral radius of $\alpha : \mathbb{F}_d \lhd (X, \mu)$, which we denote by $\rho_\eta(\alpha)$, is by definition the norm of $P$, as a bounded operator on $L^2(X, \mu)$. By [AIM19, Proposition A.11] we have that

$$\rho_\eta(\alpha) = \lim_{n \to \infty} \langle P^n(1), 1 \rangle^{1/n}$$

$$= \lim_{n \to \infty} \left( \sum_{g \in \mathbb{F}_d} \eta^n(g) (1 - H^2(\mu_0, \mu_1))^2 |g| \right)^{1/n},$$

where $|g|$ denotes the word length of a group element $g \in \mathbb{F}_d$. By [AIM19, Theorem 6.10] we then have that

$$\rho_\eta(\alpha) = \frac{(1 - H^2(\mu_0, \mu_1))^2}{2d} \left( (2d - 1) + (1 - H^2(\mu_0, \mu_1))^{-4} \right)$$

if $1 - H^2(\mu_0, \mu_1) > (2d - 1)^{-1/4}$, and

$$\rho_\eta(\alpha) = \frac{\sqrt{2d - 1}}{d}$$

if $1 - H^2(\mu_0, \mu_1) \leq (2d - 1)^{-1/4}$. Therefore, if $1 - H^2(\mu_0, \mu_1) > (2d - 1)^{-1/4}$, we have that $\rho_\eta(\alpha) > \rho_\eta(\mathbb{F}_d)$, where $\rho_\eta(\mathbb{F}_d)$ denotes the $\eta$-spectral radius of the left regular
representation. This implies that $\alpha$ is not weakly contained in the left regular representation (see, for instance, [AD03, §3.2]).

Now assume that $1 - H^2(\mu_0, \mu_1) > (2d - 1)^{-1/4}$. As in the proof of Theorem 4.2 there exist probability measures $\nu$, $\eta_0$ and $\eta_1$ on $X_0$ that are equivalent to $\mu_0$ and a number $s \in (0, 1)$ such that

$$\mu_j = (1 - s)\nu + s\eta_j \quad \text{for} \quad j = 0, 1,$$

and such that $1 - H^2(\eta_0, \eta_1) > (2d - 1)^{-1/4}$. Consider the non-singular action

$$\mathbb{F}_d \cong (X, \eta) = \prod_{e \in E(T)} (X_0, \eta_e) \quad \text{where} \quad \eta_e = \begin{cases} \eta_0 & \text{if } e \text{ is oriented towards } \rho, \\ \eta_1 & \text{if } e \text{ is oriented away from } \rho. \end{cases}$$

By Theorem 4.2 the action $\mathbb{F}_d \cong (X, \eta)$ is ergodic. Write $\rho$ for the Koopman representation associated to $\mathbb{F}_d \cong (X, \eta)$. By the claim, $\rho$ is not weakly contained in the left regular representation. Let $\lambda$ be the probability measure on $\{0, 1\}$ given by $\lambda(0) = s$. Let $\rho_0^0$ be the reduced Koopman representation of the pmp generalized Bernoulli action $\mathbb{F}_d \cong (X \times \{0, 1\}^{E(T)}, \nu^{E(T)} \times \lambda^{E(T)})$. Then $\rho_0^0$ is contained in a multiple of the left regular representation. Therefore, as $\rho$ is not weakly contained in the left regular representation, $\rho$ is not weakly contained in $\rho \otimes \rho_0^0$.

Define the map

$$\Psi: X \times X \times \{0, 1\}^{E(T)} \to X: \quad \Psi(x, y, z)_e = \begin{cases} x_e & \text{if } z_e = 0, \\ y_e & \text{if } z_e = 1. \end{cases}$$

Then $\Psi$ is $\mathbb{F}_d$-equivariant and we have that $\Psi_\mu(\eta \times \nu^{E(T)} \times \lambda^{E(T)}) = \mu$. Suppose that $\mathbb{F}_d \cong (X, \mu)$ is not strongly ergodic. Then there exists a bounded almost invariant sequence $f_n \in L^\infty(X, \mu)$ such that $\|f_n\|_2 = 1$ and $\mu(f_n) = 0$ for every $n \in \mathbb{N}$. Therefore, $\Psi_\mu(f_n)$ is a bounded almost invariant sequence for the diagonal action $\mathbb{F}_d \cong (X \times \{0, 1\}^{E(T)}, \eta \times \nu^{E(T)} \times \lambda^{E(T)})$. Let $E: L^\infty(X \times X \times \{0, 1\}^{E(T)}) \to L^\infty(X)$ be the conditional expectation that is uniquely determined by $\mu \circ E = \eta \times \nu^{E(T)} \times \lambda^{E(T)}$. By [MV20, Lemma 5.2] we have that $\lim_{n \to \infty} \|(E \circ \Psi_\mu)(f_n) - \Psi_\mu(f_n)\|_2 = 0$, and in particular we get that

$$\lim_{n \to \infty} \|(E \circ \Psi_\mu)(f_n)\|_2 = 1. \quad (4.12)$$

But just as in the proof of Theorem 3.3 we have that

$$\left\|(E \circ \Psi_\mu)\right\|_{L^2(X, \mu) \otimes C_1} < 1,$$

which is in contradiction with (4.12). We conclude that $\mathbb{F}_d \cong (X, \mu)$ is strongly ergodic.

Proposition 4.6 below complements Theorem 4.2 by considering groups $G \subset \text{Aut}(T)$ that are not closed. This is similar to [AIM19, Theorem 10.5].

**Proposition 4.6.** Let $T$ be a locally finite tree with root $\rho \in T$. Let $G \subset \text{Aut}(T)$ be an lcsc group such that the inclusion map $G \to \text{Aut}(T)$ is continuous and such that
Phase transitions for non-singular Bernoulli actions

$G \subset \text{Aut}(T)$ is not closed. Write $\delta = \delta(G \curvearrowright T)$ for the Poincaré exponent given by (1.5). Let $\mu_0$ and $\mu_1$ be non-trivial equivalent probability measures on a standard Borel space $X_0$. Consider the generalized non-singular Bernoulli action $\alpha: G \curvearrowright (X, \mu)$ given by (4.2). Let $H \subset \text{Aut}(T)$ be the closure of $G$. Then the following assertions hold.

- If $1 - H^2(\mu_0, \mu_1) > \exp(-\delta/2)$, then $\alpha$ is ergodic and its Krieger flow is determined by the essential range of the map

$$X_0 \times X_0 \rightarrow \mathbb{R} : (x, x') \mapsto \log(d\mu_0/d\mu_1)(x) - \log(d\mu_0/d\mu_1)(x')$$

as in Theorem 4.2.

- If $1 - H^2(\mu_0, \mu_1) < \exp(-\delta/2)$, then each ergodic component of $\alpha$ is of the form $G \curvearrowright H/K$, where $K$ is a compact subgroup of $H$. In particular, there exists a $G$-invariant $\sigma$-finite measure on $X$ that is equivalent to $\mu$.

**Proof.** Let $H \subset \text{Aut}(T)$ be the closure of $G$. Then $\delta(H) = \delta$ and we can apply Theorem 4.2 to the non-singular action $H \curvearrowright (X, \mu)$.

If $1 - H^2(\mu_0, \mu_1) > \exp(-\delta/2)$, then $H \curvearrowright X$ is ergodic. As $G \subset H$ is dense, we have that

$$L^\infty(X)^G = L^\infty(X)^H = \mathbb{C}1,$$

so that $G \curvearrowright X$ is ergodic. Let $H \curvearrowright X \times \mathbb{R}$ be the Maharam extension associated to $H \curvearrowright X$. Again, as $G \subset H$ is dense, we have that

$$L^\infty(X \times \mathbb{R})^G = L^\infty(X \times \mathbb{R})^H.$$

Note that the subgroup generated by the essential ranges of the maps $\log(dg^{-1}\mu/d\mu)$, with $g \in G$, is the same as the subgroup generated by the essential ranges of the maps $\log(dh^{-1}\mu/d\mu)$, with $h \in H$. Then one determines the Krieger flow of $G \curvearrowright X$ as in the proof of Theorem 4.2.

If $1 - H^2(\mu_0, \mu_1) < \exp(-\delta/2)$, the action $H \curvearrowright (X, \mu)$ is dissipative up to compact stabilizers. By [AIM19, Theorem A.29] each ergodic component is of the form $H \curvearrowright H/K$ for a compact subgroup $K \subset H$. Therefore, each ergodic component of $G \curvearrowright (X, \mu)$ is of the form $G \curvearrowright H/K$, for some compact subgroup $K \subset H$.

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**REFERENCES**


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