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Statistical aspects of mean field coupled intermittent maps

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Abstract. We study infinite systems of mean field weakly coupled intermittent maps in the Pomeau–Manneville scenario. We prove that the coupled system admits a unique 'physical' stationary state, to which all absolutely continuous states converge. Moreover, we show that suitably regular states converge polynomially.

Key words: mean field coupling, intermittent maps
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1. *Introduction*

Mean field coupled dynamics can be thought of as a dynamical system with n 'particles' with states $x_1, \ldots, x_n$ evolving according to an equation of the type

$$x_k \mapsto T\left(x_k, \varepsilon \frac{\delta_{x_1} + \cdots + \delta_{x_n}}{n}\right).$$

Here T is some transformation, $\varepsilon \in \mathbb{R}$ is the strength of coupling and δ_{x_k} are the delta functions, so $(\delta_{x_1} + \cdots + \delta_{x_n})/n$ is a probability measure describing the 'mean state' of the system.

As $n \to \infty$, it is natural to consider the evolution of the distribution of particles: if μ is a probability measure describing distribution of particles, then one looks at the operator that maps μ to the distribution of $T(x, \varepsilon\mu)$, where $x \sim \mu$ is random.

In chaotic dynamics, mean field coupled systems have been studied first when T is a perturbation of a uniformly expanding circle map by Keller [5] and followed, among others, by Bálint *et al.* [2], Blank [3], Galatolo [4], and Sélley and Tanzi [9]. The case when T is a perturbation of an Anosov diffeomorphism has been covered by Bahsoun, Liverani and Sélley [1] (see in particular [1, §2.2] for a motivation of such study). See the paper by Galatolo [4] for a general framework when the site dynamics admits exponential decay of correlations. The results of [4] also apply to certain mean field coupled random systems. We refer the reader to Tanzi [10] for a recent review on the topic and to [1] for connections with classical and important partial differential equations.

In this work, we consider the situation where T is a perturbation of the prototypical chaotic map with *non-uniform* expansion and polynomial decay of correlations: the intermittent map on the unit interval [0, 1] in the Pomeau–Manneville scenario [8]. We restrict to the case when the coupling is weak, that is, ε is small.

Our results apply to a wide class of intermittent systems satisfying standard assumptions (see §2). To keep the introduction simple, here we consider a very concrete example.

Fix $\gamma_* \in (0, 1)$ and let, for $\varepsilon \in \mathbb{R}$, $h \in L^1[0, 1]$ and $x \in [0, 1]$,

$$T_{\varepsilon h}(x) = x(1 + x^{\gamma_* + \varepsilon\gamma_h}) + \varepsilon\varphi_h(x) \bmod 1, \tag{1.1}$$

where

$$\gamma_h = \int_0^1 h(s)\sin(2\pi s)\,ds \quad \text{and} \quad \varphi_h(x) = x^2(1-x)\int_0^1 h(s)\cos(2\pi s)\,ds.$$

This way, $T_{\varepsilon h}$ is a perturbation of the intermittent map $x \mapsto x(1 + x^{\gamma_*}) \bmod 1$. Informally, γ_h changes the degree of the indifferent point at 0 and φ_h is responsible for perturbations away from 0.

We restrict to $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ with ε_0 small and to h non-negative with $\int_0^1 h(x)\,dx = 1$ (that is, h is a probability density).

Let $\mathcal{L}_{\varepsilon h} \colon L^1[0, 1] \to L^1[0, 1]$ be the transfer operator for $T_{\varepsilon h}$:

$$(\mathcal{L}_{\varepsilon h} g)(x) = \sum_{y \in T_{\varepsilon h}^{-1}(x)} \frac{g(y)}{T'_{\varepsilon h}(y)}, \tag{1.2}$$

and let

$$\mathcal{L}_\varepsilon h = \mathcal{L}_{\varepsilon h} h. \tag{1.3}$$

We call $\mathcal{L}_\varepsilon$ the *self-consistent* transfer operator. Observe that $\mathcal{L}_\varepsilon$ is nonlinear and that $\mathcal{L}_\varepsilon h$ is the density of the distribution of $T_{\varepsilon h}(x)$, if x is distributed according to the probability measure with density h.

We prove that for sufficiently small ε_0, the self-consistent transfer operator $\mathcal{L}_\varepsilon$ admits a unique *physical* (see [1, Definition 2.1]) invariant state h_ε and that $\mathcal{L}_\varepsilon^n h$ converges to h_ε in L^1 polynomially for all sufficiently regular h.

THEOREM 1.1. *There exists $\varepsilon_0 \in (0, 1 - \gamma_*)$ so that each $\mathcal{L}_\varepsilon$ with $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, as an operator on probability densities, has a unique fixed point h_ε. For every probability density h,*

$$\lim_{n\to\infty} \|\mathcal{L}_\varepsilon^n h - h_\varepsilon\|_{L^1} = 0.$$

Moreover, $h_\varepsilon \in C^\infty(0, 1]$ and there are $A, a_1, a_2, \ldots > 0$ such that for all $\ell \geq 1$ and $x \in (0, 1]$,

$$\int_0^x h_\varepsilon(s)\,ds \leq A x^{1-1/(\gamma_*+\varepsilon_0)} \quad \text{and} \quad \frac{|h_\varepsilon^{(\ell)}(x)|}{h_\varepsilon(x)} \leq \frac{a_\ell}{x^\ell}. \tag{1.4}$$

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THEOREM 1.2. *In the setup of Theorem 1.1, suppose that a probability density h is twice differentiable on $(0, 1]$ and satisfies, for some $\tilde{A}, \tilde{a}_1, \tilde{a}_2 > 0$ and all $\ell = 1, 2$ and $x \in (0, 1]$,*

$$\int_0^x h(s)\, ds \le \tilde{A} x^{1-1/(\gamma_* + \varepsilon_0)} \quad \text{and} \quad \frac{|h_\varepsilon^{(\ell)}(x)|}{h_\varepsilon(x)} \le \frac{\tilde{a}_\ell}{x^\ell}.$$

Then,

$$\|\mathcal{L}_\varepsilon^n h - h_\varepsilon\|_{L^1} \le C n^{-(1-\gamma_* - \varepsilon_0)/(\gamma_* + \varepsilon_0)}, \tag{1.5}$$

where C depends only on $\tilde{A}, \tilde{a}_1, \tilde{a}_2$ and ε_0.

Remark 1.3. The restriction $\varepsilon_0 < 1 - \gamma_*$ serves to guarantee that $\gamma_* + \varepsilon \gamma_h$ is bounded away from 1 and that the right-hand side of equation (1.5) converges to zero.

Remark 1.4. A curious corollary of Theorem 1.1 is that the density of the unique absolutely continuous invariant probability measure for the map $x \mapsto x(1 + x_*^\gamma)$ is smooth, namely $C^\infty(0, 1]$ with the bounds in equation (1.4). Our abstract framework covers such a result also for the Liverani–Saussol–Vaienti maps [7]. To the best of our knowledge, this is the first time such a result is written down. At the same time, we are aware of at least two different unwritten prior proofs which achieve similar or stronger results, one by Damien Thomine and the other by Caroline Wormell.

Remark 1.5. Another example to which our results apply is

$$T_{\varepsilon h}(x) = x(1 + x^{\gamma_*}) + \varepsilon x(1 - x) \int_0^1 h(s) \sin(\pi s)\, ds \bmod 1,$$

where $\gamma_* \in (0, 1)$ and $\varepsilon \in [0, \varepsilon_0]$. This is interesting because now each $T_{\varepsilon h}$ with $\varepsilon > 0$ is uniformly expanding, but the expansion is not uniform in ε. Thus, even for this example, standard operator contraction techniques employed in [4, 5] do not apply.

Remark 1.6. Let h_ε be as in Theorem 1.1. A natural question is to study the regularity of the map $\varepsilon \mapsto h_\varepsilon$. We expect that it should be differentiable in a suitable topology.

The paper is organized as follows. Theorems 1.1 and 1.2 are corollaries of the general results in §2, where we introduce the abstract framework and state the abstract results. The abstract proofs are carried out in §3, and in §4, we verify that the specific map in equation (1.1) fits the abstract assumptions.

2. *Assumptions and results*

We consider a family of maps $T_{\varepsilon h} : [0, 1] \to [0, 1]$, where $\varepsilon \in [-\varepsilon_*, \varepsilon_*]$, $\varepsilon_* > 0$, and h is a probability density on $[0, 1]$.

We require that each such $T_{\varepsilon h}$ is a full branch increasing map with finitely many branches, i.e. there is a finite partition of the interval $(0, 1)$ into open intervals $B_{\varepsilon h}^k$, modulo their endpoints, such that each restriction $T_{\varepsilon h} : B_{\varepsilon h}^k \to (0, 1)$ is an increasing bijection.

We assume that each restriction $T_{\varepsilon h} : B_{\varepsilon h}^k \to (0, 1)$ satisfies the following assumptions with the constants independent of ε, h or the branch.

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(a) $T_{\varepsilon h}$ is $r+1$ times continuously differentiable with $r \geq 2$.

(b) There are $c_\gamma > 0$, $C_\gamma > 1$ and $\gamma \in [0, 1)$ such that

$$1 + c_\gamma x^\gamma \leq T'_{\varepsilon h}(x) \leq C_\gamma. \tag{2.1}$$

(c) Denote $w = 1/T'_{\varepsilon h}$. There are $b_1, \ldots, b_r > 0$ and $\chi_* \in (0, 1]$ so that for all $1 \leq \ell \leq r$, $0 \leq j \leq \ell$ and each monomial $w_{\ell,j}$ in the expansion of $(w^\ell)^{(\ell-j)}$,

$$\frac{w^\ell}{\chi_\ell} \leq \frac{1}{\chi_\ell \circ T_{\varepsilon h}} - b_\ell \frac{|w_{\ell,j}|}{\chi_j}, \tag{2.2}$$

where $\chi_\ell(x) = \min\{x^\ell, \chi_*\}$. (For example, the expansion of $(w^3)''$ is $6w(w')^2 + 3w^2 w''$.)

(d) If $\partial B^k_{\varepsilon h} \not\ni 0$, that is, $B^k_{\varepsilon h}$ is not the leftmost branch, then $T_{\varepsilon h}$ has bounded distortion:

$$\frac{T''_{\varepsilon h}}{(T'_{\varepsilon h})^2} \leq C_d, \tag{2.3}$$

with $C_d > 0$.

Remark 2.1. Assumption (c) is unusual, but we did not see a way to replace it with something natural. At the same time, it is straightforward to verify and to apply. It plays the role of a distortion bound in C^r adapted to an intermittency at 0.

In addition to the above, we assume that the transfer operators corresponding to $T_{\varepsilon h}$ vary nicely in h. We state this formally in equation (2.6) after we introduce the required notation.

Define the transfer operators $\mathcal{L}_{\varepsilon h}$ and $\mathcal{L}_\varepsilon$ as in equations (1.2) and (1.3).

For an integer $k \geq 1$, let H^k denote the set of k-Hölder functions $g\colon (0, 1] \to (0, \infty)$, that is, such that g is $k-1$ times continuously differentiable with $g^{(k-1)}$ Lipschitz. Denote $\mathrm{Lip}_g(x) = \limsup_{y \to x} |g(x) - g(y)|/|x - y|$.

Suppose that $a_1, \ldots, a_r > 0$. For $1 \leq k \leq r$, let

$$\mathcal{D}^k = \biggl\{ g \in H^k : \frac{|g^{(\ell)}|}{g} \leq \frac{a_\ell}{\chi_\ell} \text{ for all } 1 \leq \ell < k, \ \frac{\mathrm{Lip}_{g^{(k-1)}}}{g} \leq \frac{a_k}{\chi_k} \biggr\}. \tag{2.4}$$

Take $A > 0$ and let

$$\mathcal{D}^k_1 = \biggl\{ g \in \mathcal{D}^k : \int_0^1 g(s)\, ds = 1, \ \int_0^x g(s)\, ds \leq A x^{1-\gamma} \biggr\}. \tag{2.5}$$

Remark 2.2. If $g \in \mathcal{D}^1_1$, then $g(x) \leq C x^{-\gamma}$, where C depends only on a_1 and A.

Now and for the rest of the paper, we fix $a_1, \ldots, a_r$ and A so that $\mathcal{D}^k$ and $\mathcal{D}^k_1$ are non-empty and invariant under $\mathcal{L}_{\varepsilon h}$. This can be done thanks to the following lemma.

LEMMA 2.3. *There are $a_1, \ldots, a_r, A > 0$ such that for all $1 \leq q \leq r$:*

(a) *$g \in \mathcal{D}^q$ implies $\mathcal{L}_{\varepsilon h} g \in \mathcal{D}^q$;*

(b) *$g \in \mathcal{D}^q_1$ implies $\mathcal{L}_{\varepsilon h} g \in \mathcal{D}^q_1$.*

Moreover, given $C > 0$, we can ensure that $\min\{a_1, \ldots, a_r, A\} > C$.

The proof of Lemma 2.3 is postponed to §3.

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Finally, we assume that there are $0 \le \beta < \min\{\gamma, 1-\gamma\}$ and $C_\beta > 0$ such that if $h_0, h_1 \in L^1$ and $v \in \mathcal{D}_1^2$, then

$$\mathcal{L}_{\varepsilon h_0} v - \mathcal{L}_{\varepsilon h_1} v = \delta(f_0 - f_1), \tag{2.6}$$

for some $f_0, f_1 \in \mathcal{D}_1^1$ with $f_0(x), f_1(x) \le C_\beta x^{-\beta}$ and $\delta \le |\varepsilon| C_\beta \|h_0 - h_1\|_{L^1}$.

Let $\mathbf{C} = (\varepsilon_*, r, c_\gamma, C_\gamma, \gamma, b_1, \ldots, b_r, \chi_*, C_d, A, a_1, \ldots, a_r, \beta, C_\beta)$ be the collection of constants from the above assumptions.

Our main abstract result is the following theorem.

THEOREM 2.4. *There exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$:*

(a) *there exists h_ε in $\mathcal{D}_1^r$ so that for every probability density h,*

$$\lim_{n\to\infty} \|\mathcal{L}_\varepsilon^n h - h_\varepsilon\|_{L^1} = 0;$$

(b) *let $\widetilde{\mathcal{D}}_1^2$ be a version of $\mathcal{D}_1^2$ with constants $\tilde{A}, \tilde{a}_1, \tilde{a}_2$ in place of A, a_1, a_2. (We do not require that $\widetilde{\mathcal{D}}_1^2$ is invariant.) Then for every $h \in \widetilde{\mathcal{D}}_1^2$,*

$$\|\mathcal{L}_\varepsilon^n h - h_\varepsilon\|_{L^1} \le C n^{1-1/\gamma},$$

where C depends only on $\mathbf{C}$ and $\tilde{A}, \tilde{a}_1, \tilde{a}_2$.

3. *Proofs*

In this section, we prove Lemma 2.3 and Theorem 2.4. The latter follows from Lemma 3.3, and Propositions 3.6 and 3.8.

Throughout, we work with maps $T_{\varepsilon h}$ as per our assumptions, in particular, ε is always assumed to belong to $[-\varepsilon_*, \varepsilon_*]$ and h is always a probability density.

3.1. *Invariance of $\mathcal{D}^q$, $\mathcal{D}_1^q$ and distortion bounds.*

We start with the proof of Lemma 2.3. Our construction of $A, a_1, \ldots, a_r$ allows them to be arbitrarily large and, without mentioning this further, we restrict the choice so that

$$x \mapsto (1-\gamma)x^{-\gamma} \quad \text{is in } \breve{\mathcal{D}}_1^r, \tag{3.1}$$

where $\breve{\mathcal{D}}_1^r$ is the version of $\mathcal{D}_1^r$ with $A/2, a_1/2, \ldots, a_r/2$ in place of $A, a_1, \ldots, a_r$. Informally, we require that $(1-\gamma)x^{-\gamma}$ is deep inside $\mathcal{D}_1^r$.

LEMMA 3.1. *There is a choice of $a_1, \ldots, a_r$ such that if $B \subset (0,1)$ is a branch of $T_{\varepsilon h}$ and $g \in \mathcal{D}^q$ with $1 \le q \le r$, then $\mathcal{L}_{\varepsilon h}(1_B g) \in \mathcal{D}^q$.*

Proof. To simplify the notation, let $T : B \to (0,1)$ denote the restriction of $T_{\varepsilon h}$ to B. Then its inverse T^{-1} is well defined. Let $w = 1/T'$ and $f = (gw) \circ T^{-1}$. We have to choose $a_1, \ldots, a_r$ to show $f \in \mathcal{D}^q$ independently of g and B.

For illustration, it is helpful to write out a couple of derivatives of f:

$$\begin{aligned} f' &= [g'w^2 + gw'w] \circ T^{-1}, \\ f'' &= [g''w^3 + 3g'w'w^2 + gw''w^2 + g(w')^2 w] \circ T^{-1}. \end{aligned}$$

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An observation that $f^{(\ell)} = (u_\ell w) \circ T^{-1}$, where $u_0 = g$ and $u_{\ell+1} = (u_\ell w)'$, generalizes the pattern:

$$f^{(\ell)} = \Big[g^{(\ell)} w^{\ell+1} + \sum_{j=0}^{\ell-1} g^{(j)} W_{\ell,j} w \Big] \circ T^{-1}. \tag{3.2}$$

Here each $W_{\ell,j}$ is a linear combination of monomials from the expansion of $(w^\ell)^{(\ell-j)}$.

By equation (2.2), for each ℓ, there is $c_\ell > 0$ depending only on $b_1, \ldots, b_{\ell-1}$, such that

$$\frac{w^\ell}{\chi_\ell} \le \frac{1}{\chi_\ell \circ T} - c_\ell \sum_{j=0}^{\ell-1} \frac{|W_{\ell,j}|}{\chi_j}.$$

Using this and the triangle inequality,

$$\begin{aligned} \bigg| \frac{g^{(\ell)}}{g} w^\ell + \sum_{j=0}^{\ell-1} \frac{g^{(j)}}{g} W_{\ell,j} \bigg| &\le \frac{|\chi_\ell g^{(\ell)}|}{g} \frac{w^\ell}{\chi_\ell} + \max_{j<\ell} \frac{|\chi_j g^{(j)}|}{g} \sum_{j=0}^{\ell-1} \frac{|W_{\ell,j}|}{\chi_j} \\ &\le \frac{|\chi_\ell g^{(\ell)}|}{\chi_\ell \circ T\ g} - \bigg[c_\ell \frac{|\chi_\ell g^{(\ell)}|}{g} - \max_{j<\ell} \frac{|\chi_j g^{(j)}|}{g} \bigg] \sum_{j=0}^{\ell-1} \frac{|W_{\ell,j}|}{\chi_j}. \end{aligned} \tag{3.3}$$

Choose $a_1 \ge c_1^{-1}$ and $a_\ell \ge c_\ell^{-1} \max_{j<\ell} a_j$ for $2 \le \ell \le r$. It is immediate that if $g \in \mathcal{D}^q$ and $1 \le \ell < q$, then the right-hand side of equation (3.3) is at most $a_\ell/\chi_\ell \circ T$, which in turn implies that $f^{(\ell)}/f \le a_\ell/\chi_\ell$. A similar argument yields $\mathrm{Lip}_{f^{(q-1)}}/f \le a_q/\chi_q$, and hence $f \in \mathcal{D}^q$ as required. $\square$

Proof of Lemma 2.3. First we show that part (a) follows from Lemma 3.1. Indeed, let $a_1, \ldots, a_r$ be as in Lemma 3.1 and suppose that $g \in \mathcal{D}^q$. Write

$$\mathcal{L}_{\varepsilon h} g = \sum_B \mathcal{L}_{\varepsilon h}(1_B g),$$

where the sum is taken over the branches of $T_{\varepsilon h}$. Each $\mathcal{L}_{\varepsilon h}(1_B g)$ belongs to $\mathcal{D}^q$ by Lemma 3.1, and $\mathcal{D}^q$ is closed under addition. Hence, $\mathcal{L}_{\varepsilon h} g \in \mathcal{D}^q$.

It remains to prove part (b) by choosing a suitable A. Without loss of generality, we restrict to $q = 1$.

Fix ε, h and denote, to simplify notation, $T = T_{\varepsilon h}$ and $\mathcal{L} = \mathcal{L}_{\varepsilon h}$. Suppose that $g \in \mathcal{D}^1$ with $\int_0^1 g(s)\, ds = 1$ and $\int_0^x g(s)\, ds \le A x^{1-\gamma}$ for all x. We have to show that if A is sufficiently large, then $\int_0^x (\mathcal{L} g)(s)\, ds \le A x^{1-\gamma}$.

Suppose that T has branches $B_1, \ldots, B_N$, where B_1 is the leftmost branch. Denote by $T_k \colon B_k \to (0, 1)$ the corresponding restrictions. Taking the sum over branches, write

$$\int_0^x (\mathcal{L} g)(s)\, ds = \sum_{k=1}^N \int_{T_k^{-1}(0,x)} g(s)\, ds. \tag{3.4}$$

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Since $T_1^{-1}(x) \leq x/(1 + cx^\gamma)$ with some c depending only on c_γ and γ,

$$\int_{T_1^{-1}(0,x)} g(s)\, ds \leq A\left(\frac{x}{1+cx^\gamma}\right)^{1-\gamma} \leq A(x^{1-\gamma} - c'x), \tag{3.5}$$

where $c' > 0$ also depends only on c_γ and γ.

Let now $k \geq 2$. Note that $T_k^{-1}(0, x) \subset (C_\gamma^{-1}, 1)$. Observe that if $g \in \mathcal{D}^1$ with $\int_0^1 g(s)\, ds = 1$, then $g(s) \leq C$ for $s \in (C_\gamma^{-1}, 1)$, where C depends only on a_1 and χ_*. Since T_k is uniformly expanding with bounded distortion in equation (2.3), $|T_k^{-1}(0, x)| \leq C'|B_k|x$ with some C' that depends only on C_d. Hence,

$$\int_{T_k^{-1}(0,x)} g(s)\, ds \leq CC'|B_k|x. \tag{3.6}$$

Assembling equations (3.4), (3.5) and (3.6), we have

$$\int_0^x (\mathcal{L}g)(s)\, ds \leq A(x^{1-\gamma} - c'x) + C''x$$

with $c', C'' > 0$ independent of A, ε and h. For each $A \geq C''/c'$, the right-hand side above is bounded by $Ax^{1-\gamma}$, as desired. □

A useful corollary of Lemma 3.1 is a distortion bound.

LEMMA 3.2. *Let $n > 0$ and $\delta > 0$. Consider maps $T_{\varepsilon h_k}$, $1 \leq k \leq n$ with some ε and h_k as per our assumptions. Choose and restrict to a single branch for every $T_{\varepsilon h_k}$, so that all $T_{\varepsilon h_k}$ are invertible and $T_{\varepsilon h_k}^{-1}$ is well defined. Denote*

$$T_n = T_{\varepsilon h_n} \circ \cdots \circ T_{\varepsilon h_1} \quad \textit{and} \quad J_n = 1/T_n' \circ T_n^{-1}.$$

Then,

$$\frac{|J_n^{(\ell)}|}{J_n} \leq \frac{a_\ell}{\chi_\ell} \quad \textit{for } 1 \leq \ell < r, \quad \textit{and} \quad \frac{\operatorname{Lip}_{J_n^{(r-1)}}}{J_n} \leq \frac{a_r}{\chi_r}. \tag{3.7}$$

In particular, for every $\delta > 0$, the bounds above are uniform in $x \in [\delta, 1]$.

Proof. Let $P_{\varepsilon h_k}$ be the transfer operator for $T_{\varepsilon h_k}$, restricted to the chosen branch:

$$P_{\varepsilon h_k} g = \frac{g}{T_{\varepsilon h_k}'} \circ T_{\varepsilon h_k}^{-1}.$$

Denote $P_n = P_{\varepsilon h_n} \cdots P_{\varepsilon h_1}$.

Let $g \equiv 1$. Clearly, $g \in \mathcal{D}^r$. By Lemma 3.1, $P_{\varepsilon h_k}\mathcal{D}^r \subset \mathcal{D}^r$ and thus $P_n g \in \mathcal{D}^r$. However, $P_n g = J_n$, and the desired result follows from the definition of $\mathcal{D}^k$. □

3.2. *Fixed point and memory loss.* Further, let h_ε be a fixed point of $\mathcal{L}_\varepsilon$ as in the following lemma; later we will show that it is unique.

LEMMA 3.3. *There exists $h_\varepsilon \in \mathcal{D}_1^r$ such that $\mathcal{L}_\varepsilon h_\varepsilon = h_\varepsilon$.*

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Proof. Suppose that $f, g \in \mathcal{D}_1^r$. Write

$$\|\mathcal{L}_{\varepsilon f} f - \mathcal{L}_{\varepsilon g} g\|_{L^1} \le \|\mathcal{L}_{\varepsilon f} f - \mathcal{L}_{\varepsilon f} g\|_{L^1} + \|\mathcal{L}_{\varepsilon f} g - \mathcal{L}_{\varepsilon g} g\|_{L^1}.$$

The first term on the right is bounded by $\|f-g\|_{L^1}$ because $\mathcal{L}_{\varepsilon f}$ is a contraction in L^1. By equation (2.6), so is the second term, up to a multiplicative constant. It follows that $\mathcal{L}_\varepsilon$ is continuous in L^1. Recall that $\mathcal{L}_\varepsilon$ preserves $\mathcal{D}_1^r$ and note that $\mathcal{D}_1^r$ is compact in the L^1 topology. By the Schauder fixed point theorem, $\mathcal{L}_\varepsilon$ has a fixed point in $\mathcal{D}_1^r$. □

Further, we use the rates of memory loss for sequential dynamics from [6].

THEOREM 3.4. *Suppose that $f, g \in \mathcal{D}_1^1$ and $h_1, h_2, \ldots$ are probability densities. Denote $\mathcal{L}_n = \mathcal{L}_{\varepsilon h_n} \cdots \mathcal{L}_{\varepsilon h_1}$. Then,*

$$\|\mathcal{L}_n f - \mathcal{L}_n g\|_{L^1} \le C_1 n^{-1/\gamma+1}.$$

More generally, if $f(x), g(x) \le C'_\gamma x^{-\gamma'}$ with $C'_\gamma > 0$ and $\gamma' \in [0,\gamma]$, then

$$\|\mathcal{L}_n f - \mathcal{L}_n g\|_{L^1} \le C_2 n^{-1/\gamma+\gamma'/\gamma}.$$

The constant C_1 depends only on $\mathbf{C}$, and C_2 depends additionally on γ', C'_γ.

Proof. In the language of [6], the family $T_{\varepsilon h_k}$ defines a non-stationary non-uniformly expanding dynamical system. As a base of 'induction', we use the whole interval $(0, 1)$. For a return time of $x \in (0, 1)$ corresponding to a sequence $T_{\varepsilon h_k}$, $k \ge n$, we take the minimal $j \ge 1$ such that $T_{\varepsilon h_k} \circ \cdots \circ T_{\varepsilon h_{j-1}}(x)$ belongs to one of the right branches of $T_{\varepsilon h_j}$, that is, not to the leftmost branch. Note that we work with the return time which is not a first return time, unlike in [6], but this is a minor issue that can be solved by extending the space where the dynamics is defined.

It is a direct verification that our assumptions and Lemma 3.2 verify [6, equations (NU:1)–(NU:7)] with tail function $h(n) = Cn^{-1/\gamma}$ with C depending only on γ and c_γ.

Further, in the language of [6], functions in $\mathcal{D}_1^1$ are densities of probability measures with a uniform tail bound $Cn^{-1/\gamma+1}$, where C depends only on $\mathbf{C}$; each $f \in \mathcal{D}_1^1$ with $f(x) \le C'_\gamma x^{-\gamma'}$ is a density of a probability measure with tail bound $Cn^{-1/\gamma+\gamma'/\gamma}$ with C depending only on $\mathbf{C}$ and γ', C'_γ.

In this setup, Theorem 3.4 is a particular case of [6, Theorem 3.8 and Remark 3.9]. □

Recall that, as a part of assumption in equation (2.6), we fixed $\beta \in [0, \min\{\gamma, 1-\gamma\})$.

LEMMA 3.5. *There is a constant $C_{\beta,\gamma} > 0$, depending only on β and γ, such that if a non-negative sequence δ_n, $n \ge 0$, satisfies*

$$\delta_n \le \xi n^{-1/\gamma+1} + \sigma \sum_{j=0}^{n-1} \delta_j (n-j)^{-1/\gamma+\beta/\gamma} \quad \textit{for all } n > 0 \tag{3.8}$$

with some $\sigma \in (0, C_{\beta,\gamma}^{-1})$ and $\xi > 0$, then

$$\delta_n \le \max\left\{\delta_0, \frac{\xi}{1-\sigma C_{\beta,\gamma}}\right\} n^{-1/\gamma+1} \quad \textit{for all } n > 0.$$

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Proof. We choose $C_{\beta,\gamma}$ which makes the following inequality true for all n:

$$\sum_{j=0}^{n-1}(j+1)^{-1/\gamma+1}(n-j)^{-1/\gamma+\beta/\gamma} \le C_{\beta,\gamma}(n+1)^{-1/\gamma+1}. \tag{3.9}$$

Let $K = \max\{\delta_0, \xi/(1-\sigma C_{\beta,\gamma})\}$. Then $\delta_0 \le K$, and if $\delta_j \le K(j+1)^{-1/\gamma+1}$ for all $j < n$, then by equations (3.8) and (3.9),

$$\delta_n \le (\xi + \sigma C_{\beta,\gamma}K)(n+1)^{-1/\gamma+1} \le K(n+1)^{-1/\gamma+1}.$$

It follows by induction that this bound holds for all n. □

PROPOSITION 3.6. *Let $\widetilde{\mathcal{D}}_1^2$ be as in Theorem 2.4. There is $\varepsilon_0 > 0$ and $C > 0$ such that for all $f, g \in \widetilde{\mathcal{D}}_1^2$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$,*

$$\|\mathcal{L}_\varepsilon^n f - \mathcal{L}_\varepsilon^n g\|_{L^1} \le Cn^{-1/\gamma+1}.$$

Proof. Without loss of generality, suppose that $g(x) = (1-\gamma)x^{-\gamma}$. By equation (3.1), $g \in \mathcal{D}_1^2$. Choose $\xi > 0$ large enough so that $(f + \xi g)/(\xi+1) \in \mathcal{D}_1^2$. Such ξ exists by Remark 2.2, equation (3.1) and the definition of $\mathcal{D}_1^2$; it depends only on A, a_1, a_2 and $\tilde{A}, \tilde{a}_1, \tilde{a}_2$.

Denote $f_n = \mathcal{L}_\varepsilon^n f$ and $g_n = \mathcal{L}_\varepsilon^n g$. Write $f_n - g_n = A_n + B_n$, where

$$\begin{aligned} A_n &= (\xi+1)\bigg(\mathcal{L}_{\varepsilon f_{n-1}} \cdots \mathcal{L}_{\varepsilon f_0}\frac{h+\xi g}{\xi+1} - \mathcal{L}_{\varepsilon f_{n-1}} \cdots \mathcal{L}_{\varepsilon f_0} g\bigg), \\ B_n &= \mathcal{L}_{\varepsilon f_{n-1}} \cdots \mathcal{L}_{\varepsilon f_0} g - \mathcal{L}_{\varepsilon g_{n-1}} \cdots \mathcal{L}_{\varepsilon g_0} g \\ &= \sum_{j=0}^{n-1} \mathcal{L}_{\varepsilon f_{n-1}} \cdots \mathcal{L}_{\varepsilon f_{j+1}}(\mathcal{L}_{\varepsilon f_j} - \mathcal{L}_{\varepsilon g_j})\mathcal{L}_{\varepsilon g_{j-1}} \cdots \mathcal{L}_{\varepsilon g_0} g. \end{aligned}$$

By the invariance of $\mathcal{D}_1^2$, the assumption in equation (2.6) and Theorem 3.4,

$$\begin{aligned} \|A_n\|_{L^1} &\le C'(\xi+2)n^{-1/\gamma+1}, \\ \|B_n\|_{L^1} &\le C'|\varepsilon| \sum_{j=0}^{n-1} \|f_j - g_j\|_{L^1}(n-j)^{-1/\gamma+\beta/\gamma}. \end{aligned}$$

Here C' depends only on **C**. Let $\delta_n = \|f_n - g_n\|_{L^1}$. Then,

$$\delta_n \le \|A_n\|_{L^1} + \|B_n\|_{L^1} \le C'n^{-1/\gamma+1} + C'|\varepsilon| \sum_{j=0}^{n-1} \delta_j (n-j)^{-1/\gamma+\beta/\gamma}.$$

By Lemma 3.5, $\delta_n \le \max\{2, C'(1-|\varepsilon|C'C_{\beta,\gamma})^{-1}\}n^{-1/\gamma+1}$ for all $n > 0$, provided that $|\varepsilon|C' < C_{\beta,\gamma}^{-1}$. □

LEMMA 3.7. *Suppose that f is a probability density on* [0, 1]. *For every $\delta > 0$, there exist $n \ge 0$ and $g \in \mathcal{D}_1^r$ such that $\|\mathcal{L}_\varepsilon^n f - g\|_{L^1} \le \delta$.*

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Proof. Denote $f_k = \mathcal{L}_\varepsilon^k f$ and $\mathcal{L}_k = \mathcal{L}_{\varepsilon f_0} \cdots \mathcal{L}_{\varepsilon f_{k-1}}$. Let $\tilde{f}$ be a C^∞ probability density with $\|f - \tilde{f}\|_{L^1} \le \delta/2$. It exists because C^∞ is dense in L^1. Then for all k,

$$\|\mathcal{L}_k f - \mathcal{L}_k \tilde{f}\|_{L^1} \le \|f - \tilde{f}\|_{L^1} \le \delta/2.$$

Choose $C \ge 0$ large enough so that $(\tilde{f} + C)/(C+1) \in \mathcal{D}_1^r$. Write

$$\mathcal{L}_k \tilde{f} - \mathcal{L}_k 1 = (C+1)\bigg[\mathcal{L}_k\bigg(\frac{\tilde{f}+C}{C+1}\bigg) - \mathcal{L}_k 1\bigg].$$

By Proposition 3.6, the right-hand side above converges to 0, in particular, $\|\mathcal{L}_n \tilde{f} - \mathcal{L}_n 1\|_{L^1} \le \delta/2$ for some n.

Take $g = \mathcal{L}_n 1$. Then $g \in \mathcal{D}_1^r$ by the invariance of $\mathcal{D}_1^r$, and $\|\mathcal{L}_\varepsilon^n f - g\| \le \delta$ by construction. □

PROPOSITION 3.8. *Suppose that f is a probability density on* $[0, 1]$. *There is* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$,

$$\lim_{n\to\infty} \|\mathcal{L}_\varepsilon^n f - h_\varepsilon\|_{L^1} = 0.$$

Proof. Choose a small $\delta > 0$. Without loss of generality, suppose that $\|f - \tilde{f}\|_{L^1} \le \delta$ with $\tilde{f} \in \mathcal{D}_1^1$. (The general case is recovered using Lemma 3.7 and replacing f with $\mathcal{L}_\varepsilon^n f$ with sufficiently large n.)

As in the proof of Proposition 3.6, denote $f_n = \mathcal{L}_\varepsilon^n f$ and $\tilde{f}_n = \mathcal{L}_\varepsilon^n \tilde{f}$, and write $f_n - \tilde{f}_n = A_n + B_n$, where

$$\begin{aligned}
A_n &= \mathcal{L}_{\varepsilon f_{n-1}} \cdots \mathcal{L}_{\varepsilon f_0} f - \mathcal{L}_{\varepsilon f_{n-1}} \cdots \mathcal{L}_{\varepsilon f_0} \tilde{f},\\
B_n &= \mathcal{L}_{\varepsilon f_{n-1}} \cdots \mathcal{L}_{\varepsilon f_0} \tilde{f} - \mathcal{L}_{\varepsilon \tilde{f}_{n-1}} \cdots \mathcal{L}_{\varepsilon \tilde{f}_0} \tilde{f}\\
&= \sum_{j=0}^{n-1} \mathcal{L}_{\varepsilon f_{n-1}} \cdots \mathcal{L}_{\varepsilon f_{j+1}} (\mathcal{L}_{\varepsilon f_j} - \mathcal{L}_{\varepsilon \tilde{f}_j}) \mathcal{L}_{\varepsilon \tilde{f}_{j-1}} \cdots \mathcal{L}_{\varepsilon \tilde{f}_0} \tilde{f}.
\end{aligned}$$

Since all $\mathcal{L}_{\varepsilon f_j}$ are contractions in L^1,

$$\|A_n\|_{L^1} \le \|f - \tilde{f}\|_{L^1} \le \delta. \tag{3.10}$$

By equation (2.6) and Theorem 3.4,

$$\begin{aligned}
\|B_n\|_{L^1} &\le C'|\varepsilon| \sum_{j=0}^{n-1} \|f_j - \tilde{f}_j\|_{L^1} (n-j)^{-1/\gamma + \beta/\gamma}\\
&\le C''|\varepsilon| \max_{j<n} \|f_j - \tilde{f}_j\|_{L^1},
\end{aligned} \tag{3.11}$$

where C' depends only on $\mathbf{C}$ and $C'' = C' \sum_{j=1}^\infty j^{-1/\gamma + \beta/\gamma}$; recall that $-1/\gamma + \beta/\gamma < -1$, so this sum is finite.

From equations (3.10) and (3.11),

$$\|f_n - \tilde{f}_n\|_{L^1} \le \|A_n\|_{L^1} + \|B_n\|_{L^1} \le \delta + C''|\varepsilon| \max_{j<n} \|f_j - \tilde{f}_j\|_{L^1}.$$

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Hence, if ε is sufficiently small so that $C''|\varepsilon| < 1$, then

$$\|f_n - \tilde{f}_n\|_{L^1} \le \delta/(1 - C''|\varepsilon|) \quad \text{for all } n.$$

Since $\delta > 0$ is arbitrary, $\|f_n - \tilde{f}_n\|_{L^1} \to 0$ as $n \to \infty$. □

4. *Example: verification of assumptions*

Here we verify that the example in equation (1.1) fits the assumptions of §2, namely assumptions (a), (b), (c), (d) and equation (2.6). The key statements are Proposition 4.1 and Corollary 4.3.

Let $\varepsilon_* > 0$, and denote $\gamma_- = \gamma_* - 2\varepsilon_*$ and $\gamma_+ = \gamma_* + 2\varepsilon_*$, so that for all ε, h,

$$\gamma_- - \varepsilon_* < \gamma_* + \varepsilon\gamma_h < \gamma_+ + \varepsilon_*.$$

Force ε_* to be small so that $0 < \gamma_- < \gamma_+ < 1$. Let $\gamma = \gamma_+$ and fix $r \ge 2$.

In Proposition 4.1, we verify assumptions (a), (b), (c) and (d) from §2, and in Corollary 4.3, we verify equation (2.6).

In this section, we use the notation $A \lesssim B$ for $A \le CB$ with C depending only on ε_*, and $A \sim B$ for $A \lesssim B \lesssim A$.

PROPOSITION 4.1. *The family of maps $T_{\varepsilon h}$ satisfies assumptions (a), (b), (c) and (d) from §2.*

Proof. It is immediate that assumptions (a), (b) and (d) hold, so we only need to justify assumption (c). Denote $\tilde{\gamma} = \gamma_* + \varepsilon\gamma_h$ and observe that, with w and each $w_{\ell,j}$ as in equation (2.2),

$$\frac{1}{x^\ell \circ T_{\varepsilon h}} - \frac{w^\ell}{x^\ell} \sim x^{\tilde{\gamma}-\ell} \quad \text{and} \quad \frac{|w_{\ell,j}|}{x^j} \lesssim x^{\tilde{\gamma}-\ell}.$$

The implied constants depend on ℓ and j but not on ε or h, and assumption (c) follows. □

It remains to verify equation (2.6). The precise expressions for γ_h and φ_h are not too important, so we rely on the following properties.

- $\varphi_h(0) = \varphi_h'(0) = \varphi_h(1) = 0$ for each h, so that, informally, φ_h has no effect on the indifferent fixed point at 0.
- The maps $h \mapsto \varphi_h$, $L^1 \to C^3$, $h \mapsto \varphi_h'$, $L^1 \mapsto C^2$, and $h \mapsto \gamma_h$, $L^1 \to \mathbb{R}$ are continuously Fréchet differentiable, that is, for each h, f,

$$\begin{aligned} \|\varphi_{h+f} - \varphi_h - \Phi_h f\|_{C^3} &= o(\|f\|_{L^1}), \\ \|\varphi_{h+f}' - \varphi_h' - \Phi_h' f\|_{C^2} &= o(\|f\|_{L^1}), \\ |\gamma_{h+f} - \gamma_h - \Gamma_h f| &= o(\|f\|_{L^1}), \end{aligned}$$

  where $\Phi_h \colon L^1 \to C^3$, $\Phi_h' \colon L^1 \to C^2$ and $\Gamma \colon L^1 \to \mathbb{R}$ are bounded linear operators, continuously depending on h.

Suppose that $f_0, f_1 \in L^1$ and $v \in \mathcal{D}_1^2$. Let $f_s = (1-s)f_0 + sf_1$ with $s \in [0, 1]$. Denote $T_s = T_{\varepsilon f_s}$ and let $\mathcal{L}_s$ be the associated transfer operator.

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PROPOSITION 4.2. $|\partial_s(\mathcal{L}_s v)| \lesssim |\varepsilon| x^{-(\gamma_+ - \gamma_-)}$ *and* $|(\partial_s(\mathcal{L}_s v))'(x)| \lesssim |\varepsilon| x^{-(\gamma_+ - \gamma_-)-1}$.

Proof. We abuse notation, restricting to a single branch of T_s, so that T_s is invertible and $\mathcal{L}_s v = (v/T_s') \circ T_s^{-1}$. Let $\zeta_s = \Phi_{f_s}(f_1 - f_0)$, $\psi_s = \Phi'_{f_s}(f_1 - f_0)$ and $\lambda_s = \Gamma_{f_s}(f_1 - f_0)$. Then,

$$\partial_s(\mathcal{L}_s v) = \left[\frac{(v' T_s' - v T_s'')\partial_s T_s}{T_s'^3} + \frac{v \partial_s T_s'}{T_s'^2}\right] \circ T_s^{-1}$$

with

$$(\partial_s T_s)(x) = \varepsilon \lambda_s x^{1+\gamma+\varepsilon\nu_{f_s}} \log x + \varepsilon \zeta_s(x),$$
$$(\partial_s T_s')(x) = \varepsilon \lambda_s x^{\gamma+\varepsilon\nu_{f_s}} [1 + (1 + \gamma + \varepsilon \nu_{f_s}) \log x] + \varepsilon \psi_s(x).$$

Observe that $|v(x)| \lesssim x^{-\gamma_+}$, $|v'(x)| \lesssim x^{-\gamma_+ - 1}$, $|\partial_s T_s| \lesssim |\varepsilon| x^{1+\gamma_-}$, $|\partial_s T_s'| \lesssim |\varepsilon| x^{\gamma_-}$, $T_s'(x) \sim 1$ and $|T_s''(x)| \lesssim x^{\gamma_- - 1}$. Hence,

$$|\partial_s(\mathcal{L}_s v)(x)| \lesssim |\varepsilon| x^{-(\gamma_+ - \gamma_-)}.$$

Differentiating in x further and observing that $|v''(x)| \lesssim x^{-\gamma_+ - 2}$, $|T_s'''(x)| \lesssim x^{\gamma_- - 2}$, $|(\partial_s T_s)'(x)| \lesssim |\varepsilon| x^{\gamma_-}$ and $|(\partial_s T_s')'(x)| \lesssim |\varepsilon| x^{\gamma_- - 1}$, we obtain

$$|(\partial_s(\mathcal{L}_s v))'(x)| \lesssim |\varepsilon| x^{-(\gamma_+ - \gamma_-)-1}.$$ □

COROLLARY 4.3. *In the setup of Proposition 4.2, we can represent*

$$\mathcal{L}_{\varepsilon f_0} v - \mathcal{L}_{\varepsilon f_1} v = \delta(g_0 - g_1),$$

where $\delta \lesssim |\varepsilon| \|f_0 - f_1\|_{L^1}$, *and* $g_0, g_1 \in \mathcal{D}_1^1$ *with* $g_0(x), g_1(x) \lesssim x^{-4\varepsilon}$.

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