THE STRUCTURE OF A SPECIAL CLASS OF NEAR-RINGS

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1. Introduction

It is well known that a Boolean ring is isomorphic to a subdirect sum of two-element fields. In [3] a near-ring \((B, +, \cdot)\) is said to be Boolean if there exists a Boolean ring \((B, +, \wedge, 1)\) with identity such that \(\cdot\) is defined in terms of \(+, \wedge,\) and 1 and, for any \(b \in B\), \(b \cdot b = b\). A Boolean near-ring \(B\) is called special if \(a \cdot b = (a \lor x) \wedge b\), where \(x\) is a fixed element of \(B\). It was pointed out that a special Boolean near-ring is a ring if and only if \(x = 0\). Furthermore, a special Boolean near-ring does not have a right identity unless \(x = 0\). It is natural to ask then whether any Boolean near-ring (which is not a ring) can have a right identity. Also, how are the subdirect structures of a special Boolean near-ring compared to those of a Boolean ring. It is the purpose of this paper to give a negative answer to the first question and to show that the subdirect structures of a special Boolean near-ring are very 'close' to those of a Boolean ring. In fact, we will investigate a class of near-rings that include the special Boolean near-rings and the Boolean semi-rings as defined in [8].

2. Preliminaries

A \((left)\) near-ring is an algebraic system \((R, +, \cdot)\) such that

(i) \((R, +)\) is a group,
(ii) \((R, \cdot)\) is a semigroup,
(iii) \(x(y + z) = xy + xz\) for all \(x, y, z \in R\).

In particular, if \(R\) contains a multiplicative semigroup \(S\) whose elements generate \((R, +)\) and satisfy.

(iv) \((x + y)s = xs + ys\), for all \(x, y \in R\) and \(s \in S\), we say that \(R\) is a distributively generated \((d.g.)\) near-ring.

The most natural example of a near-ring is given by the set \(R\) of mappings of an additive group (not necessary abelian) into itself. If the mappings are added by adding images and multiplication is iteration, then the system \((R, +, \cdot)\) is a
near-ring. If $S$ is a multiplicative semigroup of endomorphisms of $R$ and $R'$ is the subnear-ring generated by $S$, then $R'$ is a d.g. near-ring. Other examples of d.g. near-rings may be found in [5].

An element $r$ of $R$ is right (anti-right) distributive if

$$(b + c)r = br + cr \quad ((b + c)r = cr + br)$$

for all $b, c \in R$. It follows at once that an element $r$ is right distributive if and only if $(-r)$ is anti-right distributive. In particular, any element of a d.g. near-ring is a finite sum of right and anti-right distributive elements.

The kernels of near-ring homomorphisms are called ideals. Blackett [2] showed that $K$ is an ideal of a near-ring $R$ if and only if $K$ is a normal subgroup of $(R, +)$ that satisfied

(i) $RK \subseteq K$ and
(ii) $(m+k)n - mn \in K$, for all $m, n \in R$ and $k \in K$.

### 3. Subdirect sums of near-rings

The theory of subdirect sum representation for rings carries over almost word for word to near-rings [4]. A nonzero near-ring $R$ is subdirectly irreducible if and only if the intersection of all the nonzero ideals of $R$ is nonzero. The near-ring analogue of Birkhoff’s [1] fundamental result for rings can be stated as follows.

**Theorem 3.1.** [4] Every near-ring $R$ is isomorphic to a subdirect sum of subdirectly irreducible near-rings.

For a more detailed discussion of subdirect sums of near-rings, see [4]. By using the technique of subdirect sum representation, it was shown in [6] that every d.g. near-ring $R$ with the property that $x^2 = x$ for all $x$ in $R$ is a Boolean ring.

### 4. $\beta$-near-rings

**Definition 4.1.** A near-ring $R$ is called a $\beta$-near-ring if for each $x$ in $R$, $x^2 = x$ and $xyz = yxz$ for all $x, y, z \in R$.

**Example 4.2.** Let $(R, +)$ be a nontrivial group. Define multiplication by $a \cdot b = b$ for all $a, b \in R$. Then $(R, +, \cdot)$ is a $\beta$-near-ring for which $\cdot$ is not commutative and $(R, +)$ need not be of characteristic two.

**Example 4.3.** The Boolean semirings as defined in [8] are $\beta$-near-rings for which addition is commutative.

**Example 4.4.** The special Boolean near-rings as defined in [3] are $\beta$-near-rings.
Structure of near-rings

It is easily seen that if a $\beta$-near-ring $R$ has a right identity, then $R$ is a Boolean ring. In fact, we have the following much stronger result.

**Theorem 4.5.** Let $R$ be a near-ring with the property that $x^2 = x$ for all $x$ in $R$ and has a right identity $e$. Then $R$ is a Boolean ring.

**Proof.** Since $e$ is right distributive, the equation

$$(e + e)^2 = e + e$$

tells that $e + e = 0$. If $x$ is in $R$, then

$$x + x = x(e + e) = 0.$$ 

Hence every element of $(R, +)$ is of order two and consequently $(R, +)$ is commutative.

Let $w$ be an arbitrary element in $R$. Then

$$(e + w)^2 = e + w$$

yields that

$$(e + w)e + (e + w)w = e + w.$$ 

It follows that $(e + w)w = 0$ for all $w \in R$. Moreover, $0w + 0ww = 0$ implies that $0w(e + w) = 0$. Thus $0w(e + w)w = 0w$ implies that $0w0 = 0w$ and hence $0 = 0w$ for all $w \in R$.

To complete the proof we now show that $(R, \cdot)$ is commutative. This would mean that each element in $R$ is right distributive and hence $R$ is a (commutative) Boolean ring.

Let $a$ and $b$ be arbitrary elements of $R$. Then

$$(ab + ba)(ab + ba) = ab + ba,$$

$$(ab + ba)ab + (ab + ba)ba = ab + ba,$$

$$(ab + ba)ab = (ab + ba) + (ab + ba)ba,$$

$$(ab + ba)ab = (ab + ba)(e + ba).$$

Thus we have that

$$(ab + ba)abba = (ab + ba)(e + ba)ba$$

$$= (ab + ba)0$$

$$= 0.$$ 

It follows that

$$(ab + ba)abab = 0b = 0.$$ 

Similarly, expand $(ab + ba)(ba + ab) = ab + ba$ as above, we obtain that

$$(ab + ba)ba = 0.$$ 

Consequently $ab + ba = 0$. This completes the proof since every element of $(R, +)$ is of order two.

Note that Theorem 4.5 furnishes a negative answer to the first question mentioned in the introduction.
5. Subdirect structure of $\beta$-near-rings

**Definition 5.1.** A near-ring $(\mathcal{R}, +, \cdot)$ is said to be *small* if there is an element $e$ of $\mathcal{R}$ such that $e$ is a left multiplicative identity and for all $x \neq e$ in $\mathcal{R}$, either $x$ is a left identity or else $xy = 0y$ for all $y$ in $\mathcal{R}$.

It is clear that any two-element field is a small near-ring but certainly not conversely. Now we are ready to state our result which compares the subdirect structures of a $\beta$-near-ring to those of a Boolean ring.

**Theorem 5.2.** Every $\beta$-near-ring $\mathcal{R}$ is isomorphic to a subdirect sum of subdirectly irreducible near-rings $\mathcal{R}_i$ where each $\mathcal{R}_i$ is either a two-element field or a small near-ring.

To facilitate the discussion on the proof of Theorem 5.2, we first prove a few lemmas which are of interest in their own right.

**Lemma 5.3.** If $\mathcal{R}$ is a subdirectly irreducible $\beta$-near ring then $\mathcal{R}$ has a left identity.

**Proof.** For each $x$ in $\mathcal{R}$, let

$$(1) \quad A_x = \{ y \in \mathcal{R} : xy = 0 \}.$$  

By straightforward calculations, keeping in mind that $xyz = yxz$ for all $x, y, z \in \mathcal{R}$, one can easily verify that $A_x$ is an ideal of $\mathcal{R}$. If $A_x = 0$, then $x$ is a left identity since $x(xy - y) = 0$ for all $y \in \mathcal{R}$.

Now let

$$(2) \quad N = \{ x \in \mathcal{R} : A_x \neq 0 \}.$$  

Suppose $N = \mathcal{R}$. Let

$$A = \bigcap_{x \in N} A_x.$$  

Then $A$ is not zero since $\mathcal{R}$ is subdirectly irreducible. But if $w \neq 0$ is in $A$, then $w^2 = w = 0$. Thus there exists an element $e$ in $\mathcal{R}$ such that $A_e = 0$ and hence $e$ is a left identity.

**Lemma 5.4.** If $\mathcal{R}$ is a subdirectly irreducible $\beta$-near-ring and if $z \neq 0$ such that $A_z \neq 0$, then $zy = 0y$ for all $y \in \mathcal{R}$.

**Proof.** Since $A_z \neq 0$, it follows that $z \in N$ as defined in (2). Since $A \neq 0$, let $w \neq 0$ be an element in $A$. Thus $xw = 0$ for all $x \in N$. If $wy = 0$ for some $y \neq 0$ in $\mathcal{R}$, then $w \in N$ and $w^2 = w = 0$, which is a contradiction. It follows that $wy \neq 0$ for any $y \neq 0$ in $\mathcal{R}$. This means that $A_w = 0$ and hence $w$ is a left identity. Thus

$$(3) \quad zy = zwy = 0y \text{ for all } y \in \mathcal{R}.$$  

**Lemma 5.5.** If $\mathcal{R}$ is a subdirectly irreducible $\beta$-near-ring with the property that $0y = 0$ for all $y$ in $\mathcal{R}$, then each $x \neq 0$ in $\mathcal{R}$ is a left identity.
PROOF. Let $N$ be the set as defined in (2). Then Lemma 5.4 implies that if there exists an element $z \neq 0$ in $R$ such that $A_z \neq 0$, then $zy = 0y$ for all $y$ in $R$. In particular, $zz = 0z = 0$. This contradiction implies that $N = 0$. Hence each nonzero element in $R$ is a left identity.

LEMMA 5.6. If $R$ is a subdirectly irreducible $\beta$-near-ring with a nonzero right distributive element $r$, then $R$ is the two element field.

PROOF. Since $r$ is right distributive, it follows that $0r = 0$. From Lemma 5.4 and (3) with $z = r$ and $y = r$, we see that $A_r = 0$. Now let $L_r = \{y \in R : yr = 0\}$. Since $r$ is right distributive and $xyz = yxz$ for all $x, y, z \in R$, it is easily verified that $L_r$ is an ideal of $R$. Suppose that $L_r \neq 0$. Let $L = L_r \cap A$, where $A = \bigcap_{x \in N} A_x$. There exists a $w \neq 0$ in $L$ such that $xw = 0$ for each $x \in N$ and $wr = 0$. This is a contradiction since $w$ is a left identity. Thus $L_r = 0$ and we conclude that $r$ is a right identity as well as a left identity. Thus $(R, \cdot)$ is commutative and $0x = 0$ for all $x$ in $R$. By Lemma 5.5 each $x \neq 0$ in $R$ is a left identity and it follows that $x = xr = r$. Consequently $R$ is the two-element field.

We may now complete the proof of Theorem 5.2.

**PROOF OF THEOREM 5.2.** Let $R$ be a $\beta$-near-ring. By Theorem 3.1, $R$ is isomorphic to a subdirect sum of subdirectly irreducible near-rings $R_i$. Now each $R_i$ is a homomorphic image of $R$ and therefore a $\beta$-near-ring. If $R_i$ has a nonzero right distributive element then it is a two-element field by Lemma 5.6. If $R_i$ does not have a nonzero right distributive element, then $R_i$ is a small near-ring by Lemmas 5.3 and 5.4.

Since special Boolean near-rings and Boolean semi-rings as defined in [3] and [8] respectively are $\beta$-near-rings, Theorem 5.2 furnishes the subdirect structures of those near-rings as well.

An immediate corollary of Lemma 5.6 is the following characterization of Boolean rings.

**COROLLARY 5.7.** A near-ring $R$ is a Boolean ring if and only if $R$ is a $\beta$-near-ring and every nonzero homomorphic image of $R$ has a nonzero right distributive element.

Since a homomorphic image of a d.g. near-ring is again a d.g. near-ring [5], we have

**COROLLARY 5.8.** Every d.g. $\beta$-near-ring is a Boolean ring.
6. Remarks

In view of Theorem 4.5, one naturally asks that if $R$ is a near-ring with a right identity, for what positive integers $n$ such that $x^n = x$ for all $x$ in $R$ would imply that $(R, \cdot)$ is commutative. Of course it is well known that if $R$ is a ring, then $(R, \cdot)$ is commutative for all $n$. By a result in [7, Cor. 3.7] an affirmative answer for $n = n_0$ would imply that if $R$ is a near-ring with a right identity and $x^{n_0} = x$ for all $x$ in $R$, then $R$ is a commutative ring with identity. Thus it is of interest to known the answers to the questions just mentioned above.

References


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