Canad. Math. Bull. Vol. **49** (2), 2006 pp. 185–195

On the Inequality for Volume and Minkowskian Thickness

Gennadiy Averkov

Abstract. Given a centrally symmetric convex body B in $\mathbb{E}^d$, we denote by $\mathcal{M}^d(B)$ the Minkowski space (*i.e.,* finite dimensional Banach space) with unit ball B. Let K be an arbitrary convex body in $\mathcal{M}^d(B)$. The relationship between volume $V(K)$ and the Minkowskian thickness ($=$ minimal width) $\Delta_B(K)$ of K can naturally be given by the sharp geometric inequality $V(K) \geq \alpha(B) \cdot \Delta_B(K)^d$, where $\alpha(B) > 0$. As a simple corollary of the Rogers-Shephard inequality we obtain that $\binom{2d}{d}^{-1} \leq \alpha(B)/V(B) \leq 2^{-d}$ with equality on the left attained if and only if B is the difference body of a simplex and on the right if B is a cross-polytope. The main result of this paper is that for $d = 2$ the equality on the right implies that B is a parallelogram. The obtained results yield the sharp upper bound for the modified Banach–Mazur distance to the regular hexagon.

1 Introduction

By $\mathbb{E}^d, d \geq 2$, we denote the d-dimensional Euclidean space with the origin o. The volume in $\mathbb{E}^d$ is denoted by V. The abbreviations bd, int and conv stand for the boundary, interior, and convex hull, respectively. A set $K \subseteq \mathbb{E}^d$ is said to be a *convex body* if it is convex, compact and has non-empty interior, *cf.* [BF74, Sch93]. The class of convex bodies in $\mathbb{E}^d$ is denoted by $\mathcal{K}^d$, while $\mathcal{B}^d$ denotes the class of centrally symmetric convex bodies in $\mathbb{E}^d$ with center at the origin. With $K \in \mathcal{K}^d$ we associate the *support function* $h_K(u) := \max\{\langle x, u\rangle : x \in K\}$ and the *width function* $w_K(u) := h_K(u) + h_K(-u)$. The convex body $DK := \{x - y : x, y \in K\}$ is called the *difference body of* K. It is known that $h_{DK}(u) = w_K(u)$. The difference body of a triangle is said to be an *affine regular hexagon.* The classes of triangles and affine regular hexagons in $\mathbb{E}^2$ are denoted further by $\mathcal{T}$ and $\mathcal{H}$, respectively. If $o \in \operatorname{int} K$, then the convex body $K^* = \{u \in \mathbb{E}^d : h_K(u) \leq 1\}$ is called the *polar body of* K.

A finite dimensional real Banach space is called a *Minkowski space*, *cf.* [Tho96] and the surveys [MSW01, MS03]. If $B \in \mathcal{B}^d$, then by $\mathcal{M}^d(B)$ we denote the Minkowski space with unit ball B. The norm in $\mathcal{M}^d(B)$ is denoted by $\|\cdot\|_B$. Since B is a subset of the Euclidean space $\mathbb{E}^d$, we see that $\mathcal{M}^d(B)$ is equipped with an auxiliary Euclidean structure. Every measure $\alpha \cdot V$ with $\alpha > 0$ could be used as a volume in $\mathcal{M}^d(B)$, see [Tho96, §1.4]. Further on, as a volume in $\mathcal{M}^d(B)$ we will use either the measure V, which is determined from the auxiliary Euclidean structure of $\mathcal{M}^d(B)$, or the *normalized volume* $V_B(\,\cdot\,) := V(\,\cdot\,)/V(B)$, which is independent on the auxiliary Euclidean structure of $\mathcal{M}^d(B)$, *i.e.,* $V_B(K) = V_{B'}(K')$ for $K \in \mathcal{K}^d$, $B \in \mathcal{B}^d$ and $K' := A(K), B' := A(B)$, where A is a non-singular linear transformation in $\mathbb{E}^d$.

Received by the editors March 29, 2004; revised January 14, 2005.
AMS subject classification: 52A40, 46B20.
Keywords: Convex body, geometric inequality, thickness, Minkowski space, Banach space, normed space, reduced body, Banach-Mazur compactum, (modified) Banach-Mazur distance, volume ratio.

https://doi.org/10.4153/CMB-2006-019-4 Published online by Cambridge University Press

For $d = 2$ volume and normalized volume will be called *area* and *normalized area,* respectively.

Given a convex body K in $\mathcal{M}^d(B)$ and a vector u ranging over $\mathbb{E}^d \setminus \{o\}$, we introduce the *Minkowskian width function* $w_{K,B}(u) := w_K(u)/h_B(u)$. In terms of Minkowskian measures $w_{K,B}(u)$ can be given as the minimal Minkowskian distance occurring between points $p_1 \in H_1$ and $p_2 \in H_2$, where H_1 and H_2 are supporting hyperplanes of K with outward Euclidean normals u and $-u$, respectively. The *Minkowskian diameter* $\operatorname{diam}_B(K) := \max\{\|x - y\|_B : x, y \in K\}$ of K is equal to the maximum of $w_{K,B}(u)$, (for the proof, see for instance [Ave03b, Theorem 2]). The minimum of $w_{K,B}(u)$ is called the *Minkowskian thickness* $\Delta_B(K)$ of K, *cf.* [Ave03b]. In terms of the difference body of K, $\Delta_B(K)$ and $\operatorname{diam}_B(K)$ can be given by the equalities

$$\Delta_B(K) = \max\{\alpha > 0 : \alpha \cdot B \subseteq DK\}, \tag{1}$$

$$\operatorname{diam}_B(K) = \min\{\alpha > 0 : DK \subseteq \alpha \cdot B\}, \tag{2}$$

i.e., $\Delta_B(K)$ and $\operatorname{diam}_B(K)$ are Minkowskian in- and circumradii of DK, respectively (*cf.* [Ave03b, Theorems 2 and 3]).

For $K \in \mathcal{K}^d$ and $B \in \mathcal{B}^d$ we introduce the quantities

$$f(K, B) := \frac{V_B(K)}{\Delta_B(K)^d}, \tag{3}$$

$$f(B) := \min\left\{f(K, B) : K \in \mathcal{K}^d\right\}. \tag{4}$$

One can show that the functional $f(B)$ is affine invariant, *i.e.,* $f(B)$ is a well-defined quantitative characteristic of $\mathcal{M}^d(B)$. We see that $f(\alpha K, B) = f(K, B)$ for every $\alpha > 0$. Hence the quantity $f(B)$ can also be treated as the least possible normalized volume of a convex body $K \subseteq \mathcal{M}^d(B)$ of Minkowskian thickness one. On the other hand $f(B)$ is the unique positive value yielding the sharp geometric inequality

$$V_B(K) \geq f(B) \cdot \Delta_B(K)^d. \tag{5}$$

If $d = 2$ and $\mathcal{M}^d(B)$ is the Euclidean plane, then convex bodies yielding equality in (5) are precisely equilateral triangles, which was shown by Pál (see [BF74, Section 44]). Heil [Hei78] asked for the bodies yielding equality in (5) in the case when $\mathcal{M}^d(B)$ is the Euclidean space with $d \geq 3$. In [Hei78] he also constructed a convex body which might yield equality in the above mentioned case.

Now we are ready to give a precise formulation of the announced new results.

Theorem 1 *Let $\mathcal{M}^d(B)$, $d \geq 2$, be an arbitrary Minkowski space. Then*

$$\binom{2d}{d}^{-1} \leq f(B) \leq 2^{-d}. \tag{6}$$

with equality on the left attained if and only if B is the difference body of a simplex and on the right if B is a cross-polytope.

Theorem 2 *Let $B \subseteq \mathbb{E}^2$ be an arbitrary centrally symmetric planar convex body with center at the origin. Then $f(B) = \frac{1}{4}$ (i.e., $f(B)$ is maximal) if and only if B is a parallelogram.*

It should be mentioned that it is an open question whether Theorem 2 can be extended to higher dimensions, *i.e.,* it is unknown whether for $d \geq 3$ the condition $f(B) = 2^{-d}$ implies that B is a cross-polytope.

Further on, let us discuss some application of Theorems 1 and 2. In the literature on convexity and local theory of Banach spaces the class $\mathcal{B}^d$ (as well as $\mathcal{K}^d$) is often endowed with certain affine invariant distance-functions, *cf.* [MS86, TJ89, Pis89], and [Grü63, § 2]. For instance, the well-known *Banach–Mazur* distance $\mathrm{d}_1(B_1, B_2)$ between convex bodies B_1 and B_2 in $\mathcal{B}^d$ is the least possible $\alpha \geq 1$ such that for some linear image B_1' of B_1 we have $B_1' \subseteq B_2 \subseteq \alpha B_1'$, *cf.* [LM93, Section 5] and [Sza91]. Let $\mathcal{B}^d_{\mathrm{aff}}$ be the class obtained from $\mathcal{B}^b$ by identifying every two affinely equivalent bodies. Since the functional d_1 is affine invariant with respect to both its arguments, it can also be considered for the elements of $\mathcal{B}^d_{\mathrm{aff}}$. Then $(\mathcal{B}^d_{\mathrm{aff}}, \ln \mathrm{d}_1)$ is a compact metric space, *cf.* [Mac51]. The above space is usually called the *Banach–Mazur compactum* or *Minkowski compactum* (we notice that in some sources not d_1 but the metric $\ln \mathrm{d}_1$ is called the Banach–Mazur distance). Another distance often introduced on $\mathcal{B}^d_{\mathrm{aff}}$ is the *modified Banach-Mazur distance*, see [Grü63, § 2], [Khr01a, Khr01b]. The *volume ratio* of convex bodies B_1 and B_2 in $\mathcal{B}^d$ is defined by

$$\mathrm{vr}(B_1, B_2) := \left(\frac{V(B_1)}{V(B_2')} \right)^{1/d}, \tag{7}$$

where B_2' is an affine image of B_2 which is contained in B_1 and has maximal volume. Then the modified Banach–Mazur distance between B_1 and B_2 is the quantity $\mathrm{d}_2(B_1, B_2) := \mathrm{vr}(B_1, B_2)\,\mathrm{vr}(B_2, B_1)$ and again $(\mathcal{B}^d_{\mathrm{aff}}, \ln \mathrm{d}_2)$ turns out to be a compact metric space, *cf.* [Lev52]. The distance d_2 is weaker than d_1 in the sense that $\mathrm{d}_2(B_1, B_2) \leq \mathrm{d}_1(B_1, B_2)$ for $B_1, B_2 \in \mathcal{B}^d$.

Using Theorems 1 and 2 we get

Theorem 3 *Let $B \subseteq \mathbb{E}^2$ be a centrally symmetric planar convex body with center at the origin, and H be an affine regular hexagon. Then*

$$\mathrm{vr}(B, H)^2 \leq 4/3, \tag{8}$$

$$\mathrm{vr}(H, B)^2 \leq 3/2, \tag{9}$$

$$\mathrm{d}_2(B, H)^2 \leq 2. \tag{10}$$

Moreover, in each of the above three inequalities the equality is attained if and only if B is a parallelogram.

We notice that Asplund gave estimates analogous to (10) with respect to the standard *Banach–Mazur distance* d_1. More precisely, in [Asp60] he found the maximum of the Banach–Mazur distance to the regular hexagon and to the parallelogram. The diameter of $\mathcal{B}^d_{\mathrm{aff}}$ with respect to $\ln \mathrm{d}_2$ is unknown (even for $d = 2$). The diameter of $\mathcal{B}^d_{\mathrm{aff}}$ with respect to $\ln \mathrm{d}_1$ is known only for the case $d = 2$, *cf.* [Str81] and [Tho96, Problem 2.4.2].

2 Proof of Theorem 1

For a convex body $K \in \mathcal{K}^d$ the volumes of K and DK are related by the inequalities

$$\binom{2d}{d}^{-1} \leq \frac{V(K)}{V(DK)} \leq 2^{-d} \tag{11}$$

with equality on the left if and only if K is a simplex and on the right if and only if K is centrally symmetric, see [Sch93, § 7.3]. The inequality on the left-hand side of (11) is the famous *Rogers–Shephard inequality*.

A convex body $K \subseteq \mathcal{M}^d(B)$ is called *reduced* in $\mathcal{M}^d(B)$ if it does not properly contain a convex body of the same thickness in $\mathcal{M}^d(B)$, (*cf.* the papers [Las90, LasM] contain general results on reduced bodies in Euclidean and Minkowski spaces). One can easily see that the bodies yielding equality in (5) are necessarily reduced in $\mathcal{M}^d(B)$. The following theorem (proved in [LasM]) describes reduced bodies in Minkowski spaces with polytopal balls.

Theorem 4 *Let $B \subseteq \mathbb{E}^d$ be a d-dimensional convex polytope with vertices $\pm b_i$, where $i \in \{1, 2, \dots, n\}$ and $n \geq d$. Then every convex body $K \in \mathcal{K}^d$ which is reduced in $\mathcal{M}^d(B)$ can be represented by*

$$K = \operatorname{conv} \bigcup_{i=1}^{n} (\lambda[-b_i, b_i] + p_i)$$

where $\lambda := \frac{1}{2} \cdot \Delta_B(K)$ and $p_1, \dots, p_n$ are appropriately chosen points in $\mathcal{M}^d(B)$.

Proof of Theorem 1 I. Let us prove (6). We consider an arbitrary $K \in \mathcal{K}^d$ with $\Delta_B(K) = 1$. Then, in view of (1), $B \subseteq DK$, and by (11) we get $V(K)/V(B) \geq \binom{2d}{d}^{-1} V(DK)/V(B) \geq \binom{2d}{d}^{-1}$. Consequently, $f(B) \geq \binom{2d}{d}^{-1}$ with equality if and only if K is a simplex and $B = DK$. The inequality $f(B) \leq 2^{-d}$ follows from the trivial equality $f(B, B) = 2^{-d}$.

II. Now we assume that B is a cross-polytope and show that $f(B) = 2^{-d}$. Let us consider convex polytopes $P \subseteq \mathbb{E}^d$ given by

$$P = \operatorname{conv} \bigcup_{i=1}^{d} \Big(\frac{1}{2}[-u_i, u_i] + p_i \Big), \tag{12}$$

where $u_1, \dots, u_d$ is a fixed basis in $\mathbb{E}^d$ and $p_1, \dots, p_d$ are variable points in $\mathbb{E}^d$. In [McM82] and also in [Mar89] the polytopes P with minimal volume were described. Theorems given in the above mentioned papers imply that the polytope P with $p_1 = \cdots = p_d = o$ (*i.e.,* the cross-polytope) yields the minimal volume. Assume that $\pm u_i$, $i \in \{1, 2, \dots, d\}$, are (all) vertices of B. Then, in view of Theorem 4, any reduced body in $\mathcal{M}^d(B)$ having Minkowskian thickness one can be given by (12) with an appropriate choice of p_i, $i \in \{1, \dots, d\}$. Consequently, the volume of $\frac{1}{2}B$ does not exceed the volume of any reduced body in $\mathcal{M}^2(B)$ having Minkowskian thickness one, which implies that $f(B) = f(\frac{1}{2}B, B) = 2^{-d}$. ■

https://doi.org/10.4153/CMB-2006-019-4 Published online by Cambridge University Press

3 Proofs of Theorems 2 and 3

We start this section with preliminary notes on the geometry of Minkowski planes which we shall need later on in the proofs. The following lemma extends some basic properties of Euclidean one-dimensional cross-section measures in $\mathbb{E}^2$ for Minkowski planes, for the proof see [Ave03b, Theorem 4].

Lemma 5 *Let P be a convex polygon in a Minkowski plane $\mathcal{M}^2(B)$. Then*

(i) *For some vertices v_1 and v_2 of P we have $\|v_1 - v_2\|_B = \operatorname{diam}_B(P)$.*
(ii) *For some Euclidean side normal u of P we have $w_{P,B}(u) = \Delta_B(P)$.*

The *isoperimetrix* $\tilde{B}$ in a Minkowski plane $\mathcal{M}^2(B)$ is the polar body of B rotated by the angle $\frac{\pi}{2}$ about the origin, *cf.* [Tho96, Chapter 4] and [MarS]. Let T be an arbitrary triangle with Euclidean side normals $u_i,\ i \in \{1, 2, 3\}$. Then the quantities $h_i := w_{T,B}(u_i)$ are called the *Minkowskian heights of T*. By $\tilde{a}_i$ we denote the length of the side with Euclidean normal u_i measured in the Minkowski plane $\mathcal{M}^2(\tilde{B})$. The Euclidean formula *area* $= \frac{1}{2}$ *height* $\times$ *base* for the area of a triangle can be extended for Minkowski planes to the formula

$$V(T) = \frac{1}{2} h_i \tilde{a}_i, \tag{13}$$

which is discussed in [Ave03c, § 6] and [Tho96, § 4.6].

From (13) we see that for some $k \in \{1, 2, 3\}$ we have $h_k = \min\{h_i : i = 1, 2, 3\}$ and $\tilde{a}_k = \max\{\tilde{a}_i : i = 1, 2, 3\}$. But in view of Lemma 5, $h_k = \Delta_B(T)$ and $\tilde{a}_k = \operatorname{diam}_{\tilde{B}}(T)$. Thus, (13) implies

$$V(T) = \frac{1}{2}\, \Delta_B(T) \operatorname{diam}_{\tilde{B}}(T). \tag{14}$$

A triangle T is said to be *equilateral* in a Minkowski plane $\mathcal{M}^2(B)$ if all its sides have the same length in $\mathcal{M}^2(B)$. Using the well-known *monotonicity lemma* (*cf.* [MSW01, §3.5] as well as [Ave, Ave03a, AvM04]), it can be easily shown that for every direction u there exists an equilateral triangle in $\mathcal{M}^2(B)$ with a side parallel to u. Obviously, if T is an equilateral triangle in $\mathcal{M}^2(B)$ with sides of Minkowskian length one, then all vertices of the affine regular hexagon DT lie in $\operatorname{bd} B$ and all sides of DT have Minkowskian length one.

The following characterization of Minkowskian reduced triangles follows from (13) and Lemma 5. The proof of Theorem 6 can be found in [Ave03c, Theorem 7] and [CG85, §6] (equivalence (ii) $\Leftrightarrow$ (iii)) .

Theorem 6 *Let T be an arbitrary triangle in a Minkowski palne $\mathcal{M}^2(B)$. Then the following conditions are equivalent:*

(i) *T is reduced in $\mathcal{M}^2(B)$;*
(ii) *T has equal heights in $\mathcal{M}^2(B)$;*
(iii) *T is equilateral in $\mathcal{M}^2(\tilde{B})$.*

https://doi.org/10.4153/CMB-2006-019-4 Published online by Cambridge University Press

In [Ave05] the bodies $K \in \mathcal{K}^d$ yielding equality in (5) were described for the case of an arbitrary B in $\mathcal{B}^2$. In particular, the following result was obtained.

Theorem 7 *Let $\mathcal{M}^2(B)$ be an arbitrary Minkowski plane. Then there exists a Minkowskian reduced triangle T yielding equality in* (5)*, or, in other words, $f(B)$ can be given by*

$$f(B) = \min \left\{ f(T, B) : T \in \mathcal{T},\ T \text{ is reduced in } \mathcal{M}^2(B) \right\}. \tag{15}$$

Now let us examine the inequality for the area and the Minkowskian diameter of a triangle. This inequality will be needed in the proof of Theorem 2. For $B \in \mathcal{B}^2$ and a triangle T in $\mathbb{E}^2$ we introduce the following quantities

$$g(T, B) := \frac{V_B(T)}{\operatorname{diam}_B(T)^2}, \tag{16}$$

$$g(B) := \max \left\{ g(T, B) : T \in \mathcal{T} \right\}. \tag{17}$$

Analogously to the remark on the quantity $f(B)$ we state that $g(B)$ is the largest possible normalized area of a triangle $T \subseteq \mathcal{M}^d(B)$ having Minkowskian diameter one. Further on, $g(B)$ is the coefficient which determines the sharp geometric inequality

$$V_B(T) \le g(B) \cdot \operatorname{diam}_B(T)^2, \tag{18}$$

where T is an arbitrary triangle in $\mathcal{M}^2(B)$. In view of (15), $f(B)$ is the minimum of $f(T, B)$, when T ranges over $\mathcal{T}$, which shows that the quantity $g(B)$ is in some sense dual to $f(B)$. The following theorem presents basic properties of $g(B)$.

Theorem 8 *Let $\mathcal{M}^2(B)$ be an arbitrary Minkowski plane. Then the following statements hold.*

(i) *Triangles T yielding equality in* (18) *are necessarily equilateral in $\mathcal{M}^2(B)$,* i.e.,

$$g(B) = \max \left\{ g(T, B) : T \in \mathcal{T},\ T \text{ is equilateral in } \mathcal{M}^2(B) \right\}. \tag{19}$$

(ii) *We have*

$$\frac{1}{8} \le g(B) \le \frac{1}{6} \tag{20}$$

with equality on the left attained if and only if B is a parallelogram and on the right if and only if B is an affine regular hexagon.

Proof In order to prove part (i) we consider an arbitrary non-equilateral triangle T in $\mathcal{M}^2(B)$ with $\operatorname{diam}_B(T) = 1$ and then we show the existence of a triangle $T' \subseteq \mathcal{M}^2(B)$ with $\operatorname{diam}_B(T') = 1$ and strictly larger area. The latter implies the triangles T which are non-equilateral in $\mathcal{M}^2(B)$ cannot yield equality in (18). Let p_1, p_2, p_3 denote the vertices of T. Since $\operatorname{diam}_B(T) = 1$, in view of Lemma 5(i), at least one side of T has Minkowskian length one.

https://doi.org/10.4153/CMB-2006-019-4 Published online by Cambridge University Press

We start with the case when precisely one side of T, say $[p_2, p_3]$, has Minkowskian length one and the remaining two sides have Minkowskian length strictly smaller than one. Then, due to the continuity of the Minkowskian norm, we have that for those points p_1' in $\mathcal{M}^2(B)$ which are sufficiently close to p_1 the Minkowskian distance from p_1' to p_2 and p_3 is strictly less than one. But then we can take p_1' above such that $\operatorname{conv}\{p_1', p_2, p_3\} \supseteq T$, see Figure 1, and put $T' := \operatorname{conv}\{p_1', p_2, p_3\}$.

Now let us switch to the opposite case, *i.e.*, precisely two sides of T, say $[p_1, p_3]$ and $[p_2, p_3]$, have Minkowskian length one and the remaining side $[p_1, p_2]$ is of Minkowskian length less than one. We may replace T by any translate of T, and thus without loss of generality we restrict our considerations to the case when p_3 coincides with the origin. Then the boundary of the Minkowskian unit ball B contains the vertices p_1 and p_2 of T. Since $\|p_1 - p_2\|_B < 1$, the point $p_2 - p_1$ lies in the interior of B. Hence the line l through $p_2 - p_1$ and p_2 has non-empty intersection with the interior of B, see Figure 2. Consequently, l splits bd B into two boundary arcs, which intersect l precisely by their endpoints.

Let γ be the boundary arc of B with endpoints in l and such that γ and the origin lie on the different sides of l. Now let p_2' be a point on γ which is sufficiently close to p_2 but does not coincide with p_2, see Figure 2. Then $\|p_2' - p_1\|_B < 1$. Clearly, we can define the triangle T' by $T' := \operatorname{conv}\{o, p_1, p_2'\}$. Indeed, the Minkowskian diameter of T' defined like that is equal to one, which is clear from the constructions that we have performed. Since $p_2' \notin l$, the height of T' with respect to the base $[o, p_1]$ is larger than the height of T with respect to the same base, see Figure 2. Hence, T' has larger area than T. This finishes the proof in the second case.

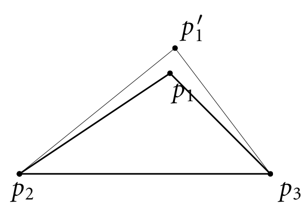

Figure 1

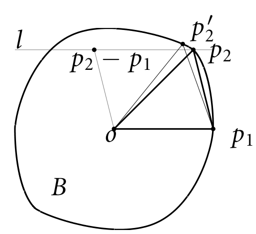

Figure 2

In [WW91] the range of the area of Minkowskian equilateral triangles with Minkowskian side length one was found. In view of (19) one can easily see that (ii) follows directly from the results presented in [WW91]. We remark in addition that the inequality $g(B) \le \frac{1}{6}$ together with the corresponding characterization of the equality case is rather obvious, while the inequality $g(B) \ge \frac{1}{8}$ (without characterization of the equality case) was first presented in [Cha66] (*cf.* also [Tho96, Lemma 4.2.6]). ■

Now we are ready to prove the main theorem of this paper.

Proof of Theorem 2 It is clear that all $K \in \mathcal{K}^2$ with $f(K,B) = f(B)$ are necessarily reduced in $\mathcal{M}^2(B)$. Let $\mathcal{T}_h(B)$ and $\mathcal{T}_s(B)$ denote the classes of triangles having equal Minkowskian heights and equal Minkowskian sides, respectively. In other words, $\mathcal{T}_h(B)$ is precisely the class of reduced triangles in $\mathcal{M}^2(B)$ (see Theorem 6), and $\mathcal{T}_s(B)$ is the class of equilateral triangles in $\mathcal{M}^2(B)$. Using (14) it is easy to verify that

$$4V(B)V(\tilde{B})f(T,B)g(T,\tilde{B}) = 1,$$

or, equivalently,

$$4V(B)V(\tilde{B})f(T,B) = \frac{1}{g(T,\tilde{B})}. \tag{21}$$

Theorem 6 states that $\mathcal{T}_h(B) = \mathcal{T}_s(\tilde{B})$. Taking in (21) the minimum over all triangles T from $\mathcal{T}_h(B) = \mathcal{T}_s(\tilde{B})$ we arrive at

$$4V(B)V(\tilde{B}) \min_{T \in \mathcal{T}_h(B)} f(T,B) = \min_{T \in \mathcal{T}_s(\tilde{B})} \frac{1}{g(T,\tilde{B})}.$$

Hence

$$4V(B)V(\tilde{B}) \min_{T \in \mathcal{T}_h(B)} f(T,B) = \frac{1}{\max_{T \in \mathcal{T}_s(\tilde{B})} g(T,\tilde{B})}.$$

Applying (15) and (19) we then obtain

$$4V(B)V(\tilde{B})f(B)g(\tilde{B}) = 1. \tag{22}$$

The quantity $V(B)V(\tilde{B})$ involved in (22) is known as the *volume-product* of B. Mahler [Mah39] proved that

$$V(B)V(\tilde{B}) \geq 8 \tag{23}$$

with equality if and only if B is a parallelogram, see also [Tho96, pp. 54–55] and [Rei86] for related higher-dimensional results.

The sufficiency need not be proved, since it is already involved in Theorem 1. The necessity holds since for any B which is not a parallelogram we have the sharp inequality $f(B) < \frac{1}{4}$. Indeed, if B is not a parallelogram, then by Theorem 8(ii) we get the sharp inequality $g(B) > \frac{1}{8}$ and by the remark to (23) we get the sharp inequality $V(B)V(\tilde{B}) > 8$. Applying these sharp inequalities together with (22) we easily arrive at the inequality $f(B) < \frac{1}{4}$. ■

Proof of Theorem 3 Let us prove (8). From (7) we obtain that

$$\operatorname{vr}(B,H)^2 = \min\left\{\frac{V(B)}{V(H')} : H' \in \mathcal{H},\ H' \subseteq B\right\}. \tag{24}$$

(We recall that $\mathcal{H}$ denotes the class of affine regular hexagons.) From (2) we can see that H' is an affine regular hexagon contained in B if and only if $H' = DT$, where

https://doi.org/10.4153/CMB-2006-019-4 Published online by Cambridge University Press

T is a triangle with $\mathrm{diam}_B(T) \le 1$. Obviously for the triangle T as above we have $6V(T) = V(H')$. Thus, (24) can be reformulated with the help of the triangles T above as follows.

$$\mathrm{vr}(B,H)^2 = \frac{1}{6}\min\left\{\frac{V(B)}{V(T)} : T \in \mathcal{T}, \mathrm{diam}_B(T) \le 1\right\}.$$

Raising the above equality to the power -1 and slightly transforming the right-hand side we come to the equality

$$\mathrm{vr}(B,H)^{-2} = 6\max\left\{\frac{V(T)}{V(B)} : T \in \mathcal{T},\ \mathrm{diam}_B(T) \le 1\right\}. \tag{25}$$

For $T \in \mathcal{T}$ with $\mathrm{diam}_B(T) < 1$ the ratio $V(T)/V(B)$ increases if we replace T by the triangle $\frac{1}{\mathrm{diam}_B(T)}T$, having Minkowskian diameter one. Consequently, the above maximum can be taken over triangles T with $\mathrm{diam}_B(T) = 1$, which implies that the right-hand side of (25) is equal to $6g(B)$. Thus, applying Theorem 8(ii), we get

$$\mathrm{vr}(B,H)^2 = \frac{1}{6g(B)} \le \frac{4}{3}. \tag{26}$$

Now let us prove (9). From (7) we deduce that

$$\mathrm{vr}(H,B)^2 = \min\left\{\frac{V(H')}{V(B)} : H' \in \mathcal{H},\ H' \supseteq B\right\}. \tag{27}$$

By (1) we see that H' is an affine regular hexagon with $H' \supseteq B$ if and only if H' is the difference body of a triangle T with $\Delta_B(T) \ge 1$. Again, for triangles T as above we have $6V(T) = V(H')$. Now we can reformulate (27) in terms of triangles as follows:

$$\mathrm{vr}(H,B)^2 = 6\min\left\{\frac{V(T)}{V(B)} : T \in \mathcal{T},\ \Delta_B(T) \ge 1\right\}. \tag{28}$$

For triangles T with $\Delta_B(T) > 1$ the ratio $V(T)/V(B)$ decreases if we replace T by the triangle $\frac{1}{\Delta_B(T)}T$, having Minkowskian thickness one. Consequently the above minimum can be taken over $T \in \mathcal{T}$ with $\Delta_B(T) = 1$. Thus, the right-hand side of (28) is equal to $6f(B)$. Therefore, applying (6) for $d = 2$, we get

$$\mathrm{vr}(H,B)^2 = 6f(B) \le \frac{3}{2}. \tag{29}$$

Inequality (10) follows directly from (8) and (9). In view of relations (29) and (26) expressing $\mathrm{vr}(H,B)^2$ and $\mathrm{vr}(B,H)^2$ by $f(B)$ and $g(B)$, respectively, we infer that the characterizations of equality cases in (8)–(10) are direct corollaries of Theorem 2 and Theorem 8(ii). ∎

Acknowledgement The author thanks the referee of this paper for pointing out the paper by E. Asplund [Asp60], which led the author to understanding the relationship of the presented results to distance-functions for classes of convex sets.

References

[Asp60] E. Asplund, *Comparision between plane symmetric convex bodies and parallelograms.* Math. Scand. **8**(1960), 171–180.

[Ave] G. Averkov, *A monotonicity lemma for bodies of constant Minkowskian width*, J. Geom., to appear.

[Ave03a] ———, *Constant Minkowskian width in terms of double normals.* J. Geom. **77**(2003), no. 1-2, 1–7.

[Ave03b] ———, *On cross-section measures in Minkowski spaces.* Extracta Math. **18**(2003), no. 2, 201–208.

[Ave03c] ———, *On the geometry of simplices in Minkowski spaces.* Stud. Univ. Žilina Math. Ser. **16** (2003), no. 1, 1–14.

[Ave05] ———, *On planar convex bodies of given Minkowskian thickness and least possible area.* Arch. Math. (Basel) **84**2005), no. 2, 183–192.

[AvM04] G. Averkov and H. Martini, *A characterization of constant width in Minkowski planes.* Aequationes Math. **68**(2004), 38–45.

[BF74] T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper.* Springer-Verlag, Berlin, 1974, Berichtigter Reprint.

[Cha66] G. D. Chakerian, *Sets of constant width.* Pacific J. Math. **19**(1966), 13–21.

[CG85] G. D. Chakerian and M. A. Ghandehari, *The Fermat problem in Minkowski spaces.* Geom. Dedicata **17**(1985), no. 3, 227–238.

[Grü63] B. Grünbaum, *Measures of symmetry for convex sets.* Proc. Sympos. Pure Math. **7**(1963), pp. 233–270.

[Hei78] E. Heil, *Kleinste konvexe Körper gegebener Dicke.* Preprint No. 453, Fachbereich Mathematik der TH Darmstadt, 1978.

[Khr01a] A. I. Khrabrov, *Distances between spaces with unconditional bases. Function theory and mathematical analysis.* J. Math. Sci. (New York) **107**(2001), no. 3, 3952–3962,

[Khr01b] ———, *Generalized volume ratios and the Banach-Mazur distance.*, transl. Math. Notes **70**(2001), no. 5-6, 838–846.

[Las90] M. Lassak, *Reduced convex bodies in the plane.* Israel J. Math. **70**(1990), no. 3, 365–379.

[LasM] M. Lassak and H. Martini, *Reduced bodies in Minkowski space.* Acta Math. Hungar. **106**(2005), no. 1-2, 17–26.

[Lev52] F. W. Levi, *Über zwei Sätze von Herrn Besicovitch.* Arch. Math. **3**(1952), 125–129.

[LM93] J. Lindenstrauss and V. D. Milman, *The local theory of normed spaces and its applications to convexity.* In: Handbook of Convex Geometry, Vol. A, B, North-Holland, Amsterdam, 1993, pp. 1149–1220.

[Mac51] A. M. Macbeath, *A compactness theorem for affine equivalence-classes of convex regions.* Canad. J. Math. **3**(1951), 54–61.

[Mah39] K. Mahler, *Ein Übertragungsprinzip für konvexe Körper.* Časopis Pěst. Mat. Fys. **68**(1939), 93–102.

[Mar89] H. Martini, *Das Volumen spezieller konvexer Polytope.* Elem. Math. **44**(1989), no. 5, 113–115.

[MarS] H. Martini and K. J. Swanepoel, *Antinorms and radon curves.* submitted.

[McM82] P. McMullen, *The volume of certain convex sets.* Math. Proc. Cambridge Philos. Soc. **91**(1982), no. 1, 91–97.

[MS86] V. D. Milman and G. Schechtman, *Asymptotic theory of finite-dimensional normed spaces.* With an appendix by M. Gromov. Lecture Notes in Mathematics 1200, Springer-Verlag, Berlin, 1986.

[MS03] H. Martini and K. J. Swanepoel, *The geometry of Minkowski spaces — a survey. II.* Expo. Math. **22**(2004), no. 2, 93–144.

[MSW01] H. Martini, K. J. Swanepoel, and G. Weiß, *The geometry of Minkowski spaces — a survey. I.* Expo. Math. **19**(2001), no. 2, 97–142.

[Pis89] G. Pisier, *The volume of convex bodies and Banach space geometry.* Cambridge Tracts in Mathematics 94, Cambridge University Press, Cambridge, 1989.

[Rei86] S. Reisner, *Zonoids with minimal volume-product.* Math. Z. **192**(1986), no. 3, 339–346.

[Sch93] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory.* Encyclopedia of Mathematics and its Applications, 44, Cambridge University Press, Cambridge, 1993.

[Str81] W. Stromquist, *The maximum distance between two-dimensional Banach spaces.* Math. Scand. **48**(1981), no. 2, 205–225.

[Sza91] S. J. Szarek, *On the geometry of the Banach-Mazur compactum.* In: Functional Analysis. Lecture Notes in Math. 1470, Springer, Berlin, 1991, pp. 48–59.

[Tho96] A. C. Thompson, *Minkowski Geometry.* Encyclopedia of Mathematics and its Applications 63, Cambridge University Press, Cambridge, 1996.

[TJ89] N. Tomczak-Jaegermann, *Banach-Mazur distances and finite-dimensional operator ideals.* Pitman Monographs and Surveys in Pure and Applied Mathematics 38, Longman Scientific & Technical, Harlow, 1989.

[WW91] M. Wellmann and B. Wernicke, *Flächeninhalte gleichseitiger Dreiecke in einer Banach-Minkowskischen Ebene.* Wiss. Z. Pädagog. Hochsch. Erfurt/Mühlhausen Math.-Natur. Reihe **27**(1991), no. 1, 21–28.

Fakultät für Mathematik
Technische Universität Chemnitz
D-09107 Chemnitz
Deutschland
e-mail: g.averkov@mathematik.tu-chemnitz.de