RINGS ALL OF WHOSE TORSION QUASI-INJECTIVE MODULES ARE INJECTIVE

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1. Introduction and background. Throughout this paper it is assumed that rings are associative, have the identity element, and all modules are left unital. \( R \) will denote a ring with identity, \( R\)-Mod the category of left \( R\)-modules, and for each left \( R\)-module \( M \), \( E(M) \) (resp. \( J(M) \)) will represent the injective hull (resp. Jacobson radical) of \( M \). Also, for a module \( M \), \( A \subseteq M \) will mean that \( A \) is an essential submodule of \( M \), and \( Z(M) \) denotes the singular submodule of \( M \). \( M \) is called singular if \( Z(M) = M \), and it is called non-singular in case \( Z(M) = 0 \). For fundamental definitions and results related to torsion theories, we refer to [12] and [14]. In this paper we shall deal mainly with Goldie torsion theory. Recall that a pair \( (G, F) \) of classes of left \( R\)-modules is known as Goldie torsion theory if \( G \) is the smallest torsion class containing all modules \( B/A \), where \( A \subseteq B \), and the torsion free class \( F \) is precisely the class of non-singular modules.

A ring \( R \) is called a left V-ring if each simple left \( R\)-module is injective. \( R \) is a left V-ring if and only if each left ideal of \( R \) is the intersection of maximal left ideals (see [13, Theorem 2.1]). In the commutative case, it is well known that \( R \) is a V-ring if and only if \( R \) is regular (in the sense of von Neumann). A module \( M \) is quasi-injective if every homomorphism from a submodule of \( M \) into \( M \) can be lifted to an endomorphism of \( M \). A ring \( R \) is called a left QI-ring if each quasi-injective left \( R\)-module is injective. These rings were originally introduced in [1], and later studied by many authors (see, for example, [2, 3, 4, 6, 7, 8, 10]). Left QI-rings are left Noetherian and left V-rings (see [1]). Thus commutative QI-rings are semisimple Artinian. The goal of this paper is to study rings all of whose (Goldie) torsion quasi-injective modules are injective. The results developed in the next section indicate that these rings provide an effective torsion theoretical generalization of left QI-rings.

2. Results. Let \( (T, F) \) be a hereditary torsion theory, and let \( \mathcal{F}(T) \) be the associated filter of left ideals of \( R \). A left ideal of \( R \) which is a member of \( \mathcal{F}(T) \) will be called an \( \mathcal{F}\)-ideal. Relative to \( (T, F) \), a left \( R\)-module \( E \) is called \( T \)-injective if \( \text{Ext}_R(R/I, E) = 0 \) for all \( I \in \mathcal{F}(T) \). This definition is equivalent to the following statement: For any \( I \in \mathcal{F}(T) \), each homomorphism \( f : I \rightarrow E \) can be extended to a homomorphism \( g : R \rightarrow E \). The following result concerning \( T \)-injective modules can be found in Golan and Teply [11, Lemma 2, p. 252].

**Lemma 1.** Let \( (T, F) \) be a hereditary torsion theory for \( R\)-Mod. Then the following are equivalent:

1. \( R \) has ACC on \( \mathcal{F}\)-ideals;
2. any direct sum of (countably many) \( T \)-injective torsion modules is \( T \)-injective.

We now consider the Goldie torsion theory \((G, F)\) for \(R\)-Mod, whose associated filter of left ideals is denoted by \(\mathcal{F}(G)\). Since all essential left ideals of \(R\) belong to \(\mathcal{F}(G)\), it follows that, relative to the Goldie torsion theory, a left \(R\)-module is \(T\)-injective if and only if it is injective. Thus the following proposition is immediate in view of the above lemma.

**Proposition 1.** Let \((G, F)\) be the Goldie torsion theory for \(R\)-Mod. Then \(R\) has ACC on \(\mathcal{F}\)-ideals if and only if each direct sum of torsion injective modules is injective.

We now phrase a definition for the sake of brevity.

**Definition.** A ring \(R\) will be called a left TQI-ring if each torsion quasi-injective left \(R\)-module is injective.

Thus every left QI-ring is left TQI. The converse, however, is not true in general (see Example 1 below). We now prove some lemmas concerning left TQI-rings. Throughout the remainder of this paper, it will be assumed that we are working in the context of Goldie torsion theory.

**Lemma 2.** Let \(R\) be a left TQI-ring. Then for each torsion left \(R\)-module \(M\), \(J(M) = 0\).

**Proof.** Let \(M\) be a torsion left \(R\)-module. Let \(x \in M\), \(x \neq 0\). Then, by Zorn's lemma, there is a submodule \(Y\) of \(M\) which is maximal among the submodules \(X\) of \(M\) with \(x \notin X\). Let \(D = Rx + Y\). Then \(x \in D\), and \(D/Y \neq (0)\). Also, \(D/Y\) is simple and torsion. Hence \(D/Y\) is injective. Therefore, \(M/Y = D/Y \oplus K/Y\), where \(K\) is a submodule of \(M\). Since \(x \notin K\), \(M/Y = D/Y\). Hence \(Y\) is a maximal submodule of \(M\). Since \(x \notin Y\), \(J(M) = (0)\).

**Corollary 1.** Let \(I\) be an \(F\)-ideal of \(R\). Then \(I\) is an intersection of maximal left ideals.

**Proof.** Since \(I\) is an \(F\)-ideal, \(R/I\) is a torsion left \(R\)-module. Therefore \(J(R/I) = (0)\). Hence \(I\) is an intersection of maximal left ideals.

**Corollary 2.** If each \(F\)-ideal is the intersection of maximal left ideals then each simple torsion left \(R\)-module is injective.

**Proof.** Let \(S\) be a simple torsion left \(R\)-module, and let \(I\) be an \(F\)-ideal. Suppose \(f \in \text{Hom}_R(I, S)\). We claim that \(f\) is extendable to an element of \(\text{Hom}_R(R, S)\). Let \(\text{Ker} f = K\). Consider the sequence: \(0 \to I/K \to R/K \to R/I \to 0\). Since \(I/K\) and \(R/I\) are torsion, \(R/K\) is torsion. Thus both \(K\) and \(I\) are \(F\)-ideals and \(I \supseteq K\). If \(I = K\) then we are done, otherwise there is a maximal left ideal \(L\) of \(R\) such that \(L\) contains \(K\), but does not contain \(I\). Thus \(L + I = R\) and \(L \cap I = K\). Hence \(R/K = L/K \oplus I/K\). Thus the above sequence splits and it can be shown that \(f\) is extendable to a map from \(R\) to \(S\). Hence \(S\) is \(T\)-injective, and so it is injective, as we are working in the Goldie torsion theory.

The proofs of Lemma 2, Corollaries 1 and 2 are adaptations from [13, Theorem 2.1].

**Lemma 3.** Let \(R\) be a left TQI-ring. Then \(R\) has ACC on \(F\)-ideals.

**Proof.** Let \(I_1 \subset I_2 \subset \ldots \subset I_n \subset \ldots\) be an ascending chain of distinct \(F\)-ideals of \(R\).
Then by Corollary 1 to Lemma 2, there are maximal left ideals \( M_k, k = 1, 2, \ldots, \) such that \( I_k \subset M_k \) but \( I_{k+1} \notin M_k \). Let \( \pi_k : R \rightarrow R/M_k \) be the natural projection. Also, let \( I = \bigcup_{k=1}^{\infty} I_k \), and define \( f : I \rightarrow \bigoplus_{k=1}^{\infty} R/M_k \) by \( f(x) = \sum_{k=1}^{\infty} \pi_k(x) \). Note that \( x \) is an element of only a finite number of the \( M_k \), for all \( x \in I \), and \( f \) is an epimorphism. Since \( \bigoplus_{k=1}^{\infty} R/M_k \) is semisimple and torsion, it is torsion quasi-injective and hence injective, as \( R \) is a left TQI-ring. Hence \( f \) extends to \( g : R \rightarrow \bigoplus_{k=1}^{\infty} R/M_k \). Since \( R \) has an identity, \( g(R) \) and hence \( g(I) \) is contained in \( \bigoplus_{k=1}^{n} R/M_k \), for some positive integer \( n \). This implies that the above chain of left \( \mathcal{F} \)-ideals is finite.

We now prove the following.

**Theorem 1.** Let \((G, F)\) be the Goldie torsion theory for \( R\text{-Mod} \). Then the following are equivalent:

1. \( R \) is left TQI;
2. each direct sum of torsion quasi-injective modules is quasi-injective.

**Proof.** Suppose \( R \) is a left TQI-ring. Let \( M = \bigoplus_{i} M_i \) be a direct sum of torsion quasi-injective modules \( M_i \). Then each \( M_i \) is injective by the hypothesis. Hence \( M \) is injective by Proposition 1. Now, assume (2). Let \( M \) be a torsion quasi-injective module. Then \( M \oplus E(M) \) is quasi-injective, by the assumption. From this it follows that \( M = E(M) \) and so \( M \) is injective.

According to a well-known result, every injective module over a left Noetherian ring is the direct sum of indecomposable injective modules. A similar decomposition property is also true for quasi-injective modules over left Noetherian rings. The next two lemmas proved below provide analogues of these results for torsion injective and torsion quasi-injective modules over rings whose \( \mathcal{F} \)-ideals satisfy the ascending chain condition.

**Lemma 4.** Let \( R \) be a ring whose \( \mathcal{F} \)-ideals have ACC. Then every torsion injective left \( R \)-module is the direct sum of indecomposable injective modules.

**Proof.** Let \( M \) be a torsion injective left \( R \)-module. Let \( x \in M, x \neq 0 \). Then \( Rx \cong R/I \), for some left ideal \( I \) of \( R \). Since \( Rx \) is torsion, \( I \) is an \( \mathcal{F} \)-ideal. Furthermore, since \( R \) has ACC on \( \mathcal{F} \)-ideals, it is easy to see that \( Rx \) is a Noetherian left \( R \)-module. Hence \( Rx \) contains a uniform submodule \( U \). Since \( E(U) \subseteq M, M \) has a torsion injective submodule which is the injective hull of its every non-zero submodule. Let \( \{ M_j \}_{i \in J} \) be a maximal independent family of submodules of \( M \) such that each \( M_j \) is the injective hull of all of its non-zero submodules, where \( J \) is an indexing set. Now let \( y \in M \) be a non-zero element. Then, as above, \( E(Ry) \) contains a submodule which is the injective hull of all of its non-zero submodules, and hence \( \bigoplus_{j \in J} M_j \cap Ry \neq (0) \). This implies that \( \bigoplus_{j \in J} M_j \subseteq M \). On the other hand, since each \( M_j \) is torsion injective and also since \( R \) has ACC on \( \mathcal{F} \)-ideals, it...
follows from Proposition 1 that $\bigoplus M_i$ is injective. Hence $\bigoplus M_i$ is a direct summand of $M$. But $\bigoplus M_i \subseteq M$. Therefore, $\bigoplus M_i = M$. Further, since the endomorphism ring of each indecomposable injective module is local, it follows from the Krull–Remak–Schmidt theorem that this decomposition is unique (up to isomorphism).

**Lemma 5.** Let $R$ be a ring whose $\mathcal{F}$-ideals have ACC. Then each torsion quasi-injective left $R$-module is the direct sum of indecomposable quasi-injective modules.

**Proof.** Let $M$ be a torsion quasi-injective left $R$-module. Then $E(M)$ is torsion, as we are dealing with Goldie torsion theory. Hence, by Lemma 4, $E(M) = \bigoplus E_i$, where each $E_i$ is an indecomposable injective module. Let $p_i : E(M) \to E_i$ be the projection map. Then since $M$ is quasi-injective, $p_i(M) \subseteq M$. Now, let $x \in M$, say $x = \sum x_i$, $x_i \in E_i$; then $x_i \in M$. Then $M = \sum M \cap E_i$, i.e. $M = \bigoplus (M \cap E_i)$. Hence $M$ is the direct sum of indecomposable quasi-injective modules.

Before we state the next theorem, we note that we shall use the notation $S(M)$ for the socle of a left $R$-module $M$. In particular, $S(R)$ will denote the socle of $R R$.

**Theorem 2.** Let $R$ be a ring with $S(R) \subseteq R$, and $(G, F)$ be the Goldie torsion theory for $R$-Mod. Then the following are equivalent:

1. $R$ is a left TQI-ring;
2. $R$ has ACC on $\mathcal{F}$-ideals, each $\mathcal{F}$-ideal is the intersection of maximal left ideals, and $R/S(R)$ is a left QI-ring.

**Proof.** (1) Let us assume that $R$ is a left TQI-ring. Then, by Lemma 3, $R$ has ACC on $\mathcal{F}$-ideals, and each $\mathcal{F}$-ideal is the intersection of maximal left ideals by Corollary 1 to Lemma 2. Furthermore, since $S(R) \subseteq R$ by the hypothesis, $S(R)$ is an $\mathcal{F}$-ideal. Hence $R/S(R)$ is semiprime by Lemma 2. Also, it follows immediately from Lemma 3, that $R/S(R)$ is Noetherian. Moreover, since $R$ is a left TQI-ring, it is straightforward to argue that $R/S(R)$ is also a left TQI-ring. Now it is a known result that every semiprime left Noetherian left TQI-ring is a left QI-ring (see [2, Corollary 9, p. 48]). Hence $R/S(R)$ is a left QI-ring.

(2) Conversely, we now assume that $R$ has ACC on $\mathcal{F}$-ideals, each $\mathcal{F}$-ideal is the intersection of maximal left ideals, and $R/S(R)$ is a left QI-ring. We prove that $R$ is a left TQI-ring. So, let $M$ be a torsion quasi-injective left $R$-module. Then, by Lemma 5, $M = \bigoplus M_\alpha$, where each $M_\alpha$ is torsion, indecomposable and quasi-injective. Consider an arbitrary but fixed $M_\alpha$. If $S(M_\alpha) \neq 0$ then there exists a simple (torsion) submodule $S$ ($\neq 0$) of $M$. Since each $\mathcal{F}$-ideal is the intersection of maximal left ideals, it follows from Corollary 2 to Lemma 2 that $S$ is injective. So, $S$ is a direct summand of $M_\alpha$. But $M_\alpha$ is an indecomposable module. Hence $M_\alpha = S$, i.e. $M_\alpha$ is injective. Let us now suppose that $S(M_\alpha) = 0$. Since $M_\alpha$ is torsion, $M_\alpha$ may be regarded as an $R/S(R)$-module. Thus $M_\alpha$ is a quasi-injective $R/S(R)$-module. Since $R/S(R)$ is a left QI-ring, $M_\alpha$ is $R/S(R)$-injective. We claim that $M_\alpha$ is $R$-injective. Let $I$ be a left ideal of $R$, and let $\alpha : I \to M_\alpha$ be an $(R)$-
homomorphism. Let us define a map $\beta : (I + S(R))/S(R) \to M_\alpha$, by the rule $\beta(x + S(R)) = \alpha(x)$, for all $x \in I$. If $x \in I \cap S(R)$ then $\alpha(x) \in S(I) \subseteq S(M_\alpha) = 0$. Hence $\beta$ is a well-defined map, which is an $(R-)$ homomorphism. Also, it is clear that $\beta$ is an $R/S(R)$-homomorphism. Since $M_\alpha$ is $R/S(R)$-injective, there is an $m \in M_\alpha$, such that $\beta(x + S(R)) = m(x + S(R))$, for all $(x + S(R)) \in (I + S(R))/S(R)$. Thus, we have $\alpha(x) = mx$, for all $x \in I$. Hence $M_\alpha$ is $(R-)$ injective. Therefore, $M = \bigoplus M_\alpha$ is a direct sum of torsion injective modules. Now, since $R$ has ACC on $\mathcal{F}$-ideals, it follows from Proposition 1 that $M$ is injective.

Now we recall a definition. $R$ is called a left $T$-ring if every non-zero left $R$-module has non-zero socle. For the purpose of the next theorem, we need a definition which is weaker than that of left $T$-rings. Namely, rings for which every non-zero torsion cyclic module has non-zero socle. Such rings will be called weakly $T$-rings.

**THEOREM 3.** Let $R$ be a weakly $T$-ring, and $(G, F)$ be the Goldie torsion theory for $R$-$\text{Mod}$. Then the following are equivalent:

1. $R$ is a left $TQI$-ring;
2. $R$ has ACC on $\mathcal{F}$-ideals and each $\mathcal{F}$-ideal is the intersection of maximal left ideals;
3. the torsion class coincides with the class of semisimple modules.

**Proof.** (1) $\Rightarrow$ (2). This follows from Corollary 1 to Lemma 2 and Lemma 3, and does not depend on the assumption that $R$ is a weakly $T$-ring.

(2) $\Rightarrow$ (3). Let us first consider a torsion quasi-injective left $R$-module $M$. By Lemma 5, $M = \bigoplus M_\alpha$, where each $M_\alpha$ is a torsion indecomposable quasi-injective left $R$-module.

Now consider an arbitrary but fixed $M_\alpha$. Let $x \in M_\alpha$, $x \neq 0$. Then $Rx$ is a non-zero torsion cyclic submodule of $M_\alpha$. Since $R$ is a weakly $T$-ring, $Rx$ has non-zero socle. So, let $S$ be a non-zero simple torsion submodule of $Rx$. By Corollary 2 to Lemma 2, it follows that $S$ is injective. Hence $S$ is a direct summand of $M_\alpha$. But $M_\alpha$ is indecomposable. So, $M_\alpha = S$. Thus $M = \bigoplus M_\alpha$ is a direct sum of torsion injective simple modules. Hence $M$ is semisimple, and also injective by Proposition 1. Since for the Goldie torsion class, every torsion module is a submodule of a torsion injective module, it follows that every torsion module is semisimple.

(3) $\Rightarrow$ (1). This is immediate.

We now give an example of a left $TQI$-ring which is not a left $QI$-ring.

**Example 1.** Let $F$ be a field, and let $R = \begin{bmatrix} F & 0 \\ F & F \end{bmatrix}$. Also, let $A = \begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ F & F \end{bmatrix}$.

In the above example, $Z(R) = 0$, and $B$ is the unique proper essential left ideal of $R$, which is also a maximal left ideal. Clearly, $R$ is an Artinian ring. Thus $R$ is a left $T$-ring.
whose \( \mathcal{F} \)-ideals have ACC and each proper essential left ideal is an intersection of maximal ideals. Now consider the Goldie torsion theory over \( R \). Since \( Z(\mathcal{F}(R)) = 0 \), the associated filter \( \mathcal{F}(G) \) consists of essential left ideals of \( R \). Hence by Theorem 2 (statement (2)), \( R \) is a left TQI-ring. On the other hand, since \( J(R) = A \cap B \neq (0) \), \( R \) is not a V-ring. Hence there exists a simple left \( R \)-module which is not injective. Since simple modules are quasi-injective, it follows that \( R \) is not a left QI-ring.

The purpose of the next theorem is to obtain a characterization of semilocal TQI-rings. By a *semilocal ring* we mean a ring which contains only a finite number of distinct maximal left ideals.

**Theorem 4.** Let \( R \) be a semilocal ring. Then the following are equivalent:

1. \( R \) is left TQI;
2. \( R \) has ACC on \( \mathcal{F} \)-ideals and each \( \mathcal{F} \)-ideal is the intersection of maximal left ideals.

**Proof.** The implication (1) \( \Rightarrow \) (2) follows Corollary 1 to Lemma 2 and Lemma 3, and it does not depend on the hypothesis that \( R \) is semilocal. Hence we need only to prove (2) \( \Rightarrow \) (1). Let \( M \) be a torsion quasi-injective left \( R \)-module. Then, by Lemma 5, \( M = \bigoplus M_\alpha \), where each \( M_\alpha \) is a (torsion) indecomposable quasi-injective left \( R \)-module. Consider an arbitrary but fixed \( M_\alpha \). Let \( x \in M_\alpha, x \neq 0 \). If \( Rx \equiv R \) then \( M_\alpha \) contains a copy of \( R \). Since \( M_\alpha \) is quasi-injective, an application of Baer's criterion implies that \( M_\alpha \) is injective. Now suppose \( Rx \equiv R/I \), for some left ideal \( I \) of \( R \). Then \( I \) is an \( \mathcal{F} \)-ideal. Hence there exist maximal left ideals \( M_1, \ldots, M_n \) of \( R \) such that \( I = M_1 \cap \ldots \cap M_n \). We claim that \( R/I \) is semisimple Artinian. Let us define a map \( \phi : R/I \rightarrow R/M_1 \oplus \ldots \oplus R/M_n \) by \( \phi([r]) = ([r_1], \ldots, [r_n]) \), where \( r_1 = r + I \) and \( r_k = r + M_k, \; k = 1, \ldots, n \). Clearly, \( \phi \) is an \( R \)-homomorphism. Also, \( \phi \) is an injection because for each non-zero \( \bar{r} \in R/I \), there is some \( M_\alpha \) such that \( r \not\in M_\alpha \). Thus \( R/I \) is a submodule of a semisimple Artinian module. Hence \( R/I \) is semisimple Artinian. As shown in Corollary 2 to Lemma 2, if each \( \mathcal{F} \)-ideal is the intersection of maximal left ideals then each torsion simple left \( R \)-module is injective. Hence \( R/I \) is injective. This implies that \( Rx \) is a direct summand of \( M_\alpha \). But \( M_\alpha \) is indecomposable. Hence \( M_\alpha \) is injective. Consequently, \( M = \bigoplus M_\alpha \) is injective, by Proposition 1, as \( R \) has ACC on \( \mathcal{F} \)-ideals.

We shall now study commutative TQI-rings. Let us first recall that a ring \( R \) is said to have SP if every left \( R \)-module splits (i.e. if the torsion submodule of each left \( R \)-module \( M \) is a direct summand of \( M \)). The next lemma is proved for not necessarily commutative TQI-rings.

**Lemma 6.** Let \( R \) be a left TQI-ring. Then every \( \mathcal{F} \)-ideal of \( R \) is idempotent.

**Proof.** Let \( I \) be an \( \mathcal{F} \)-ideal of \( R \). If \( I = I^2 \) then there is nothing to prove. So, suppose \( I \neq I^2 \). Since, for each \( r \in I, I(r + I^2) \neq 0 \) in \( I/I^2 \), i.e. \( I/I^2 \neq 0 \), it follows that \( I/I^2 \) is torsion. Now consider the exact sequence: \( 0 \rightarrow I/I^2 \rightarrow R/I^2 \rightarrow R/I \rightarrow 0 \). Since \( I/I^2 \) and \( R/I \) are torsion, \( R/I^2 \) is torsion. Hence \( I^2 \) is an \( \mathcal{F} \)-ideal. Hence by Corollary 1 to Lemma 1, both \( I \) and \( I^2 \) are intersections of maximal left ideals. Thus there is a maximal left ideal \( M \) of \( R \).
such that $I^2 \subseteq M$ but $I \not\subseteq M$. Since $M$ is a maximal ideal, $R = I + M$ and $1 = x + m$, for some $x \in I$ and $m \in M$. Then $x = x^2 + xm \in M$. Thus $1 \in M$, which is a contradiction.

**Lemma 7.** Every commutative TQI-ring is a V-ring (and hence a regular ring).

*Proof.* It is enough to show that $I = I^2$, for each ideal $I$ of $R$. If $I$ is an essential ideal then $I$ is an $\mathcal{F}$-ideal. Hence, by the above lemma, $I = I^2$. Now suppose $I$ is not an essential ideal, then there is an ideal $J$ of $R$ such that $I \cap J = (0)$ and $(I + J) \subseteq \mathcal{F} R$. Then $(I + J) = (I + J)^2 = I^2 + JJ + J^2 = I^2 + J^2$. This equation implies that $I = I^2$.

The next lemma is due to Cateforis and Sandomierski [5, Theorem 2.1, p. 156].

**Lemma 8.** For a commutative ring $R$, the following are equivalent:

1. $R$ has SP;
2. $Z(R) = 0$ and for every essential ideal $I$ of $R$, the ring $R/I$ is semisimple Artinian;
3. every $R$-module $M$ with $Z(M) = M$ is $R$-injective. In particular, if $R$ has SP then $R$ is hereditary.

**Theorem 5.** Let $R$ be a commutative ring and $(G, F)$ be the Goldie torsion theory for $R$-Mod. Then the following are equivalent:

1. $R$ is TQI;
2. $R$ has SP.

*Proof.* Let us assume (1). Then, by Lemma 7, $R$ is regular. Hence $Z(R) = 0$. Now let $I$ be an essential ideal of $R$. Then $R/I$ is a regular ring. Since $R$ is a TQI-ring, $R$ has ACC on $\mathcal{F}$-ideals by Lemma 3. Further, since every essential ideal of $R$ is an $\mathcal{F}$-ideal, it follows that $R/I$ is Noetherian, as an $R$-module and hence as an $R/I$-module. Thus $R/I$ is a regular Noetherian ring. Hence $R/I$ is a semisimple Artinian ring. Therefore, $R$ has SP by Lemma 8 (statement (2)). Let us now assume that $R$ has SP. Then $R$ is hereditary by Lemma 8. Hence $Z(R) = 0$. Thus the class of torsion modules, relative to the Goldie torsion theory for $R$-Mod, are precisely those modules $M$ for which $Z(M) = M$. But such modules are injective by Lemma 8. Hence, in particular, torsion quasi-injective modules are injective. So, $R$ is a TQI-ring.

We now give an example of a commutative TQI-ring which is not a QI-ring.

**Example 2.** [5, p. 161]. Let $K$ be a field and $A$ an infinite indexing set. Let $Q = \prod_{\alpha \in A} K^{(\alpha)}$, where $K^{(\alpha)} = K$ and $R = \sum_{\alpha \in A} \bigoplus K^{(\alpha)} + 1 \mathcal{Q}, \ 1 \in Q$. Then $R$ has only one essential ideal, namely, $I = \sum_{\alpha \in A} \bigoplus K^{(\alpha)}$, and $I$ is maximal. Since $R$ is regular, $R$ has SP by Lemma 8. Hence $R$ is a TQI-ring by Theorem 5. On the other hand, since $R$ is not semisimple Artinian and since commutative QI-rings are necessarily semisimple Artinian, it follows that $R$ is not a QI-ring.

As noted earlier, commutative TQI-rings are hereditary. Thus the next proposition is an immediate consequence of the following result, which is also due to Cateforis and Sandomierski [5, Theorem 4.1, p. 164].
**Lemma 9.** Let \( \{ R_\alpha \mid \alpha \in \Lambda \text{ and } R_\alpha \neq 0 \text{ for all } \alpha \} \) be an infinite collection of hereditary rings \( R_\alpha \) (with identity). Then \( \prod_\alpha R_\alpha \) is not a hereditary ring.

**Proposition 2.** An infinite direct product of commutative TQI-rings cannot be a TQI-ring.

We now characterize rings whose torsion quasi-injective left \( R \)-modules are \( \Sigma \)-quasi-injective. Recall that a quasi-injective module \( M \) is \( \Sigma \)-quasi-injective if \( M^{(A)} \) (the direct sum of \( \text{card } A \) copies of \( M \)) is also quasi-injective, for any set \( A \). First we state two needed results.

**Lemma 10.** [14, Prop. 4.2, p. 21]. Let \( (T, F) \) be any stable torsion theory for \( R \)-Mod (i.e. \( T \) is closed under injective hulls). Then every injective left \( R \)-module splits.

**Lemma 11.** [9, Cor. 2.2]. If \( M \) is quasi-injective and \( E(M)^{(A)} \) is injective then \( M^{(A)} \) is quasi-injective for any set \( A \).

**Theorem 6.** Let \( R \) be any ring and \( (G, F) \) be the Goldie torsion theory for \( R \)-Mod. Then the following are equivalent:

1. each torsion quasi-injective left \( R \)-module is \( \Sigma \)-quasi-injective;
2. \( R \) has ACC on \( \mathcal{F} \)-ideals.

**Proof.** (1) Suppose each torsion quasi-injective left \( R \)-module is \( \Sigma \)-quasi-injective. Let \( I_1 \subseteq I_2 \subseteq \ldots \subseteq I_k \subseteq \ldots \) be an ascending chain of \( \mathcal{F} \)-ideals of \( R \). Then we get the ascending chain

\[
(0) = I_1/I_1 \subseteq I_2/I_1 \subseteq \ldots \subseteq I_k/I_1 \subseteq \ldots
\]

of submodules of the left \( R \)-module \( \bar{R} = R/I_1 \). Let \( \bar{I}_k = I_k/I_1, \ k = 1, 2, \ldots \) Now consider the left \( R \)-modules \( \bar{R}/\bar{I}_k \). Clearly, each \( \bar{R}/\bar{I}_k \) is a torsion \( R \)-module. Let \( Q_k = E(\bar{R}/\bar{I}_k) \). Then each \( Q_k \) is a torsion left \( R \)-module. Let \( Q = \bigoplus Q_i \) and \( M = \prod_\alpha Q_\alpha \). Since \( M \) is injective, Lemma 10 implies that \( M \) splits. Suppose \( M = T \oplus K \), where \( T \) and \( K \) are the torsion and torsion free parts of \( M \), respectively. Since each \( Q_i \subseteq T \) and each \( Q_i \) is injective, we may write \( T = T_i = Q_i \oplus P_i \), for some \( P_i \subseteq T \). Hence \( \bigoplus T_i = \bigoplus Q_i \oplus \bigoplus P_i \).

Thus \( Q = \bigoplus Q_i \) is a direct summand of \( \bigoplus T_i \). But \( \bigoplus T_i \) is quasi-injective by the hypothesis. Hence \( Q \) is quasi-injective. Now let \( \bar{I} = \bigcup_{k=1}^\infty \bar{I}_k \). Then the natural \( \text{(R-)homomorphism} \ f_k : \bar{I} \to \bar{R}/\bar{I}_k \) maps \( \bar{I} \) into \( Q_k \). Note that if \( a \in \bar{I} \) then \( a \in \bar{I}_t \) for some \( t \). Thus \( f_k(a) = 0 \), for all \( k \geq t \). Let us now define a map \( f : \bar{I} \to Q \) by \( f(a) = (f_1(a), \ldots, f_i(a), \ldots) \), \( a \in \bar{I} \). This definition is meaningful since only a finite number of terms on the right-hand side are non-zero. Clearly, \( f \) is an \( R \)-homomorphism. Now \( \bar{I} \subseteq \bar{R} \subseteq Q_1 \subseteq Q \), and \( Q \) is quasi-injective. Hence there is a map \( \lambda \in \text{Hom}_R(Q, Q) \) which induces \( f \). Thus \( f(\bar{I}) \subseteq \lambda(\bar{R}) \subseteq Q \). Suppose \( \lambda(1 + I_1) = m \), where \( 1 \) is the identity of \( R \). Now...
m ∈ \sum_{i=1}^{t} Q_i, \text{ for some } t. \text{ Hence } f(\hat{I}) \subseteq Rm \subseteq \sum_{i=1}^{t} Q_i. \text{ This implies that } \hat{I}_{i+1} = \hat{I}_{i+2} = \ldots = \hat{I}.

Hence \hat{I}_{i+1} = \hat{I}_{i+2} = \ldots = \hat{I}. \text{ This proves that } R \text{ has ACC on } \mathcal{F}-\text{ideals.}

(2) Let us now suppose that \( R \) has ACC on \( \mathcal{F} \)-ideals, and let \( M \) be any torsion quasi-injective left \( R \)-module. Then \( E(M) \) is torsion, as we are working in the Goldie torsion theory. Moreover, since \( R \) has ACC on \( \mathcal{F} \)-ideals, \( E(M)^{(A)} \) is injective, for any set \( A \), by Proposition 1. Hence, by Lemma 11, \( M^{(A)} \) is quasi-injective, for any set \( A \).

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