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PROJECTIVE HOMOTOPY CLASSES OF STIEFEL MANIFOLDS

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1. Introduction. Given a homotopy class $[f]$ in $\pi_n(X)$, we say that $[f]$ is projective if and only if there is a homotopy commutative factorization

$$\begin{tikzcd} S^n \arrow[rr, "f"] \arrow[dr, "\nu"'] & & X \\ & RP^n \arrow[ur, "f'"'] & \end{tikzcd}$$

where ν is the standard double covering. We then denote by $\pi_n^{\mathrm{Proj}}(X)$ the subset of projective homotopy classes in $\pi_n(X)$.

The notion of projective homotopy classes was studied in the author's thesis [**5**], and the projective homotopy classes for spheres in the stable range, up through the 3-stem were calculated in [**6**]. The purpose of the present paper is to prove the following result:

1.1 THEOREM. *$\pi_7^{\mathrm{Proj}}(X) = \pi_7(X)$ for X equal to the Stiefel manifolds $V_{7,3}$ and $V_{7,4}$.*

The interest in projective homotopy classes of Stiefel manifolds arises from a problem concerning vector fields on spheres, studied by Zvengrowski [**7**]. In particular, he asks the following question: Is every r-field on S^{n-1} homotopic to a skew linear r-field? An r-field on S^{n-1} is defined to be a set of r-vector vector fields on S^{n-1} which are orthonormal at every point. This can be regarded as a cross section of the fibration

$$V_{n-1,r} \rightarrow V_{n,r+1} \rightarrow S^{n-1}$$

and one can then consider homotopy classes of r-fields. An r-field is said to be skew linear if and only if it is equivariant with respect to the obvious Z_2 action on $V_{n,r+1}$ and S^{n-1}.

In [**7**] Zvengrowski shows that for $r \leqq 5$, every r-field is homotopic to a skew-linear r-field. The first part of the proof makes use of a homotopy classification of r-fields on S^{n-1}; the homotopy classes of r-fields are in one-one correspondence with $(n-1)$-dimensional homotopy classes of the fibre $\pi_{n-1}(V_{n-1,r})$. In the parallelizable case, i.e., when $n = 2, 4,$ or 8, the skew linear

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r-fields are in one-one correspondence with the projective classes $\pi_{n-1}{}^{\mathrm{Proj}}(V_{n-1,r})$. Therefore one wants to know that

$$\pi_{n-1}{}^{\mathrm{Proj}}(V_{n-1,r}) = \pi_{n-1}(V_{n-1,r})$$

when $n = 2, 4$, or 8. Most of the arguments are elementary and are given in [**7**]. The concern of this paper is to handle the two non-trivial cases $\pi_7(V_{7,3})$ and $\pi_7(V_{7,4})$.

2. Construction of Postnikov systems. The principal tool in studying projective homotopy classes is the mod 2 Postnikov system for the space in question. We will first construct a Postnikov system for $V_{7,3}$. Recall (cf. [**4**]) that $H^*(V_{n,k}; Z_2)$ is the algebra over the Steenrod algebra generated by $H^*(RP^{n-1}_{n-k-1}; Z_2)$ and subject to the relation $Sq^i[j] = C_{i,j}[i+j]$ (including $[j]^2 = [2j]$), where $[j]$ denotes the generator of $H^j(RP^{n-1}_{n-k-1}; Z_2)$ (see Figure 2.1).

n	Generator of $H^n(V_{7,3}; Z_2)$
4	[4]
5	[5]
6	[6]
7	. . .
8	. . .
9	[4] [5]

Figure 2.1

Since $H^*(V_{7,3}; Z_2)$ has [4] and [5] as generators over the Steenrod algebra, to construct a Postnikov system, we begin with the map

$$[4] \times [5] : V_{7,3} \to K(Z, 4) \times K(Z_2, 5).$$

It is clear that this map induces an isomorphism on $\pi_4(-)$. If we construct a space X_5 by killing the class $Sq^2 i_4 \otimes 1$ in $H^*(K(Z, 4) \times K(Z_2, 5); Z_2)$, the map $[4] \times [5]$ will lift to X_5 and the lifting will satisfy the hypotheses of the $\mathscr{C}_p$ approximation theorem (see [**1**, p. 100]):

$$\begin{tikzcd} & X_5 \arrow[d] & \\ V_{7,3} \arrow[ur] \arrow[r] & K(Z,4) \times K(Z_2,5) \arrow[r, "Sq^2 i_4 \otimes 1"] & K(Z_2, 6). \end{tikzcd}$$

Therefore the lifting will induce a $\mathscr{C}_2$ isomorphism on $\pi_5(-)$. ($\mathscr{C}_p$ denotes the Serre class of abelian torsion groups of finite exponent such that the order of every element is prime to p.)

The cohomology of X_5 is computed in Table 1 using the Serre exact sequence. The symbol "$Sq^{i,j}$" in the table denotes $Sq^i Sq^j$. The arrows indicate transgression from the cohomology of the fibre to the cohomology of the base. The Greek letters denote classes which pull back to the appropriate $Sq^i Sq^j$ of the fundamental class of the fibre. For example, $\alpha(2)$ in $H^*(X_5; Z_2)$ pulls back to

TABLE 1

n	$H^*(K(Z,4)\times K(Z_2,5))$	$H^*(K(Z_2,5))$	$H^*(X_5)$	$H^*(K(Z_2,6)\times K(Z_2,6))$	$H^*(X_6)$
4	$i_4\otimes 1$		$i_4\otimes 1$		$i_4\otimes 1$
5	$1\otimes i_5$	i_5	$1\otimes i_5$		$1\otimes i_5$
6	$Sq^2i_4\otimes 1$ $1\otimes Sq^1i_5$	Sq^1i_5	 $1\otimes Sq^1i_5$	$i_6\otimes 1$ $1\otimes i_6$	 $1\otimes Sq^1i_5$
7	$Sq^3i_4\otimes 1$ $1\otimes Sq^2i_5$	Sq^2i_5	$1\otimes Sq^2i_5$ $\alpha(2)$	$Sq^1i_6\otimes 1$ $1\otimes Sq^1i_6$	
8	$i_4^2\otimes 1$ $1\otimes Sq^3i_5$ $1\otimes Sq^{2,1}i_5$	Sq^3i_5 $Sq^{2,1}i_5$	$i_4^2\otimes 1$ $\beta(3)$ $1\otimes Sq^3i_5$ $\gamma(2,1)$ $1\otimes Sq^{2,1}i_5$	$Sq^2i_6\otimes 1$ $1\otimes Sq^2i_6$	$i_4^2\otimes 1$ $1\otimes Sq^{2,1}i_5$ $\bar{\gamma}(2,1)$
9	$1\otimes Sq^4i_5$ $1\otimes Sq^{3,1}i_5$ $i_4\otimes i_5$ etc.	Sq^4i_5 $Sq^{3,1}i_5$	$1\otimes Sq^4i_5$ $1\otimes Sq^{3,1}i_5$ $i_4\otimes i_5$ $\delta(3,1)$ etc.		
10	$Sq^{4,2}i_4\otimes 1$ etc.				

Sq^2i_5 in $H^*(K(Z_2, 5); Z_2)$; $\gamma(2, 1)$ pulls back to $Sq^{2,1}i_5$. In dimensions through eight, at least, we are in the range of Serre's exact sequence, and transgression is defined on all elements in the cohomology of the fibre.

The succeeding stages of the Postnikov system are constructed by deciding which class or classes must be killed in order that the hypotheses of the $\mathscr{C}_p$ approximation theorem be satisfied at the next stage (see Table 1). The Postnikov system we then get is displayed in Figure 2.2.

$$\begin{array}{lllll}
K(Z,7)\times K(Z_4,7)\times K(Z_4,7) \to & X_7 & & & \\
& \downarrow & & & \\
K(Z_2,6)\times K(Z_2,6) \to & X_6 & \to K(Z,8)\times K(Z_4,8)\times K(Z_4,8) & & \\
& \downarrow & & & \\
K(Z_2,5) \to & X_5 & \to K(Z_2,7)\times K(Z_2,7) & & \\
& \downarrow & & & \\
& K(Z,4)\times K(Z_2,5) & \to K(Z_2,6) & &
\end{array}$$

Figure 2.2

The Postnikov invariants are $Sq^2 i_4 \otimes 1$, $(1 \otimes Sq^{2,1} i_5) \times \alpha(2)$, and $(i_4{}^2 \otimes 1) \times (1 \otimes Sq^{2,1} i_5) \times \bar{\gamma}(2, 1)$, in that order. ($\bar{\gamma}(2, 1)$ denotes the image of $\gamma(2, 1)$ in $H^*(X_6; Z_2)$.) It is clear that the class $i_4{}^2 \otimes 1$ in the cohomology of X_6 is the mod 2 reduction of an integral class. To see that $1 \otimes Sq^{2,1} i_5$ and $\bar{\gamma}(2, 1)$ are both reductions of Z_4 classes but not Z_8 classes, we need certain information about the squaring operations in $H^*(X_5; Z_2)$.

2.1 Lemma. *The classes $\alpha(2)$, $\gamma(2, 1)$, and $\delta(3, 1)$ in $H^*(X_5; Z_2)$ can be chosen so that $Sq^2\, \alpha(2) = \delta(3, 1)$ and $Sq^1\, \gamma(2, 1) = \delta(3, 1)$.*

Proof. $\delta(3, 1)$ is any class which pulls back to $Sq^3 Sq^1 i_5$, so by the Adem relation, $Sq^2 Sq^2 = Sq^3 Sq^1$, it is clear that the two equalities hold modulo the image of p^*, where p is the fibre map. To show that these classes can be chosen so that strict equality holds, we use a naturality argument. We consider the 2-stage Postnikov system

$$\begin{array}{c} X_5{}' \\ \downarrow \\ K(Z, 4) \xrightarrow{Sq^2 i_4} K(Z_2, 6). \end{array}$$

The projection onto the first factor of $K(Z, 4) \times K(Z_2, 5)$ induces a map from the first Postnikov system to the second:

$$\begin{array}{ccc} X_5 & \xrightarrow{\varphi} & X_5{}' \\ \downarrow & & \downarrow \\ K(Z, 4) \times K(Z_2, 5) & \longrightarrow & K(Z, 4). \end{array}$$

It is clear that $H^*(X_5{}'; Z_2)$ is identical to $H^*(X_5; Z_2)$ except that there are no classes of the form $1 \otimes Sq^I i_5$ (see Table 1). In particular, the image of p^* is zero in $H^9(X_5{}'; Z_2)$. Therefore we have that

$$Sq^1\, \gamma'(2, 1) = \delta'(3, 1) = Sq^2\, \alpha'(2)$$

in $H^9(X_5{}'; Z_2)$. Then we simply choose $\alpha(2)$, $\gamma(2, 1)$, and $\delta(3, 1)$ in $H^*(X_5; Z_2)$ to be $\varphi^*(\alpha'(2))$, $\varphi^*(\gamma'(2, 1))$, and $\varphi^*(\delta'(3, 1))$, respectively. The result now follows by naturality.

We then use the well-known Bockstein lemma, whose proof is given in [**1**, p. 106]:

2.2 Lemma. *Let p: $E \to B$ be a Serre fibration with fibre F. Let d_i denote the ith Bockstein homomorphism and let r denote the inclusion of the fibre into the total space. Suppose that a class u in $H^n(F; Z_2)$ transgresses to $d_i v$ for some v in $H^n(B; Z_2)$. Then $d_{i+1} p^*(v)$ is defined in $H^{n+1}(E; Z_2)$ and $r^*(d_{i+1} p^*(v)) = d_1 u$.*

We apply this to the fibration $K(Z_2, 6) \times K(Z_2, 6) \to X_6 \to X_5$, taking u to be $Sq^2 i_6 \otimes 1$, v to be $1 \otimes Sq^{2,1} i_5$, and i to be 1. Then $d_2(1 \otimes Sq^{2,1} i_5)$ is defined and non-zero in $H^*(X_6; Z_2)$. We then use the fact that a class w is the mod 2 reduction of a Z_{2^k} class but not a $Z_{2^k} + 1$ class if and only if $d_k w$ is defined and non-zero. Similarly for $\bar{\gamma}(2, 1)$, we take u to be $1 \otimes Sq^2 i_6$, v to be $\gamma(2, 1)$, and i to be 1. We note that $u = 1 \otimes Sq^2 i_6$ transgresses to $Sq^2\alpha(2)$, which by Lemma 2.1 is precisely $\delta(3, 1)$, and $\delta(3, 1) = d_1\gamma(2, 1)$.

We can therefore conclude that $d_2\bar{\gamma}(2, 1)$ is defined and non-zero in $H^*(X_6; Z_2)$. This finishes the argument that $1 \otimes Sq^{2,1}i_5$ and $\bar{\gamma}(2, 1)$ are reductions of Z_4 but not Z_8 classes.

It can now be verified that the map

$$[4] \times [5]: V_{7,3} \to K(Z, 4) \times K(Z_2, 5)$$

lifts to each X_k and that each lifting $V_{7,3} \to X_k$ induces a $\mathscr{C}_2$ isomorphism on $\pi_i(-)$, $i \leqq k$.

According to the calculations of Paechter [**2**], $\pi_i(V_{7,3})$ is 2-primary except in dimension 7, where $\pi_7(V_{7,3}) = Z + Z_4 + Z_{12}$. Therefore the lifting $V_{7,3} \to X_7$ induces an isomorphism on $\pi_i(-)$ for $i < 7$ and an epimorphism for $i = 7$. For any *CW* complex K, this implies that the induced map $[K, V_{7,3}] \to [K, X_7]$ is bijective if $\dim K < 7$ and surjective if $\dim K = 7$ (see [**3**, Corollary 7.6.23]). In particular, $[RP^7, V_{7,3}] \to [RP^7, X_7]$ is surjective, so

$$\pi_7^{\mathrm{Proj}}(V_{7,3}) \to \pi_7^{\mathrm{Proj}}(X_7)$$

is also surjective. Therefore $\pi_7^{\mathrm{Proj}}(X_7)$ is equal to the 2-primary component of $\pi_7^{\mathrm{Proj}}(V_{7,3})$.

Next we construct a Postnikov system for $V_{7,4}$. The Z_2 cohomology of $V_{7,4}$ is given in Figure 2.3.

n	Generator of $H^n(V_{7,4}; Z_2)$
3	[3]
4	[4]
5	[5]
6	$[6] = ([3])^2$
7	[3] [4]
8	[3] [5]
9	$[3]\,[6] = ([3])^3$, [4] [5]

Figure 2.3

We begin with the map

$$[3]: V_{7,4} \to K(Z_2, 3),$$

which induces an isomorphism on $\pi_3(-)$. To construct the next stage, Y_5, the class $Sq^2Sq^1i_3$ in $H^*(K(Z_2, 3); Z_2)$ must be killed. The cohomology of Y_5 is computed in Table 2. We note that we are not in the range of Serre's exact sequence; in particular, there is $i_3 \otimes i_5$ in $H^3(B; H^5(F))$ which could be hit by $Sq^2 i_5$. However, i_5 transgresses to $Sq^{2,1}i_3$, so $Sq^2 i_5$ transgresses to $Sq^{2,2,1}i_3$ which is 0 by the Adem relations. In particular, $Sq^2 i_5$ survives. Furthermore, $i_3 \otimes i_5$ will hit $i_3 \otimes Sq^{2,1}i_3$, adding nothing new to $H^8(Y_5, Z_2)$. Therefore, the

TABLE 2

n	$H^*(K(Z_2, 3))$	$H^*(K(Z_2, 5))$	$H^*(Y_5)$	$H^*(K(Z_2, 6))$	$H^*(Y_6)$
3	i_3		i_3		i_3
4	$Sq^1 i_3$		$Sq^1 i_3$		$Sq^1 i_3$
5	$Sq^2 i_3$	i_5	$Sq^2 i_3$		$Sq^2 i_3$
6	$i_3{}^2$ $Sq^{2,1} i_3$	$Sq^1 i_5$	$i_3{}^2$	i_6	$i_3{}^2$
7	$i_3 Sq^1 i_3$ $Sq^{3,1} i_3$	$Sq^2 i_5$	$i_3 Sq^1 i_3$ $\alpha(2)$	$Sq^1 i_6$	$i_3 Sq^1 i_3$
8	$i_3 Sq^2 i_3$ $(Sq^1 i_3)^2$	$Sq^3 i_5$ $Sq^{2,1} i_5$	$i_3 Sq^2 i_3$ $(Sq^1 i_3)^2$ $\beta(3)$ $\gamma(2, 1)$	$Sq^2 i_6$	$i_3 Sq^2 i_3$ $(Sq^1 i_3)^2$ $\bar{\gamma}(2, 1)$
9	$(Sq^1 i_3)(Sq^2 i_3)$ $i_3{}^3$, $i_3 Sq^{2,1} i_3$ $Sq^{4,2} i_3$ etc.	$Sq^4 i_5$ $Sq^{3,1} i_5$	$(Sq^1 i_3)(Sq^2 i_3)$ $i_3{}^3$, $i_3 Sq^{2,1} i_3$ $Sq^{4,2} i_3$ $\delta(3, 1)$ etc.	$Sq^3 i_6$ $Sq^{2,1} i_6$	
10	$Sq^{4,2,1} i_3$ etc.				

fact that we are not in the range of Serre's exact sequence does not complicate matters, at least through dimension eight.

As before, the succeeding stages are constructed by deciding which classes must be killed in order that the hypotheses of the $\mathscr{C}_p$ approximation theorem be satisfied. The Postnikov system for $V_{7,4}$ is shown in Figure 2.4.

$$
\begin{array}{ccccc}
K(Z, 7) \times K(Z_4, 7) & \rightarrow & Y_7 & & \\
 & & \downarrow & & \\
K(Z_2, 6) & \rightarrow & Y_6 & \rightarrow & K(Z, 8) \times K(Z_4, 8) \\
 & & \downarrow & & \\
K(Z_2, 5) & \rightarrow & Y_5 & \rightarrow & K(Z_2, 7) \\
 & & \downarrow & & \\
 & & K(Z_2, 3) & \rightarrow & K(Z_2, 6)
\end{array}
$$

Figure 2.4

The Postnikov invariants are $Sq^2 Sq^1 i_3$, $\alpha(2)$, and $(Sq^1 i_3)^2 \times \bar{\gamma}(2, 1)$, in that order.

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It follows from the universal coefficient theorem that $Sq^1 i_3$ is the mod 2 reduction of an integral class, so the same is true for $(Sq^1 i_3)^2$. To show that $\bar{\gamma}(2, 1)$ is the mod 2 reduction of a Z_4 class and not a Z_8 class, we need the following fact, analogous to Lemma 2.1:

2.3 LEMMA. *The classes* $\alpha(2)$, $\gamma(2, 1)$, *and* $\delta(3, 1)$ *in* $H^*(Y_5; Z_2)$ *can be chosen such that* $Sq^2\alpha(2) = \delta(3, 1)$ *and* $Sq^1\gamma(2, 1) = \delta(3, 1)$.

Proof. The projection $V_{7,4} \to V_{7,3}$ (dropping the last row of a 4×7 matrix) induces a map between the mod 2 Postnikov systems. Letting $\varphi: Y_5 \to X_5$ denote the map induced on the fifth stage, it is easy to show that we can take the $\alpha(2)$ in $H^*(Y_5; Z_2)$ to be $\varphi^*(\alpha(2))$, and similarly for $\gamma(2, 1)$ and $\delta(3, 1)$. The conclusion now follows from Lemma 2.1.

We can now apply the Bockstein lemma to show that $d_2\bar{\gamma}(2, 1)$ is defined and non-zero in $H^*(Y_6; Z_2)$. This implies that $\bar{\gamma}(2, 1)$ is the reduction of a Z_4 class but not a Z_8 class.

The calculations of Paechter [**2**] show that $\pi_i(V_{7,4})$ is only 2-primary for $i \leqq 7$, so the lifting $V_{7,4} \to Y_7$ induces an isomorphism on $\pi_i(-)$ for $i \leqq 7$ and an epimorphism for $i = 8$ (note that $\pi_8(Y_7) = 0$). From this we can conclude that $\pi_7^{\text{Proj}}(V_{7,4})$ is isomorphic to $\pi_7^{\text{Proj}}(Y_7)$.

3. Computation of $\pi_7^{\text{Proj}}(X)$**.** Several propositions will lead to the proof of Theorem 1.1. The first result is that the sums of certain projective classes are again projective (Proposition 3.2). This will follow from the fact that each fibration in the Postnikov system is principal:

3.1 LEMMA. *Let* (H, e) *be an H-space acting on the left of a space* (X, x_0). *Suppose that there is a map* $r: H \to X$ *preserving base points such that the following diagrams are homotopy commutative*:

$$\begin{tikzcd} H \times H \arrow[r, "M"] \arrow[d, "id \times r"'] & H \arrow[d, "r"] \\ H \times X \arrow[r, "M_1"] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\cong"] \arrow[drr, "id"'] & \{e\} \times X \arrow[r, "\subset"] & H \times X \arrow[d, "M_1"] \\ & & X \end{tikzcd}$$

where M *is the multiplication of* H *and* M_1 *is the action of* H *on* X. *Let* $n > 1$, $f: S^n \to X$, $g': S^n \to H$, $g = r \circ g': S^n \to X$. *Then we have that*

$$[f] + [g] = [M_1 \circ (f \times g') \circ \Delta]$$

in $\pi_n(X)$.

This lemma generalizes a familiar result on H-spaces and is proved in [**7**].

3.2 PROPOSITION. *Let* $F \to E \to B$ *be a principal fibration with fibre map* p *and inclusion map* r. *Let* $[f], [g] \in \pi_n^{\text{Proj}}(E)$ *such that* $[g]$ *factors as a projective class* *through the fibre; i.e., there is a* $[g'] \in \pi_n^{\mathrm{Proj}}(F)$ *such that* $[g] = [r \circ g']$. *Then* $[f] + [g] \in \pi_n^{\mathrm{Proj}}(E)$.

Proof. By definition there are maps $f':RP^n \to E$ and $g'':RP^n \to F$ such that $f \simeq f' \circ \nu$ and $g' \simeq g'' \circ \nu$. Since the fibration is principal, F is an H-space acting on the left of E in such a way that the hypotheses of Lemma 3.1 are satisfied. Therefore

$$\begin{aligned}[f] + [g] &= [M_1 \circ (f \times g') \circ \Delta] \\ &= [M_1 \circ (f' \circ \nu \times g'' \circ \nu) \circ \Delta] \\ &= [M_1 \circ (f \times g') \circ \Delta \circ \nu],\end{aligned}$$

and this is clearly projective.

3.3 PROPOSITION. *Let* X *be an* $(n - 1)$ *connected space where* n *is odd. Then* $\pi_n^{\mathrm{Proj}}(X) = 2 \cdot \pi_n(X)$.

Proof. Since X is $(n - 1)$ connected, any map $RP^n \to X$ factors (up to homotopy) through RP^n/RP^{n-1}. Thus $S^n \to X$ is projective if and only if it factors as

$$S^n \to RP^n \to RP^n/RP^{n-1} \cong S^n \to X.$$

Since $S^n \to RP^n \to S^n$ has degree 2 when n is odd, the conclusion now follows.

We now prove several results that are more technical. The symbols $\mathfrak{k}_1$, $\mathfrak{k}_2$, and $\mathfrak{k}_3$ will denote the classes $i_4^2 \otimes 1$, $1 \otimes Sq^{2,1}i_5$, and $\bar{\gamma}(2, 1)$, respectively, in the cohomology of X_6, with the appropriate coefficients (see Figure 2.2). Similarly, the symbols $\mathfrak{l}_1$ and $\mathfrak{l}_2$ will denote the classes $(Sq^1 i_3)^2$ and $\bar{\gamma}(2, 1)$ in the cohomology of Y_6 (see Figure 2.4).

3.4 PROPOSITION. *Let* $g:RP^7 \to X_6$ *be any map (Figure 2.2) and let* g' *denote the unique extension of* g *to* RP^8 *(since* $\pi_7(X_6) = \pi_8(X_6) = 0$, *a unique extension exists by the Puppe sequence). Then there is a lifting* $f:RP^7 \to X_7$ *of* g *satisfying* $[f \circ \nu] = (a_1, a_2, a_3)$ in $\pi_7(X_7) = Z + Z_4 + Z_4$, *where* $a_i = 0$ *if* $g'^*(\mathfrak{k}_i) = 0$ *and* a_i *is odd if* $g'^*(\mathfrak{k}_i) \neq 0$, $i = 1, 2, 3$.

Similarly, let $g:RP^7 \to Y_6$ *(Figure 2.4) and let* g' *denote its unique extension to* RP^8. *Then there is a lifting* $f:RP^7 \to Y_7$ *of* g *satisfying* $[f \circ \nu] = (b_1, b_2)$ *in* $\pi_7(Y_7) = Z + Z_4$, *where* $b_i = 0$ *if* $g'^*(\mathfrak{l}_i) = 0$ *and* b_i *is odd if* $g'^*(\mathfrak{l}_i) \neq 0$, $i = 1, 2$.

Proof. We shall prove only the first statement since the proof of the second is identical. We construct the space X_7 from X_6 by first killing those $\mathfrak{k}_i$'s whose images under g'^* are zero (call the resulting space X_7') and then killing (the images of) the remaining $\mathfrak{k}_i$'s in X_7'. It is easy to see that this gives the same space as the one obtained by killing all of the $\mathfrak{k}_i$'s at the same time. The lifting of f is now gotten simply by lifting g' to X_7', then restricting this lifting to RP^7, and finally lifting the restriction to X_7 (see Figure 3.1):

$$\begin{tikzcd}
 & & X_7 \arrow[d] \\
 & & X_7' \arrow[d] \\
RP^7 \arrow[r, hook, "\subset"] \arrow[urruu, "f"] & RP^8 \arrow[r, "g'"] \arrow[ur] & X_6
\end{tikzcd}$$

Figure 3.1

By construction, $p' \circ f$ has an extension to RP^8, where p' is the fibre map $X_7 \to X_7'$. Therefore $p' \circ f \circ \nu$ is null homotopic by the Puppe sequence. This says precisely that $a_i = 0$ for those i's satisfying $g'^*(\mathfrak{k}_i) = 0$.

Now assume $g'^*(\mathfrak{k}_i) \neq 0$ for some i. We may consider $\mathfrak{k}_i$ as a Z_2 class since $g'^*(\mathfrak{k}_i) \neq 0$ if and only if g'^* of the mod 2 reduction of $\mathfrak{k}_i$ is non-zero. We now construct a space W_7 by killing only $\mathfrak{k}_i$, and killing this as a Z_2 class rather than as a Z or Z_4 class:

$$\begin{array}{ccl} K(Z_2, 7) \to & W_7 & \\ & \downarrow & \\ & X_6 & \to K(Z_2, 8) \end{array}$$

where $X_6 \to K(Z_2, 8)$ represents the mod 2 reduction of $\mathfrak{k}_i$. By naturality of induced fibrations, there is a map $\varphi: X_7 \to W_7$ such that the induced map on $\pi_7(-)$ sends (a_1, a_2, a_3) in $Z + Z_4 + Z_4$ to the mod 2 reduction of a_i If $[f \circ \nu] = (a_1, a_2, a_3)$, then a_i is odd if and only if $[\varphi \circ f \circ \nu]$ is non-zero. The Puppe sequence says that $[\varphi \circ f \circ \nu]$ is non-zero if and only if $\varphi \circ f$ is not extendable to RP^8. But $g'^*(\mathfrak{k}_i)$ with $\mathfrak{k}_i$ considered as a Z_2 class is precisely the obstruction to lifting g' to W_7. Any extension of $\varphi \circ f$ to RP^8 would constitute a lifting of g', so the condition $g'^*(\mathfrak{k}_i) \neq 0$ implies that $\varphi \circ f$ has no such extension. Therefore $[\varphi \circ f \circ \nu]$ is non-zero, or a_i is odd.

3.5 PROPOSITION. *Let $h': RP^8 \to K(Z, 4) \times K(Z_2, 5)$ be any map (Figure 2.2). Then there are liftings $h_1', h_2': RP^8 \to X_5$ of h' such that $h_1'^*(\gamma(2, 1)) = 0$ and $h_2'^*(\gamma(2, 1)) \neq 0$.*

An identical result holds for any map $h': RP^8 \to K(Z_2, 3)$ (Figure 2.4).

Proof. Since $H^*(RP^8; Z_2) = Z_2[u]/(u^{n+1})$, the only possible obstruction to lifting h' is Sq^2u^4, which is zero. So h' lifts to a map g_1', say. We construct another lifting, g_2', by using the action of the fibre on the total space to "add" g_1' to the map $u^5: RP^8 \to K(Z_2, 5)$ (compare Lemma 3.1). The sum is represented by $[M_1 \circ (u^5 \times g_1') \circ \Delta]$, and it is clearly a lifting of h'. By the commutativity of the diagram

$$\begin{array}{ccc} K(Z_2, 5) \times K(Z_2, 5) & \xrightarrow{M} & K(Z_2, 5) \\ {\scriptstyle id \times r}\downarrow & & \downarrow{\scriptstyle r} \\ K(Z_2, 5) \times X_5 & \xrightarrow{M_1} & X_5 \end{array}$$

where M is the H-space multiplication and M_1 is the action of the fibre on X_5. we get that

$$M_1^*(\gamma(2, 1)) = 1 \otimes \gamma(2, 1) + Sq^{2,1}i_5 \otimes 1$$

(recall that $r^*(\gamma(2, 1)) = Sq^{2,1}i_5$). This implies that

$$g_2'^*(\gamma(2, 1)) = Sq^{2,1}u_5 + g_1'^*(\gamma(2, 1)).$$

Since $Sq^{2,1}u_5$ is non-zero, it follows that if $g_1'^*(\gamma(2, 1)) = 0$, then $g_2'^*(\gamma(2, 1)) \neq 0$, and vice-versa. Therefore h_1' and h_2' can be chosen from g_1' and g_2'.

3.6 PROPOSITION. *Given any map $g:RP^7 \to X_5$ or Y_5, then $g^*(\alpha(2)) = 0$.*

Proof. We extend g to $g':RP^9 \to X_5$ or Y_5 and note that $g^*(\alpha(2)) = 0$ if and only if $g'^*(\alpha(2)) = 0$. Supposing to the contrary that $g'^*(\alpha(2)) = u^7$, we have by Lemmas 2.1 and 2.3 that

$$u^9 = Sq^2 g'^*(\alpha(2)) = g'^*(\delta(3, 1)) = Sq^1 g'^*(\gamma(2, 1)).$$

But $Sq^1 g'^*(\gamma(2, 1))$ must be zero since $g'^*(\gamma(2, 1))$ is in dimension eight. Therefore $g^*(\alpha(2)) = 0$.

Proof of Theorem 1.1. We begin with $X = V_{7,3}$. The object is to construct eight types of projective classes, namely classes of the form (a, b, c) in $\pi_7(X_7) = Z + Z_4 + Z_4$ where each one of the a, b, and c is specified to be either zero or an odd number. Since the fibrations in the Postnikov system are principal, we can then add to these any projective class of X_7 which is the image of a projective class of the fibre, and the sum is again projective (Proposition 3.2). By Proposition 3.3 the images of the projective classes of the fibre are precisely those (a, b, c) in $Z + Z_4 + Z_4$ where each of a, b, and c is even. So by adding such classes, any element of $\pi_7(X_7)$ can be realized as a projective class.

To construct the eight types of projective classes described above, we first consider the composition

$$RP^7 \xrightarrow{h} K(Z, 4) \xrightarrow{j} K(Z, 4) \times K(Z_2, 5)$$

where h is non-trivial (note that $[RP^7, K(Z, 4)] = H^4(RP^7; Z) = Z_2$) and j is the inclusion into the first factor. It follows from Proposition 3.5 that there are liftings h_1 and h_2 of $j \circ h$ to X_5 such that $h_1'^*(\gamma(2, 1)) = 0$ and $h_2'^*(\gamma(2, 1)) \neq 0$ (h_i' denotes the unique extension of h_i to RP^8). By Proposition 3.6, h_1 and h_2 can be lifted to X_6; we denote the liftings by g_1 and g_2, respectively. It follows that $g_i'^*(i_4^2 \otimes 1) \neq 0$ and $g_i'^*(1 \otimes Sq^{2,1}i_5) = 0$ for $i = 1, 2$, and $g_i'^*(\gamma(2, 1)) = 0$ or u^8 in $H^*(RP^8; Z_2)$ according as $i = 1$ or 2. Therefore, by Proposition 3.4 there are liftings f_i of g_i to X_7 which satisfy $[f_1 \circ \nu] = (\text{odd}, 0, 0)$ and $[f_2 \circ \nu] = (\text{odd}, 0, \text{odd})$.

Next we consider the composition

$$RP^7 \xrightarrow{u^5} K(Z_2, 5) \xrightarrow{k} K(Z, 4) \times K(Z_2, 5)$$

where k is the inclusion into the second factor. Following the same procedure, we obtain liftings f_5 and f_6 to X_7 which satisfy $[f_5 \circ \nu] = (0, \text{odd}, 0)$ and $[f_6 \circ \nu] = (0, \text{odd}, \text{odd})$.

Next we consider the map

$$h \times u^5 : RP^7 \to K(Z, 4) \times K(Z_2, 5)$$

where h is non-trivial. Then $h \times u^5$ has liftings f_5 and f_6 to X_7 such that $[f_5 \circ \nu] = (\text{odd}, \text{odd}, 0)$ and $[f_6 \circ \nu] = (\text{odd}, \text{odd}, \text{odd})$.

Finally, we consider the composition

$$RP^7 \xrightarrow{u^5} K(Z_2, 5) \to X_5.$$

This lifts to X_7 (the obstruction is $Sq^2u^5 = 0$) and since its extension to RP^8 pulls the class $\gamma(2, 1)$ back to u^8, the lifting f_7 can be chosen so that it satisfies $[f_7 \circ \nu] = (0, 0, \text{odd})$.

The maps $f_i \circ \nu$, $i = 1, \ldots, 7$, together with the trivial map yield the eight types of projective homotopy classes we require. We therefore conclude that $\pi_7^{\text{Proj}}(X_7) = \pi_7(X_7)$.

This shows that only the 2-primary part of $V_{7,3}$ is projective. To show that all of $\pi_7(V_{7,3}) = Z + Z_4 + Z_{12}$ is projective (including the 3-primary part and sums of 3-primary classes with 2-primary classes), we consider a Postnikov system for $V_{7,3}$ (not mod 2). The last stage will look like

$$\begin{array}{ccc} K(Z + Z_4 + Z_4 + Z_3) & \to & W_7 \\ & & \downarrow \\ & & W_6 \end{array}$$

Since $2 \cdot Z_3 = Z_3$, any class in the 3-primary part comes from a projective class of the fibre. By Proposition 3.2 these classes add projectively to other projective classes, and so $\pi_7^{\text{Proj}}(V_{7,3}) = \pi_7(V_{7,3})$.

We now prove Theorem 1.1 for $X = V_{7,4}$. Again, the object is to construct projective classes (a, b) in $\pi_7(Y_7) = Z + Z_4$ where both a and b are specified to be either zero or an odd number. Then it will follow from Propositions 3.2 and 3.3 that $\pi_7(Y_7)$ consists entirely of projective classes.

First, we consider the map

$$u^3 : RP^7 \to K(Z_2, 3).$$

By Proposition 3.5, u^3 has liftings h_1 and h_2 to Y_5 which satisfy $h_1'^*(\gamma(2, 1)) = 0$ and $h_2'^*(\gamma(2, 1)) \neq 0$. We can then lift h_1 and h_2 to Y_6, and by Proposition 3.4 these maps will have liftings f_1 and f_2 to Y_7 which will satisfy $[f_1 \circ \nu] = (\text{odd}, 0)$ and $[f_2 \circ \nu] = (\text{odd}, \text{odd})$.

Secondly, we consider the composition

$$RP^7 \xrightarrow{u^5} K(Z_2, 5) \to Y_5$$

which lifts to Y_7 (the obstruction is $Sq^2 u^5 = 0$). Since the extension of this composition to RP^8 pulls the class $\gamma(2, 1)$ back to u^8, the lifting f_3 can be chosen so that $[f_3 \circ \nu] = (0, \text{odd})$.

The maps $f_i \circ \nu$, $i = 1, 2, 3$, together with the trivial map give the four types of projective classes specified above. We conclude that $\pi_7^{\text{Proj}}(Y_7) = \pi_7(Y_7)$, and since $\pi_7(V_{7,4})$ is only 2-primary, it follows that $\pi_7^{\text{Proj}}(V_{7,4}) = \pi_7(V_{7,4})$.

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