Monotone operators and dentability

S.P. Fitzpatrick

P.S. Kenderov has shown that every monotone operator on an Asplund Banach space is continuous on a dense \( G_\delta \) subset of the interior of its domain. We prove a general result which yields as special cases both Kenderov's Theorem and a theorem of Collier on the Fréchet differentiability of weak* lower semicontinuous convex functions.

Let \( E \) be a real Banach space with dual \( E^* \). A multivalued mapping \( T : E \to E^* \) is called a monotone operator on \( E \) if \( \langle x^*-y^*, x-y \rangle \geq 0 \) whenever \( x^* \in Tx \) and \( y^* \in Ty \). It is called maximal monotone if, in addition, its graph

\[
G(T) = \{ (x, x^*) : x \in E, x^* \in Tx \}
\]

is not properly contained in the graph of any other monotone operator on \( E \).

We say that a monotone operator \( T \) on \( E \) is locally bounded at \( x \in E \) if there is a neighborhood \( U \) of \( x \) such that \( T(U) = U(Ty : y \in U) \) is a bounded subset of \( E^* \). We define the domain of \( T \) to be \( D(T) = \{ x \in E : Tx \neq \emptyset \} \), and we say that \( T \) is continuous at a point \( x \in D(T) \) if, whenever \( x_n \to x \), \( x^*_n \in Tx_n \), and \( x^* \in Tx \), we have \( \|x^*_n - x^*\| \to 0 \). This is the same as being single-valued and norm-to-norm upper semicontinuous at \( x \), where \( T \) is said to be upper semi-continuous at \( x \in E \) if given any neighborhood \( V \) of \( 0 \) in \( E^* \), there is a neighborhood

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of $x$ such that $T(U) \subset Tx + V$. (We will always be using the norm topology in $E$ and the norm or the weak* topology in $E^*$.)

Let $F$ be a norm closed subspace of $E^*$. Then an $F$-slice of a nonempty subset $C$ of $E$ is a set of the form

$$S(f, \alpha, C) = \{x \in C : \langle f, x \rangle > M(f, C) - \alpha\} ,$$

where $M(f, C) = \sup\{\langle f, x \rangle : x \in C\}$, $\alpha > 0$, and $f \in F$. We say $C$ is $F$-dentable if for every $\varepsilon > 0$ there is an $F$-slice of $C$ of diameter less than $\varepsilon$. There are only two choices for $F$ of interest to us; when $F$ is all of $E^*$ (in which case an $F$-slice is simply called a "slice"), or where $E$ is itself a dual space and $F$ is its predual; that is, when $E = F^*$, so $F \subset F^{**} = E^*$ (and $F$-slices are called "weak* slices"). In these two cases, we use the terms "dentability" and "weak* dentability". A space $E$ has the Radon-Nikodym property if every bounded subset of $E$ is dentable.

Let $f : E \to (-\infty, \infty]$ be a lower semicontinuous convex function. The subdifferential $\partial f$ of $f$ is defined by setting, for $x \in E$,

$$\partial f(x) = \{x^* \in E^* : \langle x^*, y-x \rangle \leq f(y) - f(x) \text{ for all } y \in E\} .$$

It is easy to see that $\partial f$ is a monotone operator. Minty [9] showed that the subdifferential of a continuous convex function is maximal monotone, and Rockafellar [14] showed the same for arbitrary proper lower semicontinuous functions (see also Taylor [15]). Note that if $f$ is continuous on an open convex set $C \subset E$, then by the separation theorem $C \subset \text{int } D(\partial f)$.

We call $E$ an Asplund space if every lower semicontinuous convex function on $E$ is Fréchet differentiable on a dense $G_\delta$ subset of the set of points where it is continuous. Asplund [1] showed that if $E$ is an Asplund space, then every bounded subset of $E^*$ is weak* dentable, and Namioka and Phelps [10] proved the converse. The subdifferential $\partial f$ of a convex function $f$ is continuous at a point $x$ of its domain if and only if $f$ is Fréchet differentiable at $x$ (see Asplund and Rockafellar [2]), so the following result of Kenderov [7] generalizes that of Namioka and Phelps.

**Theorem 1** (Kenderov). If $E$ is an Asplund space, then every mono-
Monotone operator \( T \) on \( E \) is continuous on a dense \( G_\delta \) subset of \( \text{int} \, D(T) \).

Special cases of this result were obtained earlier by Robert [12], Fitzpatrick [5], and Kenderov and Robert [8]. We will prove the following result, which yields Theorem 1 when \( F = E^* \) and \( C = \text{int} \, D(T) \).

**THEOREM 2.** Let \( F \) be a closed subspace of \( E^* \) such that every bounded subset of \( F \) is \( E \)-dentable. Let \( T \) be a monotone operator on \( E \) and \( C \) an open subset of \( D(T) \). If \( Tx \cap F \neq \emptyset \) for \( x \) in a dense subset of \( C \), then \( T \) is continuous on a dense \( G_\delta \) subset of \( C \).

Note that if \( f : E^* \to (-\infty, \infty] \) is a weak* lower semicontinuous convex function then \( \partial f(x) \cap E \) is nonempty for a dense set of \( x \) in \( C \) where \( C \) equals the domain of (norm) continuity of \( f \). (This follows from Phelps [11], or Brøndsted and Rockafellar [3].) Applying Theorem 2 to \( T = \partial f \) with \( E \) considered as a subspace of the dual of \( E^* \), we get the following result.

**COROLLARY 3** (Collier [4]). Let \( E \) have the Radon-Nikodym property and let \( f \) be a weak* lower semicontinuous convex function on \( E^* \). Then \( f \) is Fréchet differentiable on a dense \( G_\delta \) subset of its domain of continuity.

To prove Theorem 2 we need some preliminary results about maximal monotone operators.

**PROPOSITION 4** (Rockafellar [13]). Let \( T \) be a maximal monotone operator on \( E \) with \( \text{int} \,(\text{co} \, D(T)) \neq \emptyset \). Then \( \text{int} \, D(T) \) is convex, \( D(T) = \text{int} \, D(T) \), and \( T \) is locally bounded at each point of \( \text{int} \, D(T) \).

The next result follows readily from local boundedness.

**PROPOSITION 5** (Kenderov [6]). If \( T \) is a maximal monotone operator then \( T \) is norm-to-weak* upper semicontinuous at each point of \( \text{int} \, D(T) \).

Now with \( F \subset E^* \), \( T \), and \( C \) as in Theorem 2, we can assume without loss of generality that \( T \) is maximal monotone. We write \( \overline{\text{co}} \, A \) for the weak* closed convex hull of a subset \( A \) of \( E^* \). Define \( T_F \) by

\[
T_F(x) = \bigcap_{\epsilon > 0} \overline{\text{co}} \left( \{T[b(x, \epsilon)] \cap F \} \subset E^* \mid x \in C \right),
\]

where \( b(x, \epsilon) \) denotes the closed ball \( \{y \in E : \|y-x\| \leq \epsilon \} \). It is clear
from Proposition 4 and our assumptions on $T$ that $T_p(x)$ is a nonempty weak* compact convex subset of $E^*$ for all $x \in C$.

**Lemma 6.** The set valued map $T_p$ is monotone and $T_p x = Tx$ for each $x$ in the open subset $C$ of $D(T)$.

**Proof.** Let $x \in C$ and suppose $x^* \in T_p x \setminus Tx$. By maximality of $T$ there is $y \in E$ and $y^* \in Ty$ such that $(x^* - y^*, x - y) = \delta < 0$. By definition of $T_p x$, for each $n \geq 1$ we can find $x_n \in B(x, n^{-1})$ and $x_n^* \in Tx_n \cap F$ such that $(x_n^* - y^*, x_n - y) < \delta/2$. By local boundedness of $T$ at $x$, for large $n$ we have $(x_n^* - y^*, x_n - y) < \delta/3 < 0$, which contradicts the monotonicity of $T$. So $T_p x \subseteq Tx$; hence $T_p$ is monotone.

Now suppose $x \in C$ and $x^* \in Tx \setminus T_p x$. By the separation theorem, there is $z \in E$, $\|z\| = 1$, such that $(x^*, z) > M\{z, T_p x\}$. So there is $\epsilon > 0$ such that $B(x, \epsilon) \subseteq C$ and $(x^*, z) > M\{z, \overline{co}(T[B(x, \epsilon)] \cap F)\}$.

Now if $w^* \in T_p \{x + (\epsilon/2)z\}$, then monotonicity of $T$ yields

$$0 \leq (x^* - z^*, (x + (\epsilon/2)z) - x) = (w^* - z^*, (\epsilon/2)z).$$

Since $B(x + (\epsilon/2)z, \epsilon/2) \subseteq B(x, \epsilon)$, any such $w^*$ is in $\overline{co}(T[B(x, \epsilon)] \cap F)$, which contradicts

$$(w^*, z) \geq (x^*, z) > M\{z, \overline{co}(T[B(x, \epsilon)] \cap F)\}.$$  

Hence $T_p x = Tx$ for all $x \in C$.

Now we use a modification of the main idea of Kenderov's proof [7] to complete the proof of Theorem 2.

Let $V_n = \bigcup \text{int}\{x \in C : Tx \subseteq B(y^*, n^{-1})\}$ and let $G = \cap \bigcap V_n$. Clearly $G$ is the set of points of $C$ where $T$ is continuous, and $V_n$ is open for each $n$; so we only need to show that each $V_n$ is dense in $G$.

Suppose $x \in C$ and $\epsilon > 0$. By Proposition 4, there is an open convex neighborhood $U$ of $x$, $U \subseteq B(x, \epsilon) \subseteq C$, such that $T(U)$ is
bounded. Let \( A = T(U) \cap F \), which is by assumption nonempty and bounded. It follows that there is a slice \( S = S(z, \alpha, A) \) of \( A \) with diameter less than \((2\varepsilon)^{-1}\) and \( z \in E \). Let \( v^* \in S \), \( v^* \in T v \) with \( v \in U \). For sufficiently small \( \beta > 0 \), the point \( w = v + \beta z \) is in \( U \). If \( w^* \in Tw \), we have
\[
0 \leq \langle w^*-v^*, w-v \rangle = \beta \langle w^*-v^*, z \rangle,
\]
so that \( \langle w^*, z \rangle \geq \langle v^*, z \rangle > M(z, A) - \alpha \). By Proposition 5, there is an open neighborhood \( W \) of \( w \), \( W \subset U \), such that
\[
T(W) \subset Tw + \{ y^* \in E^* : |\langle y^*, z \rangle| < \langle v^*, z \rangle - M(z, A) + \alpha \} ;
\]
so \( T(W) \cap F \subset S(z, \alpha, A) \). Since \( S \) is contained in some closed ball \( B(y^*, n^{-1}) \) with \( y^* \in F \), the set \( T_p(W) \) is contained in \( B(y^*, n^{-1}) \) (since the ball is weak* closed). By Lemma 6, \( T(W) \subset B(y^*, n^{-1}) \), so \( w \in V_n \). Hence \( V_n \) is dense, which completes the proof.

References


Department of Mathematics,  
University of Washington,  
Seattle,  
Washington,  
USA.