ON SCHOENEBERG'S THEOREM

by C. MACLACHLAN

(Received 3 May, 1972; revised 15 August 1972)

Let S be a compact Riemann surface of genus $g \geqq 2$ and σ an automorphism (conformal self-homeomorphism) of S of order n. Let $S^* = S/\langle\sigma\rangle$ have genus g^*. In [5], Schoeneberg gave a sufficient condition that a fixed point $P \in S$ of σ should be a Weierstrass point of S, i.e., that S should support a function that has a pole of order less than or equal to g at P and is elsewhere regular.

THEOREM (Schoeneberg). *P is a Weierstrass point of S provided that $g^* \neq [g/n]$. ($[x]$ denotes the integral part of x.)*

By the uniformization theorem, S can be represented as a quotient surface U/K, where U denotes the upper half-plane and K a Fuchsian group isomorphic to the fundamental group of S. Furthermore, G will be a (finite) group of automorphisms of U/K if and only if $G \cong \Gamma/K$, where Γ is a Fuchsian group with compact quotient space U/Γ. Such groups are known to have a presentation of the following form:

$$\left.\begin{aligned}&\text{Generators: } x_1, x_2, \ldots, x_r, a_1, b_1, \ldots, a_{g^*}, b_{g^*}.\\ &\text{Relations:}\quad x_i^{m_i} = 1\ (i = 1, 2, \ldots, r),\ \prod_{j=1}^{r} x_j \prod_{k=1}^{g^*} [a_k, b_k] = 1.\end{aligned}\right\} \tag{1}$$

If the presentation is (1), the group is said to have signature $(g^*; m_1, \ldots, m_r)$. Such a group has a fundamental polygon F_Γ in U with hyperbolic area

$$\mu(F_\Gamma) = 2(g^* - 1) + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right). \tag{2}$$

If K is of index n in Γ, then

$$n\mu(F_\Gamma) = \mu(F_K), \tag{3}$$

and combining (2) and (3) gives a form of the Riemann-Hurwitz relation.

Here we sharpen Schoeneberg's condition to criteria on the signature of the corresponding Fuchsian group. Our method uses results of Lewittes [3] which we have employed before [4]. (For other applications of similar methods, see [2].) In the proof of the theorem below we shall use the notation and results of [4].

THEOREM. *Let σ be an automorphism of order n of a compact Riemann surface $S = U/K$ of genus $g \geqq 2$. Let Γ be a Fuchsian group such that $\langle\sigma\rangle \cong \Gamma/K$. Let σ have a fixed point $P \in S$. If P is not a Weierstrass point, then Γ has signature of one of the following forms:*

(i) $\left(\dfrac{g}{n};\ n, n\right)$,

(ii) $\left(\dfrac{g-(n-1)}{n};\ n, n, n, n\right)$,

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(iii) $(g^*;\ n, m_1, m_2)$,

where $2ng^* = 2g-1-n+\dfrac{m_1+m_2}{(m_1,m_2)}$ *and the least common multiple of* m_1, m_2 *is* n.

Proof. Let P have gap sequence $\{\gamma_1, \gamma_2, \ldots, \gamma_g\}$ and choose a local parameter z at P such that locally σ^{-1} is $z \to \varepsilon z$, where ε is a primitive nth root of unity. Letting σ act on the g-dimensional space of abelian differentials on S of the first kind, one obtains, with respect to a suitable basis, a diagonal representation of σ with entries $\{\varepsilon^{\gamma_1}, \varepsilon^{\gamma_2}, \ldots, \varepsilon^{\gamma_g}\}$ [**3**].

Assume that Γ has the presentation (1). Let $\pi : \Gamma \to Z_n$ be the natural projection combined with the isomorphism $\sigma \leftrightarrow 1$, where we write elements of Z_n as residues modulo n. Since the kernel of π contains no elements of finite order, each m_i divides n. Assume that $m_1 = n$ and adjust π so that, locally at P, σ^{-1} is $z \to \varepsilon z$ where $\varepsilon = \exp[(2\pi/n)i]$. Now suppose that $\pi(x_\mu) = \xi_\mu\ (\mu = 1, 2, \ldots, r)$. Then, if N_ν denotes the multiplicity of $\exp[2\pi\nu/n)i]$ as an eigenvalue of σ, we have

$$\left.\begin{aligned} N_0 &= g^*, \\ N_\nu &= g^*-1+\sum_{\substack{\mu=1 \\ \nu\,.\,\xi_\mu \not\equiv 0 \,(\mathrm{mod}\, n)}}^{r} \left(1-\left\langle \frac{\nu\,.\,\xi_\mu}{n}\right\rangle\right), \end{aligned}\right\} \tag{4}$$

where $\langle x \rangle$ denotes the fractional part of x. (See [**4**].)

As already noted above, $\xi_1 = 1$. Also, from the relations (1), $\sum_{i=1}^{r} \xi_i \equiv 0 \,(\mathrm{mod}\, n)$. Let $\sum_{i=1}^{r} \xi_i = an$. Then, from (4),

$$N_1 = g^*-1+r-a, \quad N_{n-1} = g^*-1+a.$$

Now suppose that P is not a Weierstrass point. Then σ has eigenvalues $\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^g$ where $\varepsilon = \exp[(2\pi/n)i]$. Thus, writing $g = nk+l$, where $0 \leqq l < n$, we have $N_0 = k$, $N_1 = k+1$, $N_2 = k+1, \ldots, N_l = k+1$, $N_{l+1} = k, \ldots, N_{n-1} = k$. Hence $g^* = k$ and we consider three cases.

(i) $l = 0$. Then $g^* = \dfrac{g}{n}$. $N_1 = N_{n-1} = g^*$. Thus $a = 1$ and $r = 2$ and, from the Riemann-Hurwitz relation, $m_2 = n$.

(ii) $l = n-1$. Then $g^* = \dfrac{g-(n-1)}{n}$. $N_1 = N_{n-1} = g^*+1$. Thus $a = 2$, $r = 4$ and, from the Riemann-Hurwitz relation, $m_2 = m_3 = m_4 = n$.

(iii) $l \neq 0$, $n-1$. $N_1 = g^*+1$, $N_{n-1} = g^*$. Thus $a = 1$, $r = 3$. By the Riemann-Hurwitz relation, we have

$$\frac{2(g-1)}{n} = 2(g^*-1)+\left(1-\frac{1}{n}\right)+\left(1-\frac{1}{m_2}\right)+\left(1-\frac{1}{m_3}\right).$$

But the least common multiple of m_2 and m_3 must be n [**1**]. So $\dfrac{n}{m_2} = \dfrac{m_2}{(m_2, m_3)}$ and (iii) follows.

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Finally, we note that the conditions given in the theorem are, with a small number of exceptions for low values of g and n, not generally necessary for P to be a Weierstrass point.

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UNIVERSITY OF ABERDEEN