WEYL’S THEOREM HOLDS FOR $p$-HYPONORMAL OPERATORS

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1. Introduction. Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. Let $K(\mathcal{H})$ be the algebra of all compact operators of $B(\mathcal{H})$. For an operator $T \in B(\mathcal{H})$, let $\sigma(T)$, $\sigma_p(T)$, $\sigma_\pi(T)$ and $\pi_{oo}(T)$ denote the spectrum, the point spectrum, the approximate point spectrum and the set of all isolated eigenvalues of finite multiplicity of $T$, respectively. We denote the kernel and the range of an operator $T$ by $\ker(T)$ and $R(T)$, respectively. For a subset $\mathcal{Y}$ of $\mathcal{H}$, the norm closure of $\mathcal{Y}$ is denoted by $\overline{\mathcal{Y}}$. The Weyl spectrum $\omega(T)$ of $T \in B(\mathcal{H})$ is defined as the set

$$\omega(T) = \bigcap_{K \in K(\mathcal{H})} \sigma(T + K).$$

We say that Weyl’s theorem holds for $T$ if the following equality holds;

$$\omega(T) = \sigma(T) - \pi_{oo}(T).$$

An operator $T \in B(\mathcal{H})$ is said to be $p$-hyponormal if $(T^*T)^p \succeq (TT^*)^p$. Especially, when $p = 1$ and $p = \frac{1}{2}$, $T$ is called hyponormal and semi-hyponormal, respectively. It is well known that a $p$-hyponormal operator is $q$-hyponormal for $q \leq p$ by Löwner’s Theorem. In [8], Coburn showed that Weyl’s theorem holds for hyponormal operators. In this paper, we shall prove the following results.

**Theorem 0.** Let $T$ be a $p$-hyponormal operator on $\mathcal{H}$ where $0 < p < 1$. Then Weyl’s theorem holds for $T$.

2. Proof of Theorem 0. Throughout this section, let $p$ satisfy $0 < p < 1$. First in [2] Baxley proved the following result.

**Theorem A** (Lemma 3 of [2]). Let $T \in B(\mathcal{H})$. Suppose that $T$ satisfies the following condition $C-1$.

$C-1.$ If $\{\lambda_n\}$ is a infinite sequence of distinct points of the set of eigenvalues of finite multiplicity of $T$ and $\{x_n\}$ is any sequence of corresponding normalized eigenvectors, then the sequence $\{x_n\}$ does not converge.

Then

$$\sigma(T) - \pi_{oo}(T) \subset \omega(T).$$

Chô and Huruya proved the following result.

**Theorem B** (Corollary 5 of [5]). Let $T$ be $p$-hyponormal. Let $\alpha, \beta \in \sigma_p(T)$ where $\alpha \neq \beta$. If $x$ and $y$ are eigenvectors of $\alpha$ and $\beta$, respectively, then $(x, y) = 0$.

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By Theorem B, it follows that if $T$ is $p$-hyponormal, then $T$ satisfies C-1. Hence it is clear that if $T$ is $p$-hyponormal, then

$$\sigma(T) - \pi_{oo}(T) \subseteq \omega(T).$$

For the proof of the converse inclusion relation, we shall prove the following result.

**Theorem 1.** Let $T$ be $p$-hyponormal. If $\lambda$ is an isolated point of $\sigma(T)$, then $\lambda \in \sigma_p(T)$.

Since the theorem holds for $\lambda \neq 0$, by Theorem 1 of [7], we need only prove the case $\lambda = 0$.

For this proof, we need the Aluthge transform (cf. [1]). Let $U |T|$ be the polar decomposition of $T \in B(\mathcal{H})$. Then Aluthge introduced the transform

$$T \rightarrow \tilde{T} = |T|^{1/2} U |T|^{1/2},$$

and proved the following result.

**Theorem C (Theorems 1 and 2 of [1]).** Let $T$ be $p$-hyponormal. Then $\tilde{T} = |T|^{1/2} U |T|^{1/2}$ is $(p + \frac{1}{2})$-hyponormal.

Though the operator $U$ in Aluthge's paper is unitary, it is easy to check that Theorem C holds for any $p$-hyponormal operator.

We need some further results.

**Lemma 1.** Let $T = U |T|$ be $p$-hyponormal. Then $\sigma(T) = \sigma(\tilde{T})$, where $\tilde{T} = |T|^{1/2} U |T|^{1/2}$.

**Proof.** To see this write $T = (U |T|^{1/2}) |T|^{1/2}$ and consider separately $\lambda = 0$ and $\lambda \neq 0$.

**Lemma 2.** Let $T$ be semi-hyponormal. If $0$ is an isolated point of $\sigma(T)$, then $0 \in \sigma_p(T)$.

**Proof.** Let $T = U |T|$ be the polar decomposition of $T$ and $\tilde{T} = |T|^{1/2} U |T|^{1/2}$. Since $0$ belongs to the boundary of $\sigma(T)$, by Lemma 1 it follows $0 \in \sigma(\tilde{T}) = \sigma(T)$. Therefore, $0$ is an isolated point of $\sigma(\tilde{T})$. Since, by Theorem C, $\tilde{T}$ is hyponormal, from a Stampfli result (Theorem 2 of [10]) it follows that $0$ is an eigenvalue of $\tilde{T}$. Hence there exists a nonzero $x_0 \in \mathcal{H}$ such that $\tilde{T}x_0 = 0$. Since $|T|^{1/2} U |T|^{1/2} x_0 = 0$, we have $U |T|^{1/2} x_0 \in \ker(|T|^{1/2})$. Since, by Lemma 1 of [5], $\ker(T) \subseteq \ker(T^*)$, it follows that

$$T^* (U |T|^{1/2} x_0) = |T|^{3/2} x_0 = 0.$$

Hence $|T| x_0 = 0$. Therefore we have $0 \in \sigma_p(T)$.

**Proof of Theorem 1 for $\lambda = 0$ and $0 < p < \frac{1}{2}$.** Let $T = U |T|$ be the polar decomposition of $T$ and $\tilde{T} = |T|^{1/2} U |T|^{1/2}$. By Lemma 1, it follows that $0 \in \sigma(\tilde{T})$ and $0$ is an isolated point of $\sigma(\tilde{T})$. Since, by Theorem C, $\tilde{T}$ is semi-hyponormal, by Lemma 2 it follows that $0 \in \sigma_p(\tilde{T})$. Hence also it follows that $0 \in \sigma_p(T)$ on the analogy of the proof of Lemma 2.

**Proof of the inclusion relation.** $\omega(T) \subseteq \sigma(T) - \pi_{oo}(T)$.

Let $\lambda \in \pi_{oo}(T)$. By Theorem 4 of [5] or Theorem 2 of [9], we have

$$\ker(T - \lambda) \supseteq \ker((T - \lambda)^*) = (R(T - \lambda))^\perp.$$
Hence we have the following decomposition of $T - \lambda$:

$$T - \lambda = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \text{ on } \ker(T - \lambda) \oplus \overline{R((T - \lambda)^*)}.$$  

Since

$$T = \begin{pmatrix} \lambda & 0 \\ 0 & S + \lambda \end{pmatrix},$$

$S + \lambda$ is a $p$-hyponormal operator on $\overline{R((T - \lambda)^*)}$. If $\lambda \in \sigma(S + \lambda)$, by Theorem 1 we have $\lambda \in \sigma_p(S + \lambda)$ because $\lambda$ is an isolated point of $\sigma(S + \lambda)$. This is a contradiction. Hence $\lambda \notin \sigma(S + \lambda)$. Therefore $0 \notin \sigma(S)$. Let

$$K = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$  

Then $K \in \mathcal{H}(\mathcal{H})$ and

$$T + K - \lambda = \begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}$$

is an invertible operator. Therefore $\lambda \notin \omega(T)$. Hence we have

$$\omega(T) \subset \sigma(T) - \pi_{oo}(T)$$

and the proof of the theorem is complete.

3. Application.

**Corollary 1.** Let $T$ be $p$-hyponormal. If $\pi_{oo}(T) = \emptyset$, then for every $K \in \mathcal{H}(\mathcal{H})$

$$\|T\| \leq \|T + K\|.$$  

**Proof.** By Corollary 10 of [5], we have that $r(T) = \|T\|$, where $r(T)$ is the spectral radius of $T$. Hence from Theorem 1 it follows that $\|T\| \leq \|T + K\|$ for every $K \in \mathcal{H}(\mathcal{H})$.

**Corollary 2.** Let $T$ be $p$-hyponormal. Then there exist orthogonal reducing subspaces $\mathcal{M}$ and $\mathcal{N}$ for $T$ such that $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$, $T|_\mathcal{M}$ is a normal operator on $\mathcal{M}$ and

$$\omega(T|_\mathcal{N}) = \sigma(T|_\mathcal{N}).$$

**Proof.** For $\lambda \in \sigma_p(T)$, let

$$\mathcal{M}_\lambda = \{x \mid Tx = \lambda x\}.$$  

Then, by Theorem 4 of [5], $\mathcal{M}_\lambda$ is a reducing subspace for $T$. Let

$$\mathcal{M} = \bigoplus_{\lambda \in \sigma_p(T)} \mathcal{M}_\lambda \text{ and } \mathcal{N} = \mathcal{M}^\perp.$$  

Then $\mathcal{M}$ reduces $T$ and $T|_\mathcal{M}$ is normal. Let $S = T|_\mathcal{N}$. Then $S$ is a $p$-hyponormal operator on $\mathcal{N}$. By Theorem 0, Weyl's theorem holds for $S$. Since $\pi_{oo}(S) = \emptyset$, it follows that $\omega(S) = \sigma(S)$. 

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Corollary 3. Let $T$ be $p$-hyponormal. Then

$$
\|T^*T^p - (TT^*)^p\| \leq \frac{p}{\pi} \int_{\sigma(T)} r^{2p-1} \, dr \, d\theta.
$$

Proof. Let $\mu$ be planar Lebesgue measure. Then we have $\mu(\pi_{\infty}(T)) = 0$. Hence the result follows from Theorem 5 of [6].

Corollary 4. Let $T$ be $p$-hyponormal. Then, for every $K \in \mathcal{H}(\mathcal{H})$,

$$
\|(T^*T)^p - (TT^*)^p\| \leq \frac{p}{\pi} \int_{\sigma(T+K)} r^{2p-1} \, dr \, d\theta.
$$

Proof. Since $\omega(T) \subset \sigma(T + K)$ for every $K \in \mathcal{H}(\mathcal{H})$, the result follows from Corollary 3.

REFERENCES