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BLASCHKE PRODUCTS IN LIPSCHITZ SPACES

MIROLJUB JEVTIĆ

Matematički Fakultet, Univerzitet u Beogradu, PP 550, 11000 Belgrade, Serbia (jevtic@matf.bg.ac.yu)

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Abstract We study the membership of Blaschke products in Lipschitz spaces, especially for interpolating Blaschke products and for those whose zeros lie in a Stolz angle. We prove several theorems that complement or extend the earlier works of Ahern and the author.

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1. Introduction

One of the central questions about Blaschke products is that of their membership in some classical function spaces. In this paper we study their membership in Lipschitz spaces. We prove several theorems that complement or extend the earlier works of Ahern and the author.

If f is a function analytic in the unit disc D, $f \in H(D)$, and $0 \leqslant r < 1$, we set

$$M_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p \,dt\right)^{1/p}, \quad 0 < p < \infty,$$

and

$$M_\infty(r, f) = \sup_{|z|=r} |f(z)|.$$

For $0 < p \leqslant \infty$, the Hardy space H^p is the set of all functions $f \in H(D)$ satisfying

$$\|f\|_{H^p} := \sup_{0<r<1} M_p(r, f) < \infty.$$

The weighted Bergman space $A^{p,\alpha}$, $0 < p < \infty$, $-1 < \alpha < \infty$, is the space of all functions $f \in H(D)$ such that

$$\|f\|_{A^{p,\alpha}} := \left(\int_D |f(z)|^p (1 - |z|)^\alpha \,dA(z)\right)^{1/p} < \infty,$$

where $\mathrm{d}A(z) = \mathrm{d}x\,\mathrm{d}y$ denotes the Lebesgue area measure in D. We mention [**3**, **5**] as general references for the theory of Hardy spaces and [**4**, **9**] for the theory of Bergman spaces.

If $f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k$ is analytic in D and $\alpha > 0$, we define the fractional derivative of f of order α by

$$D^{\alpha} f(z) = \sum_{k=0}^{\infty} (k+1)^{\alpha} \hat{f}(k) z^k.$$

An inner function in the unit disc D is a bounded analytic function whose radial limits have modulus 1 almost everywhere. It can be factored as a product of a Blaschke product and a singular inner function. A Blaschke product B has the form

$$B(z) = \prod_{k=1}^{\infty} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \bar{z}_k z},$$

where $\sum_{k=1}^{\infty}(1 - |z_k|) < \infty$. A singular inner function S has the representation

$$S(z) = \exp\biggl\{ - \int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}\theta} + z}{\mathrm{e}^{\mathrm{i}\theta} - z} \, \mathrm{d}\mu(\theta) \biggr\},$$

where μ is a positive singular Borel measure on the interval $[0, 2\pi]$ (see [**3**, Chapter II]).

We denote by $\{z_k\}$ the zero sequence (each zero according to its multiplicity and $|z_k| \leqslant |z_{k+1}|$, $k = 1, 2, \dots$) of an inner function φ and by $\{z_k(a)\}$ the zero sequence of its Möbius transform

$$\varphi_a(z) = \frac{\varphi(z) - a}{1 - \bar{a}\varphi(z)}, \quad a \in D.$$

Let

$$\nu_n = \operatorname{card}\{k : 2^{-n-1} < d_k \leqslant 2^{-n}\}, \qquad n = 0, 1, \dots,$$

and

$$\nu_n(a) = \operatorname{card}\{k : 2^{-n-1} < d_k(a) \leqslant 2^{-n}\}, \quad n = 0, 1, \dots,$$

where $d_k = 1 - |z_k|$ and $d_k(a) = 1 - |z_k(a)|$.

We are now ready to state our first result.

Theorem 1.1. *Suppose that B is a Blaschke product. Let $\{z_k\}$ be the zero sequence of B and let $d_k = 1 - |z_k|$, $k = 1, 2, \dots$. Then*

(i) *If $d_k = O(a^k)$ for some $0 < a < 1$, then*

$$M_p^p(r, D^{1/p}B) = O\biggl(\log \frac{1}{1-r}\biggr), \quad 0 < p < \infty.$$

(ii) *If*

$$M_p^p(r, D^{1/p}B) = O\biggl(\log \frac{1}{1-r}\biggr), \quad 0 < p \leqslant 2,$$

then $d_k = O(a^k)$ for some $0 < a < 1$.

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We do not know whether statement (ii) remains true for $2 < p < \infty$.

If φ is an inner function, let

$$\Delta(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi}(1-|\varphi(re^{i\theta})|^2)\,d\theta, \quad 0 \leqslant r < 1.$$

A motivation for our Theorem 1.1 is the following.

Theorem A (Ahern [1, Theorem 3.3, p. 329]). *Let B be a Blaschke product with zeros $\{z_k\}$ and let $d_k = 1-|z_k|$, $k = 1,2,\dots$. Then*

$$\Delta(r,B) = O\biggl((1-r)\log\frac{1}{1-r}\biggr)$$

if and only if $d_k = O(a^k)$, for some $0 < a < 1$.

By the Schwarz–Pick lemma we have

$$M_1(r,B') = \frac{1}{2\pi}\int_0^{2\pi}|B'(re^{i\theta})|\,d\theta \leqslant \frac{1}{2\pi}\int_0^{2\pi}\frac{1-|B(re^{i\theta})|^2}{1-r}\,d\theta = \frac{\Delta(r,B)}{1-r}.$$

Hence, if

$$\Delta(r,B) = O\biggl((1-r)\log\frac{1}{1-r}\biggr),$$

then

$$M_1(r,B') = O\biggl(\log\frac{1}{1-r}\biggr).$$

On the other hand, the inequality

$$1-|B(re^{i\theta})| \leqslant \int_r^1 |B'(\rho e^{i\theta})|\,d\rho$$

is valid for almost all θ, since B is an inner function. Therefore, if

$$M_1(r,B') = O\biggl(\log\frac{1}{1-r}\biggr),$$

we find that

$$\Delta(r,B) = O\biggl((1-r)\log\frac{1}{1-r}\biggr).$$

It is easy to see that

$$M_1(r,B') = O\biggl(\log\frac{1}{1-r}\biggr)$$

if and only if

$$M_1(r,D^1B) = O\biggl(\log\frac{1}{1-r}\biggr).$$

Thus, our Theorem 1.1 is an extension of Theorem A.

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If $0 < p \leqslant \infty$ and $0 \leqslant \alpha < \infty$, then a function $f \in H(D)$ is said to belong to the Lipschitz space $\Lambda^{p,\alpha}$ if

$$\|f\|_{p,\alpha} = \sup_{0<r<1} (1-r) M_p(r, D^{1+\alpha} f) < \infty.$$

The subspace $\lambda^{p,\alpha}$ consists of those $f \in \Lambda^{p,\alpha}$ for which

$$\lim_{r\to 1} (1-r) M_p(r, D^{1+\alpha} f) = 0.$$

Theorem 1.2. *Let B be a Blaschke product with zero sequence $\{z_k\}$. Suppose that $\max(0, 1/p - 1) < \alpha < 1/p$. If*

$$\sup_n 2^{-n(1-\alpha p)} \nu_n < \infty, \tag{1.1}$$

then

$$B \in \Lambda^{p,\alpha}. \tag{1.2}$$

Moreover,

$$\|B\|_{p,\alpha}^p \leqslant C \sup_n 2^{-n(1-\alpha p)} \nu_n,$$

where C is a constant depending only on p and α.

Note that Verbitsky [**14**, Theorem 2 (a)] proved that the condition $\sup_n n d_n^{1-\alpha p} < \infty$, equivalent to (1.1), implies (1.2) if $1 \leqslant p < \infty$ and $0 < \alpha < 1/p$. The proof of Theorem 1.2 will be given in § 3. A similar argument gives the 'o' version of Theorem 1.2. Details will be omitted.

Theorem 1.3. *Let B be a Blaschke product with zero sequence $\{z_k\}$. Suppose that $\max(0, 1/p - 1) < \alpha < 1/p$. If*

$$\lim_{n\to\infty} 2^{-n(1-\alpha p)} \nu_n = 0, \tag{1.3}$$

or equivalently

$$\lim_{n\to\infty} n d_n^{1-\alpha p} = 0, \tag{1.4}$$

then

$$B \in \lambda^{p,\alpha}. \tag{1.5}$$

For interpolating Blaschke products, the condition in (1.1) (respectively, (1.3)) is also necessary in order that $B \in \Lambda^{p,\alpha}$, (respectively, $B \in \lambda^{p,\alpha}$).

Recall that a sequence $\{z_k\}$ in D is called interpolating sequence if

$$\inf_j \prod_{k: z_k \neq z_j} \left| \frac{z_k - z_j}{1 - \bar{z}_k z_j} \right| > 0.$$

A sequence $\{z_k\}$ is called a Carleson–Newman sequence if it can be represented as a union of finitely many interpolating sequences.

Theorem 1.4. *Let B be a Blaschke product with Carleson–Newman zero sequence $\{z_k\}$. Suppose that* $\max(0, 1/p-1) < \alpha < 1/p$. *If* $B \in \Lambda^{p,\alpha}$, *then* $\sup_n 2^{-n(1-\alpha p)}\nu_n < \infty$.

Verbitsky [**14**, Theorem 2 (b)] proved that if $B \in \Lambda^{p,\alpha}$, $1 \leqslant p < \infty$, $0 < \alpha < 1/p$, then $\sup_n n d_n^{1-\alpha p} < \infty$.

Theorem 1.5. *Let B be a Blaschke product with Carleson–Newman zero sequence $\{z_k\}$. Suppose that* $\max(0, 1/p-1) < \alpha < 1/p$. *If* $B \in \lambda^{p,\alpha}$, *then*

$$\lim_{n\to\infty} 2^{-n(1-\alpha p)}\nu_n = 0, \tag{1.6}$$

or, equivalently,

$$\lim_{n\to\infty} n d_n^{1-\alpha p} = 0. \tag{1.7}$$

Given $\xi \in \partial D$ (the unit circle) and $\sigma \in (1, \infty)$ we set

$$\Omega_\sigma(\xi) = \{z \in D : |1 - \bar{\xi} z| < \sigma(1 - |z|)\}.$$

Any such domain $\Omega_\sigma(\xi)$, $1 < \sigma < \infty$, will be called a Stolz angle with vertex ξ. The domain $\Omega_\sigma(1)$ will be simply denoted by Ω_σ.

We denote by $\mathcal{B}$ the class of all Blaschke products whose zeros $\{z_k\}$ lie in a fixed Stolz angle Ω_σ.

It is well known that $\mathcal{B} \subset \Lambda^{p,1/(2p)}$, $0 < p < \infty$ [**10**, Theorem 2.2]. In this paper we improve this result.

Theorem 1.6. $\mathcal{B} \subset \lambda^{p,1/(2p)}$ *for* $\frac{1}{2} < p < \infty$.

We do not know whether $\mathcal{B} \subset \lambda^{p,1/(2p)}$ for $0 < p \leqslant \frac{1}{2}$.

2. Blaschke products in $\Lambda^{p,1/p}$, $0 < p < \infty$, and in some related spaces

If f is an analytic function in D such that

$$M_p(r, D^{1+(1/p)} f) = O\left(\frac{1}{1-r}\right),$$

i.e. $f \in \Lambda^{p,1/p}$, then

$$M_p^p(r, D^{1/p} f) = O\left(\log \frac{1}{1-r}\right) \quad \text{if } 0 < p \leqslant 2$$

and

$$M_p^2(r, D^{1/p} f) = O\left(\log \frac{1}{1-r}\right) \quad \text{if } 2 \leqslant p < \infty$$

(see [**6**]). In this section we prove Theorem 1.1 and as an application we get a better estimate for Blaschke products. We also improve [**7**, Theorem 4].

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Proof of Theorem 1.1. (i) It follows from [**2**, Lemma 2.1] that

$$M_q^q(r, D^{1/q}B) \leqslant C M_p^p(r, D^{1/p}B), \quad 0 < p < q < \infty. \tag{2.1}$$

Thus, it is sufficient to show that

$$M_{p_n}^{p_n}(r, D^{1/p_n}B) = O\biggl(\log \frac{1}{1-r}\biggr) \tag{2.2}$$

for a sequence p_n going to zero. We take $p_n = 1/n$, $n = 1, 2, \dots$, and we will show that

$$M_{1/n}^{1/n}(r, B^{(n)}) = O\biggl(\log \frac{1}{1-r}\biggr). \tag{2.3}$$

It is easy to see that (2.2) is equivalent to (2.3) for $p_n = 1/n$.

From [**11**, Lemma 3.4] it follows that

$$|B^{(n)}(z)| \leqslant C \sum \prod_{j=2}^{n+1} f_j^{\alpha_j}(z),$$

where the sum is over the finite set of all n-tuples $(\alpha_2, \dots, \alpha_{n+1})$ of non-negative integers such that

$$\sum_{j=1}^{n} j\alpha_{j+1} = n \quad \text{and} \quad f_j(z) = \sum_{k=1}^{\infty} \frac{d_k}{|1 - \bar{z}_k z|^j}.$$

Since $1/n \leqslant 1$ we see that

$$|B^{(n)}(z)|^{1/n} \leqslant C \sum \prod_{j=2}^{n+1} f_j^{\alpha_j/n}(z)$$

and

$$M_{1/n}^{1/n}(r, B^{(n)}) \leqslant C \sum \int_0^{2\pi} \biggl(\prod_{j=2}^{n+1} f_j^{\alpha_j/n}(r\mathrm{e}^{\mathrm{i}\theta})\biggr) \,\mathrm{d}\theta.$$

We will show that each term of this finite sum is $O(\log 1/(1-r))$. For $j = 2, 3, \dots, n+1$, we let $\alpha_j \beta_j = n/(j-1)$. Then

$$\sum_{j=2}^{n+1} \frac{1}{\beta_j} = 1,$$

so it follows from Hölder's inequality that

$$\frac{1}{2\pi} \int_0^{2\pi} \biggl(\prod_{j=2}^{n+1} f_j^{\alpha_j/n}(r\mathrm{e}^{\mathrm{i}\theta})\biggr) \,\mathrm{d}\theta \leqslant \prod_{j=2}^{n+1} M_{\beta_j}(r, f_j^{\alpha_j/n}).$$

Thus, it is sufficient to show that

$$M_{\beta_j}(r, f_j^{\alpha_j/n}) \leqslant C\biggl(\log \frac{1}{1-r}\biggr)^{1/\beta_j}, \quad j = 2, \dots, n+1. \tag{2.4}$$

Using the standard estimate [**3**, Lemma, p. 65]

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\theta}{|1-\bar{z}_k r\mathrm{e}^{\mathrm{i}\theta}|^{j/(j-1)}} \leqslant \frac{C}{(d_k+1-r)^{1/(j-1)}}, \quad j=2,\dots,n+1,$$

we find that

$$\begin{aligned} M_{\beta_j}^{\beta_j}(r, f_j^{\alpha_j/n}) &= \frac{1}{2\pi}\int_0^{2\pi}\biggl(\sum_{k=1}^{\infty}\frac{d_k}{|1-\bar{z}_k r\mathrm{e}^{\mathrm{i}\theta}|^{j}}\biggr)^{1/(j-1)}\mathrm{d}\theta \\ &\leqslant \sum_{k=1}^{\infty} d_k^{1/(j-1)}\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\theta}{|1-\bar{z}_k r\mathrm{e}^{\mathrm{i}\theta}|^{j/(j-1)}} \\ &\leqslant C\sum_{k=1}^{\infty}\frac{d_k^{1/(j-1)}}{(d_k+1-r)^{1/(j-1)}} \\ &\leqslant C\biggl(\sum_{d_k\geqslant 1-r} 1 + (1-r)^{-1/(j-1)}\sum_{d_k<1-r} d_k^{1/(j-1)}\biggr). \end{aligned} \tag{2.5}$$

Now suppose that $d_k \leqslant Ca^k$ for some $C>0$ and $0<a<1$. If $d_k \geqslant 1-r$, then we have $1-r \leqslant Ca^k$, which implies that $k = O(\log 1/(1-r))$. Hence,

$$\sum_{d_k\geqslant 1-r} 1 = O\biggl(\log\frac{1}{1-r}\biggr). \tag{2.6}$$

Now, for any $\varepsilon>0$,

$$\sum_{d_k<\varepsilon} d_k^{1/(j-1)} = \sum_{n=0}^{\infty}\sum_{2^{-(n+1)}\varepsilon\leqslant d_k<2^{-n}\varepsilon} d_k^{1/(j-1)} \leqslant \sum_{n=0}^{\infty}(2^{-n}\varepsilon)^{1/(j-1)}N(n),$$

where $N(n)$ is the number of d_k which are greater than or equal to $2^{-(n+1)}\varepsilon$. If $d_k \geqslant 2^{-(n+1)}\varepsilon$, then $2^{-(n+1)}\varepsilon \leqslant Ca^k$, which implies that $k = O(n+\log(1/\varepsilon))$, i.e. that

$$N(n) = O\biggl(n+\log\frac{1}{\varepsilon}\biggr).$$

Thus,

$$\sum_{d_k<\varepsilon} d_k^{1/(j-1)} = O\biggl(\varepsilon^{1/(j-1)}\log\frac{1}{\varepsilon}\biggr).$$

In particular, setting $\varepsilon = 1-r$, we get

$$(1-r)^{-1/(j-1)}\sum_{d_k<1-r} d_k^{1/(j-1)} = O\biggl(\log\frac{1}{1-r}\biggr). \tag{2.7}$$

Now (2.4) follows from (2.5)–(2.7).

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(ii) Let

$$B(z)=\sum_{k=0}^{\infty}\hat{B}_k z^k,\quad A_n=\sum_{k=n}^{\infty}|\hat{B}_k|^2\quad\text{and}\quad s_n=\sum_{k=1}^{n}k|\hat{B}_k|^2.$$

Now let

$$M_p^p(r,D^{1/p}B)=O(\log 1/(1-r)),\quad 0<p\leqslant 2.$$

Then

$$M_2^2(r,D^{1/2}B)=O(\log 1/(1-r)),$$

by (2.1). Hence,

$$\sum_{k=1}^{\infty}k|\hat{B}_k|^2 r^{2k}=O\biggl(\log\frac{1}{1-r}\biggr).$$

So,

$$r^{2N}\sum_{k=1}^{N}k|\hat{B}_k|^2\leqslant C\log\frac{1}{1-r}$$

for any $0<r<1$. Now let $r=1-1/N$. Since

$$\biggl(1-\frac{1}{N}\biggr)^{2N}\geqslant m>0,$$

if $N\geqslant 2$, we find that

$$s_N=\sum_{k=1}^{N}k|\hat{B}_k|^2\leqslant C\log N. \tag{2.8}$$

Using the identity

$$\sum_{n=N}^{M}|\hat{B}_n|^2=\sum_{n=N}^{M-1}s_n(n^{-1}-(n+1)^{-1})+s_M M^{-1}-s_{N-1}N^{-1}$$

and letting $M\to\infty$, we obtain

$$A_N=\sum_{n=N}^{\infty}|\hat{B}_n|^2\leqslant C\sum_{n=N}^{\infty}\frac{\log n}{n^2}\leqslant C\frac{\log N}{N}. \tag{2.9}$$

Since $\sum_{k=0}^{\infty}|\hat{B}_k|^2=1$, we find that

$$\begin{aligned}\Delta(r,B)&=\sum_{k=1}^{\infty}|\hat{B}_k|^2(1-r^{2k})\\&=\sum_{k=1}^{N}|\hat{B}_k|^2(1-r^{2k})+\sum_{k=N+1}^{\infty}|\hat{B}_k|^2(1-r^{2k})\\&\leqslant(1-r^2)\sum_{k=1}^{N}k|\hat{B}_k|^2+A_{N+1}.\end{aligned} \tag{2.10}$$

For given $0 < r < 1$, choose N so that $1 - 1/N \leqslant r < 1 - 1/(N+1)$. Then using (2.8)–(2.10) we find that

$$\Delta(r, B) = O\biggl((1-r)\log\frac{1}{1-r}\biggr).$$

Hence, $d_k = O(a^k)$, for some $0 < a < 1$, by Theorem A. $\square$

If φ is an inner function that belongs to $\lambda^{p,1/p}$, $0 < p < \infty$, then φ must be a finite Blaschke product. This follows from [**2**, Theorem 3.2].

Sequences $\{z_k\}$ in D which satisfy the condition

$$1 - |z_{k+1}| \leqslant c(1 - |z_k|), \quad k = 1, 2, \dots,$$

where $0 < c < 1$, are called exponential sequences.

The Lipschitz space $\Lambda^{p,1/p}$, $0 < p < \infty$, contains infinite Blaschke products. This follows from the next theorem (see also [**15**]).

Theorem B (Ahern and Jevtić [2]). *Let B be a Blaschke product with zeros $\{z_k\}$. Then the following statements are equivalent:*

(i) *$B \in \Lambda^{p,1/p}$, for some p, $0 < p < \infty$;*

(ii) *$B \in \Lambda^{p,1/p}$ for all p, $0 < p < \infty$;*

(iii) *$\sum_{k=n}^{\infty}(1 - |z_k|) \leqslant C(1 - |z_n|)$ for all $n = 1, 2, \dots$;*

(iv) *$\{z_k\}$ is a finite union of exponential sequences.*

Proposition 2.1. *Let B be a Blaschke product with zeros $\{z_k\}$. If*

$$M_p(r, D^{1+(1/p)}B) = O\biggl(\frac{1}{1-r}\biggr), \quad 0 < p < \infty,$$

then

$$M_p^p(r, D^{1/p}B) = O\biggl(\log\frac{1}{1-r}\biggr).$$

Proof. If $M_p(r, D^{1+(1/p)}B) = O(1/(1-r))$, i.e. $B \in \Lambda^{p,1/p}$, $0 < p < \infty$, then $1 - |z_k| = O(a^k)$ for some $0 < a < 1$, by Theorem B. Now, $M_p^p(r, D^{1/p}B) = O(\log 1/(1-r))$, by Theorem 1.1 (i). $\square$

Proposition 2.2. *Let B be an interpolating Blaschke product whose zero sequence $\{z_k\}$ lies in a Stolz angle. Then $B'' \in A^{p,p-1}$ for all $p \in (0, 1)$.*

Proof. The sequence $\{z_k\}$ is a finite union of exponential sequences. See [**13**, Theorem 3]. Therefore,

$$M_p(r, D^{1+(1/p)}B) = O\biggl(\frac{1}{1-r}\biggr), \quad 0 < p < \infty, \quad \text{by Theorem B.}$$

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On the other hand,

$$\int_0^1 (1-r)^{p-1} M_p^p(r, B'')\,\mathrm{d}r < \infty$$

if and only if

$$\int_0^1 M_p^p(r, D^{1+(1/p)}B)\,\mathrm{d}r < \infty.$$

Thus, $B'' \in A^{p,p-1}$ for all p, $0 < p < 1$. □

By a well-known theorem of Littlewood–Paley type, if $B'' \in A^{p,p-1}$, $0 < p \leqslant 2$, then $B' \in H^p$ [**12**]. Therefore, Proposition 2.2 is an improvement of [**7**, Theorem 4].

3. Blaschke products in $\Lambda^{p,\alpha}$, $\max(0, (1/p) - 1) < \alpha < 1/p$

For a proof of Theorem 1.2 we need the following.

Proposition 3.1. *If f is a Bloch function, i.e. $f \in \Lambda^{\infty,0}$, then $f \in \Lambda^{p,\alpha}$, $0 < p, \alpha < \infty$, implies that $f \in \Lambda^{pt,\alpha/t}$, $1 \leqslant t < \infty$.*

See also [**10**, Lemma 2.2, p. 22].

Proof of Theorem 1.2. Without loss of generality assume that $B(0) = 0$. Suppose first that $0 < \alpha < 1$ and hence that $\frac{1}{2} < p < \infty$. It follows from the interpolation property of the spaces $\Lambda^{p,\alpha}$ (Proposition 3.1) that it suffices to prove (1.2) for $\frac{1}{2} < p < 1$. We use the trivial estimate

$$|B'(z)| \leqslant \sum_{k=1}^{\infty} \frac{1 - |z_k|^2}{|1 - \bar{z}_k z|^2}$$

of the derivative of a Blaschke product and the inequality

$$\left(\sum_{k=1}^{\infty} a_k\right)^p \leqslant \sum_{k=1}^{\infty} a_k^p, \quad a_k \geqslant 0,\ 0 < p \leqslant 1,$$

to get

$$\int_0^{2\pi} |B'(r\mathrm{e}^{\mathrm{i}\theta})|^p\,\mathrm{d}\theta \leqslant C \sum_{k=1}^{\infty} \frac{d_k^p}{(d_k + 1 - r)^{2p-1}}.$$

If $0 < \alpha < 1$, then the norm $\|B\|_{p,\alpha}$ is comparable to

$$|B(0)| + \sup_r (1-r)^{1-\alpha} M_p(r, B').$$

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Thus,

$$\begin{aligned}\|B\|_{p,\alpha}^p &\leqslant C\sup_n 2^{-np(1-\alpha)}\sum_{k=1}^{\infty}\frac{d_k^p}{(d_k+2^{-n})^{2p-1}}\\ &\leqslant C\sup_n 2^{-np(1-\alpha)}\biggl(\sum_{d_k\leqslant 2^{-n}}2^{n(2p-1)}d_k^p+\sum_{d_k>2^{-n}}d_k^{1-p}\biggr)\\ &\leqslant C\sup_n 2^{-np(1-\alpha)}\biggl(2^{n(2p-1)}\sum_{k=n}^{\infty}2^{-kp}\nu_k+\sum_{k=0}^{n-1}2^{-k(1-p)}\nu_k\biggr).\end{aligned}$$

Since $\nu_k\leqslant C2^{k(1-\alpha p)}$, we find that

$$\begin{aligned}\|B\|_{p,\alpha}^p &\leqslant C\sup_k\{2^{-k(1-\alpha p)}\nu_k\}\\ &\qquad\times\sup_n\biggl\{2^{-np(1-\alpha)}\biggl(2^{n(2p-1)}\sum_{k=n}^{\infty}2^{-kp}2^{k(1-\alpha p)}+\sum_{k=0}^{n-1}2^{-k(1-p)}2^{k(1-\alpha p)}\biggr)\biggr\}\\ &\leqslant C\sup_k 2^{-k(1-\alpha p)}\nu_k.\end{aligned}$$

Now let $1\leqslant\alpha<2$. It follows that $\frac{1}{3}<p<1$. By Proposition 3.1 it suffices to prove (1.2) for $\frac{1}{3}<p\leqslant\frac{1}{2}$. Now $\|B\|_{p,\alpha}^p$ is comparable to

$$|B(0)|^p+\sup_r(1-r)^{p(2-\alpha)}\int_0^{2\pi}|B''(re^{i\theta})|^p\,d\theta.$$

Thus,

$$\begin{aligned}\|B\|_{p,\alpha}^p &\leqslant C\sup_r\biggl\{(1-r)^{p(2-\alpha)}\\ &\qquad\times\int_0^{2\pi}\biggl(\biggl(\sum_{k=1}^{\infty}\frac{1-|z_k|^2}{|1-re^{i\theta}\bar z_k|^2}\biggr)^{2p}+\biggl(\sum_{k=1}^{\infty}\frac{1-|z_k|^2}{|1-re^{i\theta}\bar z_k|^3}\biggr)^{p}\biggr)d\theta\biggr\}\\ &\leqslant C\sup_r\biggl\{(1-r)^{p(2-\alpha)}\\ &\qquad\times\biggl(\int_0^{2\pi}\biggl(\sum_{k=1}^{\infty}\frac{d_k^{2p}}{|1-re^{i\theta}\bar z_k|^{4p}}+\sum_{k=1}^{\infty}\frac{d_k^{p}}{|1-re^{i\theta}\bar z_k|^{3p}}\biggr)d\theta\biggr)\biggr\}.\end{aligned}$$

Since $|1-\bar z_k re^{i\theta}|\geqslant 1-r|z_k|\geqslant d_k$, the first term on the right-hand side of the preceding inequality is majorized by the second one. Thus,

$$\|B\|_{p,\alpha}^p\leqslant C\sup_n 2^{-np(2-\alpha)}\sum_{k=1}^{\infty}\frac{d_k^p}{(d_k+2^{-n})^{3p-1}}.$$

The rest of the proof is completed into the same way as for $0<\alpha<1$, taking into account the fact that $1\leqslant\alpha<2$ and $\frac{1}{3}<p$.

In the case when $N \leqslant \alpha < N+1$, $N > 2$, we can assume that

$$\frac{1}{N+2} < p < \frac{1}{N+1}$$

and proceed in the same way, using estimates of higher derivatives given in § 2 (see p. 694). □

Proof of Theorem 1.4. Let B be a Blaschke product in $\Lambda^{p,\alpha}$,

$$\max\left(0, \frac{1}{p} - 1\right) < \alpha < \frac{1}{p},$$

for which the zero sequence $\{z_k\}$ is a Carleson–Newman sequence. Again, we may assume that $B(0) = 0$.

By Proposition 3.1 we may suppose $0 < \alpha < 1$ and $p \geqslant 1$. Note that if the zero sequence $\{z_k\}$ of a Blaschke product B is a Carleson–Newman sequence, then

$$1 - |B(z)|^2 \geqslant C \sum_{k=1}^{\infty} \frac{(1-|z|^2)(1-|z_k|^2)}{|1-\bar{z}_k z|^2}. \tag{3.1}$$

Now we use the fact that $\|B\|_{p,\alpha}^p$ is comparable to

$$\sup_r (1-r)^{-p\alpha} \int_0^{2\pi} (|B(\mathrm{e}^{\mathrm{i}\theta})| - |B(r\mathrm{e}^{\mathrm{i}\theta})|)^p \,\mathrm{d}\theta. \tag{3.2}$$

Using (3.1), (3.2) and taking into account the fact that $|B(\mathrm{e}^{\mathrm{i}\theta})| = 1$ almost everywhere, we find that

$$\begin{aligned}
\infty > \|B\|_{p,\alpha}^p &\geqslant \sup_r (1-r)^{-p\alpha} \int_0^{2\pi} \left(\sum_{k=1}^{\infty} \frac{(1-r)(1-|z_k|)}{|1-\bar{z}_k r\mathrm{e}^{\mathrm{i}\theta}|^2}\right)^p \mathrm{d}\theta \\
&\geqslant C \sup_r (1-r)^{-\alpha p} \int_0^{2\pi} \left(\sum_{k=1}^{\infty} \frac{(1-r)^p(1-|z_k|)^p}{|1-\bar{z}_k r\mathrm{e}^{\mathrm{i}\theta}|^{2p}}\right) \mathrm{d}\theta \\
&\geqslant C \sup_r (1-r)^{-\alpha p+p} \sum_{k=1}^{\infty} \frac{d_k^p}{(d_k+1-r)^{2p-1}} \\
&\geqslant C \sup_n 2^{-n(p-\alpha p)} \sum_{k=1}^{\infty} \frac{d_k^p}{(d_k+2^{-n})^{2p-1}} \\
&\geqslant C \sup_n 2^{-n(p-\alpha p)} \sum_{d_k > 2^{-n}} d_k^{1-p} \\
&\geqslant C \sup_n 2^{-n(p-\alpha p)} \sum_{k=0}^{n-1} 2^{-k(1-p)} \nu_k \\
&\geqslant C \sup_n 2^{-n(p-\alpha p)} 2^{-n(1-p)} \nu_{n-1} \\
&\geqslant C \sup_n 2^{-n(1-\alpha p)} \nu_n.
\end{aligned}$$

□

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4. Blaschke products with zeroes in a non-tangential region

In [**8**, Corollary 2.3, p. 324] it is shown that if $B \in \mathcal{B}$, then $M_1(r, B') = o((1-r)^{-1/2})$ or, equivalently, $M_1(r, D^{3/2}B) = o(1/(1-r))$. Thus, $\mathcal{B} \subset \lambda^{1,1/2}$.

The 'o' version of Proposition 3.1 is as follows.

Proposition 4.1. *If $f \in \Lambda^{\infty,0}$, then $f \in \lambda^{p,\alpha}$ implies that $f \in \lambda^{pt,\alpha/t}$, $1 \leqslant t < \infty$.*

As a corollary we have $\mathcal{B} \subset \lambda^{p,1/(2p)}$, $1 \leqslant p < \infty$.

To show that $\mathcal{B} \subset \lambda^{p,1/(2p)}$ for $\frac{1}{2} < p < \infty$, we need the following.

Theorem 4.2. *Let $0 < p < \infty$ and $\max(0, 1/p - 1) < \alpha < 1/p$. Let f be an inner function. Then $f \in \lambda^{p,\alpha}$ if and only if, for any $0 < \delta < \frac{1}{2}$,*

$$\lim_{n\to\infty} 2^{-n(1-\alpha p)} \int_{K_\delta} \nu_n(a)\, \mathrm{d}A(a) = 0,$$

where $K_\delta = \{z \in D : \delta \leqslant |z| \leqslant 1-\delta\}$.

Corollary 4.3. *If f is an inner function in $\lambda^{p,\alpha}$, $\max(0, 1/p - 1) < \alpha < 1/p$, then $f \in \lambda^{pt,\alpha/t}$, for $1/p - \alpha < t < 1$.*

Since $\mathcal{B} \subset \lambda^{1,1/2}$, we have that $\mathcal{B} \subset \lambda^{p,1/(2p)}$ for $\frac{1}{2} < p < 1$. Thus, we have proved Theorem 1.6. It remains to prove Theorem 4.2.

The following lemmas will be needed.

Lemma 4.4. *Let f be an inner function and let $\{z_k\}$ be its zero sequence. Then*

$$\log \frac{1}{|f(z)|^2} \geqslant \sum_{k=1}^{\infty} \frac{(1-|z|^2)(1-|z_k|^2)}{|1-\bar{z}_k z|^2}.$$

See [**5**, Chapter 7, Lemma 1.2].

Corollary 4.5. *For any $p \geqslant 1$ and ρ, $2^{-n} < 1-\rho < 2^{-n+1}$,*

$$\int_0^{2\pi} \log^p \frac{1}{|f(\rho \mathrm{e}^{\mathrm{i}\theta})|}\, \mathrm{d}\theta \geqslant C_p 2^{-n} \nu_n.$$

Proof. By Lemma 4.4,

$$\log^p \frac{1}{|f(\rho \mathrm{e}^{\mathrm{i}\theta})|^2} \geqslant C_p \sum_{k=1}^{\infty} \frac{(1-|z_k|^2)^p (1-\rho^2)^p}{|1-\bar{z}_k \rho \mathrm{e}^{\mathrm{i}\theta}|^{2p}}.$$

Thus,

$$\int_0^{2\pi} \log^p \frac{1}{|f(\rho \mathrm{e}^{\mathrm{i}\theta})|}\, \mathrm{d}\theta \geqslant C_p \sum_{k=1}^{\infty} \frac{d_k^p (1-\rho)^p}{(d_k + (1-\rho))^{2p-1}} \geqslant C_p \nu_n 2^{-n}.$$

□

Lemma 4.6. *Suppose that $0 < q < \infty$, $0 < \delta < \frac{1}{2}$ and $z \in D$. Then*

$$\int_{K_\delta} \log^q \left| \frac{1 - \bar{a}z}{z - a} \right| \mathrm{d}A(a) \leqslant C_{q,\delta}(1 - |z|)^q. \tag{4.1}$$

Proof. If $|z| < \frac{1}{2}\delta$, then $|a - z| \geqslant |a| - |z| > \frac{1}{2}\delta$ for $a \in K_\delta$, and

$$\left| \frac{1 - \bar{a}z}{z - a} \right| \leqslant \frac{4}{\delta}.$$

Since $1 - |z| > \frac{1}{2}\delta$, it follows that (4.1) holds.

In the case when $1 - \frac{1}{2}\delta < |z| < 1$, we use the identity

$$\left| \frac{1 - \bar{a}z}{z - a} \right|^2 = \frac{(1 - |z|^2)(1 - |a|^2)}{|z - a|^2} + 1. \tag{4.2}$$

It is clear that $|a - z| \geqslant |z| - |a| > \frac{1}{2}\delta$ for $a \in K_\delta$, and

$$\log \left| \frac{1 - \bar{a}z}{z - a} \right|^2 \leqslant \log \left(1 + \frac{4}{\delta}(1 - |z|^2) \right) \leqslant C_\delta(1 - |z|).$$

From this it follows that (4.1) holds.

It remains to consider the case when $\frac{1}{2}\delta \leqslant 1 - |z| \leqslant 1 - \frac{1}{2}\delta$. Let $z = \rho \mathrm{e}^{\mathrm{i}\theta}$. Applying (4.2), we see that

$$\int_0^{2\pi} \log^q \left| \frac{1 - \bar{a}z}{z - a} \right|^2 \mathrm{d}\theta \leqslant \int_0^{2\pi} \log^q \left(1 + \frac{1}{2\delta^2 \sin^2 \frac{1}{2}\theta} \right) \mathrm{d}\theta \leqslant C_{q,\delta} < \infty.$$

This implies (4.1). Thus, the lemma is proved. □

Suppose that f is an inner function. Putting $f(z)$ in place of z in (4.1), we obtain the following estimate.

Corollary 4.7. *Suppose that $0 < q < \infty$, $0 < \delta < \frac{1}{2}$ and $z \in D$. If f is an inner function, then*

$$\int_{K_\delta} \log^q \frac{1}{|f_a(z)|} \mathrm{d}A(a) \leqslant C_{q,\delta}(1 - |f(z)|)^q.$$

Proof of Theorem 4.2. Let $f \in \lambda^{p,\alpha}$. Without loss of generality let $f(0) = 0$. Using Proposition 4.1, we conclude that we may suppose that $1 \leqslant p < \infty$ and $0 < \alpha < 1$. In this case we have

$$\lim_{r \to 1} (1 - r)^{-\alpha p} \int_0^{2\pi} (1 - |f(r\mathrm{e}^{\mathrm{i}\theta})|)^p \, \mathrm{d}\theta = 0.$$

Using Corollary 4.7 we find that

$$\lim_{r \to 1} (1 - r)^{-\alpha p} \int_0^{2\pi} \int_{K_\delta} \log^p \frac{1}{|f_a(r\mathrm{e}^{\mathrm{i}\theta})|} \mathrm{d}A(a) \, \mathrm{d}\theta = 0$$

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and

$$\lim_{n\to\infty} 2^{n\alpha p}\int_{K_\delta} 2^{-n}\nu_n(a)A(a)=0$$

by Corollary 4.5.

Assume that $0<\alpha<1$ and $\frac{1}{2}<p<1$. Then we have

$$\begin{aligned}(1-r)^{p(1-\alpha)}&\int_0^{2\pi}|f'(r\mathrm{e}^{\mathrm{i}\theta})|^p\,\mathrm{d}\theta\\ &\leqslant C(1-r)^{p(1-\alpha)}\int_{K_\delta}\int_0^{2\pi}|f_a'(r\mathrm{e}^{\mathrm{i}\theta})|^p\,\mathrm{d}\theta\,\mathrm{d}A(a)\\ &\leqslant C(1-r)^{p(1-\alpha)}\int_{K_\delta}\biggl(\sum_{k=1}^{\infty}\frac{d_k(a)^p}{(d_k(a)+1-r)^{2p-1}}\biggr)\,\mathrm{d}A(a).\end{aligned}$$

Thus,

$$\begin{aligned}\|f\|_{p,\alpha}^p&\leqslant C\sup_n 2^{-np(1-\alpha)}\int_{K_\delta}\biggl(\sum_{k=1}^{\infty}\frac{d_k(a)^p}{(d_k(a)+2^{-n})^{2p-1}}\biggr)\,\mathrm{d}A(a)\\ &\leqslant C\sup_n 2^{-np(1-\alpha)}\int_{K_\delta}\biggl(2^{n(2p-1)}\sum_{d_k\leqslant 2^{-n}}d_k(a)^p+\sum_{d_k(a)>2^{-n}}d_k(a)^{1-p}\biggr)\,\mathrm{d}A(a)\\ &\leqslant C\sup_n 2^{-np(1-\alpha)}\\ &\qquad\times\biggl(2^{n(2p-1)}\sum_{k=n}^{\infty}2^{-kp}\int_{K_\delta}\nu_k(a)\,\mathrm{d}A(a)+\sum_{k=0}^{n-1}2^{-k(1-p)}\int_{K_\delta}\nu_k(a)\,\mathrm{d}A(a)\biggr).\end{aligned}$$

Using the fact that

$$\sup_n 2^{-n(1-\alpha p)}\int_{K_\delta}\nu_n(a)\,\mathrm{d}A(a)\leqslant C<\infty$$

we find that

$$\|f\|_{p,\alpha}^p\leqslant C\sup_n 2^{-n(1-\alpha p)}\int_{K_\delta}\nu_n(a)\,\mathrm{d}A(a).$$

Also, if

$$\lim_{n\to\infty}2^{-n(1-\alpha p)}\int_{K_\delta}\nu_n(a)\,\mathrm{d}A(a)=0,$$

then the above consideration shows that

$$\lim_{r\to 1}(1-r)^{p(1-\alpha)}\int_0^{2\pi}|f'(r\mathrm{e}^{\mathrm{i}\theta})|^p\,\mathrm{d}\theta=0,$$

or, equivalently,

$$\lim_{r\to 1}(1-r)M_p(r,D^{1+\alpha}f)=0.$$

The rest of the proof is analogous to that of Theorem 1.2. In the case when $\alpha\geqslant 1$, we use similar estimates of higher derivatives of the Blaschke products $f_a(z)$. □

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In [**7**] it is proved that if $B \in \mathcal{B}$, then $B' \in H^p$ for all $p \in (0, \frac{1}{2})$. The following is an improvement of this result.

Proposition 4.8. *If $B \in \mathcal{B}$, then $B'' \in A^{p,p-1}$ for all $p \in (0, \frac{1}{2})$.*

Proof. First, $B'' \in A^{p,p-1}$ if and only if

$$\int_0^1 (1-r)^{-1/2} M_p^p(r, D^{1+1/2p}B)\,\mathrm{d}r < \infty.$$

Since

$$M_p^p(r, D^{1+1/2p}B) \leqslant \frac{C}{(1-r)^p},$$

we see that the last integral is finite if $0 < p < \frac{1}{2}$. □

As seen above, if $B'' \in A^{p,p-1}$, $0 < p \leqslant 2$, then $B' \in H^p$.

As a final remark we note that the characterization of Blaschke products, with zeros in a fixed non-tangential region, in $\Lambda^{p,\alpha}$, for $0 < p < \infty$, $1/2p < \alpha < 1/p$, is given in [**10**]. Theorem 2.1 in [**10**] states that if $B \in \mathcal{B}$, then $B \in \Lambda^{p,\alpha}$, $0 < p < \infty$, $1/2p < \alpha < 1/p$, if and only if

$$\{d_k^{1/p-\alpha} k^\alpha\} \in l^\infty,$$

or, equivalently,

$$\{2^{-k(1/p-\alpha)} \nu_k^\alpha\} \in l^\infty.$$

Analogously, if $B \in \mathcal{B}$, then $B \in \lambda^{p,\alpha}$, $0 < p < \infty$, $1/2p < \alpha < 1/p$, if and only if

$$\lim_{n\to\infty} d_n^{1-\alpha p} n^{\alpha p} = 0,$$

or, equivalently,

$$\lim_{n\to\infty} 2^{-n(1-\alpha p)} \nu_n^{\alpha p} = 0.$$

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