THE TOPOLOGICAL DEGREE OF A-PROPER MAPPING
IN THE MENERG PN-SPACE (II)

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In the paper "The topological degree of A-proper mapping in the Menger PN-space (I)", the new concept of A-proper topological degree has been given. Now, utilising the new concept, we give the corresponding definitions of convex A-proper, $P_2$-compact and $P_7$-compact in Menger PN-space. As an application of these new concepts, we prove the existence of solution for some equations.

1. INTRODUCTION

In this paper, utilising A-proper properties, we discuss the existence of solution for some equations. For the sake of convenience, we recall some definitions and properties of PN-space.

DEFINITION 1: (Chang [1].) A probabilistic normed space (shortly a PN-space) is an ordered pair $(E, F)$, where $E$ is a real linear space, $F$ is a mapping of $E$ into $D$ ($D$ is the set of all distribution functions. We shall denote the distribution function $F(x)$ by $F_x$. $F_x(t)$ denotes the value $F_x$ for $t \in R$.) satisfying the following conditions:

(PN-1) $F_x(0) = 0$;

(PN-2) $F_x(t) = H(t)$ for all $t \in R$ if and only if $x = \theta$, where $H(t) = 0$ when $t \leq 0$, and $H(t) = 1$ when $t > 0$;

(PN-3) For all $x \neq 0$, $F_{\alpha x}(t) = F_x(t/|\alpha|)$;

(PN-4) For any $x, y \in E$ and $t_1, t_2 \in R$, if $F_x(t_1) = 1$ and $F_y(t_2) = 1$, then we have $F_{x+y}(t_1 + t_2) = 1$.

LEMMA 1. (Chang [1].) Let $(E, F, \Delta)$ be a Menger PN-space with a continuous $t$-norm $\Delta$, then $x_n \subset E$ is said to be convergent to $x \in E$ if for any $t > 0$, we have $\lim_{n \to \infty} F_{x_n - x}(t) = H(t)$.

LEMMA 2. The generalised topological degree $\text{Deg}(f, \Omega, p)$ has the following properties:

\begin{itemize}
  \item[(i)] $\text{Deg}(I, \Omega, p) = 1, \forall p \in \Omega$, where $I$ is an identity operator;
\end{itemize}
(iii) If $\text{Deg}(f, \Omega, p) \neq \{0\}$, then the equation $f(x) = p$ has a solution in $\Omega$;

(iii) If $L : [0, 1] \times \overline{\Omega} \to E$ is continuous and for any fixed $t \in [0, 1]$, $L(t, \cdot) : \overline{\Omega} \to E$ is an $A$-proper mapping satisfying

$$\lim_{t \to t_0} \inf_{x \in \overline{\Omega}} F_{L(t,x)-L(t_0,x)}(\varepsilon) = H(\varepsilon), \quad \forall \varepsilon > 0.$$  

Let $p \notin h_t(\partial \Omega)$, $0 \leq t \leq 1$, where $h_t(x) = L(t, x)$, then we have

$$\text{Deg}(h_t, \Omega, p) = \text{Deg}(h_0, \Omega, p), \quad \forall 0 \leq t \leq 1;$$

(iv) If $\Omega_0$ is an open subset of $\Omega$ and $p \notin f(\overline{\Omega} \setminus \Omega_0)$, then we have

$$\text{Deg}(f, \Omega, p) = \text{Deg}(f, \Omega_0, p);$$

(v) If $\Omega(1)$ and $\Omega(2)$ are two disjoint open subsets of $\Omega$ and

$$p \notin f(\overline{\Omega} \setminus (\Omega(1) \cup \Omega(2))),$$

then

$$\text{Deg}(f, \Omega, p) \subseteq \text{Deg}(f, \Omega(1), p) + \text{Deg}(f, \Omega(2), p).$$

If either $\text{Deg}(f, \Omega(1), p)$ or $\text{Deg}(f, \Omega(2), p)$ is single-valued, then

$$\text{Deg}(f, \Omega, p) = \text{Deg}(f, \Omega(1), p) + \text{Deg}(f, \Omega(2), p);$$

(vi) If $p \notin f(\partial \Omega)$, then $\text{Deg}(f, \Omega, p) = \text{Deg}(f - p, \Omega, \theta);$  

(vii) If $p$ varies on every connected component of $E \setminus f(\partial \Omega)$, then $\text{Deg}(f, \Omega, p)$ is a constant.

2. MAIN RESULTS

**Lemma 3.** Let $(E, F, \Delta)$ be a projected complete Menger PN-space, $\Delta$ is a continuous $t$-norm, and $f : \overline{\Omega} \to E$ is an $A$-proper mapping. Then $\lambda f$ is also an $A$-proper mapping ($\lambda \neq 0$).

**Proof:** For any sequence $\{x_{n_k}\} \in \overline{\Omega}_{n_k}$, we have

$$\lim_{k \to \infty} F_{Q_{\lambda n_k}f(x_{n_k})-Q_{\lambda n_k}(p)}(t) = H(t), \quad \forall t > 0.$$ 

Because $y \in E$, $E$ is a linear space and $\lambda \neq 0$, then we have $y/\lambda \in E$. Hence the above is equal to

$$\lim_{k \to \infty} F_{Q_{\lambda n_k}f(x_{n_k})-Q_{\lambda n_k}(y/\lambda)}(t/\lambda) = H(t) = H(t/\lambda)$$

Because $f$ is an $A$-proper mapping, then by the definition, there exists a convergent subsequence $\{x_{n_{i_k}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{i_k}} \to x \in \overline{\Omega}$, $f(x) = y/\lambda$. So $\lambda f(x) = y$. Therefore $\lambda f$ is an $A$-proper mapping.
**Lemma 4.** Let \((E, F, \Delta)\) be a projected complete Menger PN-space, \(\Delta\) be a continuous t-norm, \(f : \Omega \rightarrow E\) be an A-proper mapping, \(C : \Omega \rightarrow E\) be a continuous compact mapping, and \((f + C)(x) = f(x) + C(x)\). Then \(f + C\) is an A-proper mapping.

**Proof:** If for any sequence \(\{x_{nk}\} \in \Omega_{nk}\), we have

\[
\lim_{k \to \infty} F_{Q_{nk}} f(x_{nk}) = H(t) \quad \forall t > 0
\]

Because \(C\) is a continuous compact mapping and \(x_{nk} \in \Omega_{nk} \subset \Omega\), then there exists a subsequence (shortly, we assume that it is \(\{x_{nk}\}\) itself) such that \(C(x_{nk}) \to y_0 \in E\). Because \(Q_{nk}\) is continuous and linear, then we have \(Q_{nk} C(x_{nk}) \to Q_{nk} (y_0)\). Because

\[
F_{Q_{nk}} f(x_{nk}) - Q_{nk} (y_0)(t) = F_{Q_{nk}} f(x_{nk}) + Q_{nk} C(x_{nk}) - Q_{nk} C(x_{nk}) - Q_{nk} (y_0)(t)
\]

\[
\Delta \left( F_{Q_{nk}} f(x_{nk}) + Q_{nk} C(x_{nk}) - Q_{nk} (y_0)(t) \right)
\]

Taking limit between the two sides, we have

\[
\lim_{k \to \infty} F_{Q_{nk}} f(x_{nk}) - Q_{nk} (y_0)(t) = H \left( \frac{t}{2} \right) = H(t)
\]

By the A-proper properties of \(f\), there must exist a convergent subsequence \(\{x_{nk}\}\) of \(\{x_{nk}\}\) such that \(x_{nk} \to x \in \Omega\) and \(f(x) = y - y_0\). By the continuity of \(C\), we have \(C(x_{nk}) \to C(x)\). Therefore \(C(x) = y_0\) and \(f(x) + C(x) = y\). Hence \(f + C\) is an A-proper mapping.

**Lemma 5.** \(I\) is an A-proper mapping.

**Proof:** If for any sequence \(\{x_{nk}\} \in \Omega_{nk}\), we have \(\lim_{k \to \infty} F_{Q_{nk}} f(x_{nk}) - Q_{nk} (y)(t) = H(t)\), then \(\lim_{k \to \infty} Q_{nk} (x_{nk}) - Q_{nk} (y) = 0\). Because \(Q_{nk} x \to x_n (n \to \infty)\), then we have \(\lim_{k \to \infty} x_{nk} - y = 0\) that is, \(\lim_{k \to \infty} x_{nk} = y\), hence there exists a subsequence \(\{x_{nk}\}\) of \(\{x_{nk}\}\) such that \(x_{nk} \to y\). Because \(I(y) = y\), then \(I\) is an A-proper mapping.

**Lemma 6.** \(A\) and \(B\) are two nonempty number sets. If for any \(x \in A\) and \(y \in B\), we have \(x < y\), then \(\text{Sup} A \leq \text{inf} B\). In particular, when \(A = \{a\}\) the conclusion still holds.

**Proof:** It is obvious.

**Theorem 1.** Let \((E, F, \Delta)\) be a projected complete Menger PN-space, \(\Delta\) be a continuous t-norm, \(f : \Omega \rightarrow E\) be an A-proper mapping, \(C : \Omega \rightarrow E\) be a continuous compact mapping and \((f - C)(x) = f(x) - C(x)\), \(p \notin f(\partial \Omega)\), \(p \notin C(\partial \Omega)\). For any \(\varepsilon > 0\), \(\lambda \in R\), \(x \in \partial \Omega\), \(F_{f(x)} \lambda C(x)(\varepsilon) > F_p(\varepsilon)\). Then \(\text{Deg}(f, \Omega, p) = \text{Deg}(C, \Omega, p)\).

**Proof:** Let \(L(t, x) = tf(x) + (1 - t)C(x)\). Because \(f\) and \(C\) are continuous, then \(L(t, x)\) is also continuous. Because \(f\) is an A-proper mapping, by Lemma 3, \(tf\) is an A-proper mapping. Because \(C\) is continuous and compact, then \((1 - t)C\) is still continuous
and compact. By Lemma 4, \( tf(x) + (1 - t)C(x) \) is an A-proper mapping. Because \( f \) is an A-proper mapping and \( C \) is continuous and compact, then \( f - C \) is continuous and bounded. Hence when \( t \to t_0 \), we have \( t(f(x) - C(x)) \to t_0(f(x) - C(x)) \). Hence

\[
\lim_{t \to t_0} F_{t(f(x) - C(x)) - t_0(f(x) - C(x))}(\varepsilon) = H(\varepsilon), \quad \forall \varepsilon > 0, \quad \forall x \in \bar{\Omega}.
\]

Then for any \( \lambda > 0 \), we have \( F_{t(f(x) - C(x)) - t_0(f(x) - C(x))}(\varepsilon) > 1 - \lambda \) \((t \to t_0)\). By Lemma 6, we have \( \inf_{x \in \bar{\Omega}} F_{t(f(x) - C(x)) - t_0(f(x) - C(x))}(\varepsilon) \geq 1 - \lambda \). By the arbitrariness of \( \lambda \), we have

\[
\inf_{x \in \bar{\Omega}} F_{t(f(x) - C(x)) - t_0(f(x) - C(x))}(\varepsilon) = 1 \quad (t \to t_0).
\]

Therefore

\[
\lim_{t \to t_0} \inf_{x \in \bar{\Omega}} F_{t(f(x) - C(x)) - t_0(f(x) - C(x))}(\varepsilon) = H(\varepsilon).
\]

Next, we prove \( p \notin h_t(\partial \Omega) \). Using reduction to absurdity, we assume there exist \( x_0 \in \partial \Omega \) and \( t_0 \in [0,1] \) such that \( h_{t_0}(x_0) = p \). Because \( p \notin f(\partial \Omega) \), then \( t_0 \neq 1 \). Because \( p \notin C(\partial \Omega) \), then \( t_0 \neq 0 \). Hence \( t_0 \in (0,1) \). By \( p = t_0f(x_0) + (1 - t_0)C(x_0) \), we have \( p/t_0 = f(x_0) + (1 - t_0)/(t_0)C(x_0) \). Taking \( \lambda = (1 - t_0)/(t_0) \), we have

\[
F_{f(x_0) + \lambda C(x_0)}(\varepsilon) = F_{f(x_0) + (1 - t_0)/(t_0)C(x_0)}(\varepsilon) = F_{(p/t_0)}(\varepsilon) = F_{p}(\varepsilon) < F_{p}(\varepsilon).
\]

It contradicts known conditions. Hence we have \( p \notin h_t(\partial \Omega) \). By the Lemma 2 (iii), we have \( \text{Deg}(h_1, \Omega, p) = \text{Deg}(h_0, \Omega, p) \). Hence \( \text{Deg}(f, \Omega, p) = \text{Deg}(C, \Omega, p) \). \( \square \)

**Definition 2:** Let \((E, F, \Delta)\) be a projected complete Menger PN-space. \( \Delta \) is a continuous t-norm. \( \Omega \) is the bounded open set of \( E \):

(i) If for any \( \lambda \geq 0 \), \( f + \lambda I : \bar{\Omega} \to E \) is A-proper, then \( f \) is called \( P_\lambda \)-compact mapping;

(ii) For given \( \gamma > 0 \), if for any \( \lambda \geq \gamma \), \( f - \lambda I : \bar{\Omega} \to E \) is A-proper, then \( f \) is called \( P_\gamma \)-compact mapping.

**Definition 3:** Let \((E, F, \Delta)\) be a projected complete Menger PN-space. \( \Delta \) is a continuous t-norm. \( \Omega \) is a bounded open neighbourhood of \( E \), which is symmetric about \( 0 \in \Omega \). \( f : \bar{\Omega} \to E \) is said to be a convex A-proper mapping, if for \( L : [0,1] \times \bar{\Omega} \to E \), we have \( L(t, x) = h_t(x) = (1/t + t)f(x) - (t/1 + t)f(-x) \) is A-proper.

**Theorem 2:** Let \((E, F, \Delta)\) be a projected complete Menger PN-space. \( \Delta \) is a continuous t-norm. \( \Omega \) is a bounded open set of \( E \), \( f : \bar{\Omega} \to E \) is a \( P_\lambda \)-compact mapping, \( \theta \in \Omega \), we assume that \( F_{x-f}(\varepsilon) > F_{x}(\varepsilon), \forall \varepsilon > 0, \forall x \in \partial \Omega \), then there must exist an \( x^* \in \Omega \) such that \( f(x^*) = \theta \).

**Proof:** Let \( h_t(x) = L(t, x) = (1-t)f(x) + tx \), then \( L : [0,1] \times \bar{\Omega} \to E \) is continuous. When \( t \neq 1 \), we have \( h_t = (1-t)f + tI = (1-t)(f + (t/1-t)I). \) Because \( (t/1-t) \geq 0 \), by the \( P_\lambda \)-compact property of \( f \), \( h_t : \bar{\Omega} \to E \) is A-proper. Because \( h_1 = I \) is A-proper,
then for any \( t \in [0,1] \), \( h_t \) is A-proper. Because \( f \) is a \( P_\ast \)-compact mapping, then \( f(x) \) is bounded. Because \( \overline{\Omega} \) is a bounded closed set, \( I(x) \) is bounded. Hence \( f(x) - x \) is bounded.

Thus for any \( t, t_0 \in [0,1] \), \( x \in \overline{\Omega} \), when \( t \to t_0 \), we have \( t(f(x) - x) \to t_0(f(x) - x) \). Thus

\[
\lim_{t \to t_0} F_t(f(x) - x) - t_0(f(x) - x)(\varepsilon) = H(\varepsilon).
\]

By Lemma 6, we have

\[
\lim \inf_{t \to t_0} F_t(f(x) - x) - t_0(f(x) - x)(\varepsilon) = H(\varepsilon).
\]

In the following, we prove \( \theta \notin h_t(\partial \Omega) \) \( (t \in [0,1]) \). Assuming there exist an \( x_0 \in \partial \Omega \) and a \( t_0 \in [0,1] \) such that \( h_{t_0}(x_0) = \theta \) that is, \( (1 - t_0)f(x_0) + t_0x_0 = \theta \). Because \( \theta \in \Omega \), then \( t_0 \neq 1 \). Thus

\[
f(x_0) = \frac{t_0}{t_0 - 1}x_0, \quad F_{x_0}(f(x_0)(\varepsilon)) = F_{x_0-(t_0/t_0-1)x_0}(\varepsilon) = F_{(1/1-t_0)x_0}(\varepsilon) = F_{x_0}((1 - t_0)x_0).
\]

Because \( t_0 \in [0,1] \), then \( (1 - t_0)x_0 \leq \varepsilon \). By the properties of distribution function, we have \( F_{x_0}((1 - t_0)x_0) \leq F_{x_0}(\varepsilon) \). It contradicts known conditions. Thus \( \theta \notin h_t(\partial \Omega) \). By the Lemma 2 (iii), we have

\[
\text{Deg}(f, \Omega, \theta) = \text{Deg}(h_0, \Omega, \theta) = \text{Deg}(h_1, \Omega, \theta) = \text{Deg}(I, \Omega, \theta) = \{1\}.
\]

Therefore, there exists an \( x^* \in \Omega \) such that \( f(x^*) = \theta \).

**Theorem 3.** Let \( (E, F, \Delta) \) be a projected complete Menger PN-space. \( \Delta \) is a continuous t-norm. \( \Omega \) is a bounded open neighbourhood of \( E \), which is symmetric about \( \theta \in \Omega \). If \( f : \overline{\Omega} \to E \) is an A-proper mapping and

\[
f(-x) = -f(x), \quad f(x) \neq \theta, \quad \forall x \in \partial \Omega,
\]

then there exists an \( x_0 \in \Omega \) such that \( f(x_0) = \theta \).

**Proof:** Because \( f(-x) = -f(x) \), then \( f(-x) + f(x) = 0 \). Because \( Q_{n_k} \) is continuous and linear, then \( Q_{n_k}(f(-x) + f(x)) = 0 \) and \( Q_{n_k}f(-x) = -Q_{n_k}f(x) \). Hence \( Q_{n_k}f \) is an odd mapping. By the properties of topological degree in finite dimensional space, \( \text{deg}(Q_{n_k}f, \Omega_{n_k}, \theta) \) is an odd integer (see Chang [2]). By the definition \( \text{Deg}(f, \Omega, \theta) = \{\gamma \in \mathbb{Z}^* \mid \text{there exists a subsequence } \{n_k\} \text{ of } \{n\} \text{ such that } \text{deg}(Q_{n_k}f, \Omega_{n_k}, \theta) \rightarrow \gamma\} \), then there exists an \( x_0 \in \Omega \) such that \( f(x_0) = \theta \).

**References**
