THE TOPOLOGICAL DEGREE OF A-PROPER MAPPING
IN THE MENGER PN-SPACE (II)

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In the paper "The topological degree of A-proper mapping in the Menger PN-space (I)", the new concept of A-proper topological degree has been given. Now, utilising the new concept, we give the corresponding definitions of convex A-proper, $P_1$-compact and $P_2$-compact in Menger PN-space. As an application of these new concepts, we prove the existence of solution for some equations.

1. INTRODUCTION

In this paper, utilising A-proper properties, we discuss the existence of solution for some equations. For the sake of convenience, we recall some definitions and properties of PN-space.

DEFINITION 1: (Chang [1].) A probabilistic normed space (shortly a PN-space) is an ordered pair $(E, F)$, where $E$ is a real linear space, $F$ is a mapping of $E$ into $D$ ($D$ is the set of all distribution functions. We shall denote the distribution function $F(x)$ by $F_x$, $F_x(t)$ denotes the value $F_x$ for $t \in \mathbb{R}$.) satisfying the following conditions:

(PN-1) $F_x(0) = 0$;

(PN-2) $F_x(t) = H(t)$ for all $t \in \mathbb{R}$ if and only if $x = \theta$, where $H(t) = 0$ when $t \leq 0$, and $H(t) = 1$ when $t > 0$;

(PN-3) For all $\alpha \neq 0$, $F_{\alpha x}(t) = F_x(t/|\alpha|)$;

(PN-4) For any $x, y \in E$ and $t_1, t_2 \in \mathbb{R}$, if $F_x(t_1) = 1$ and $F_y(t_2) = 1$, then we have $F_{x+y}(t_1 + t_2) = 1$.

LEMMA 1. (Chang [1].) Let $(E, F, \Delta)$ be a Menger PN-space with a continuous $t$-norm $\Delta$, then $x_n \subset E$ is said to be convergent to $x \in E$ if for any $t > 0$, we have $\lim_{n \to \infty} F_{x_n-x}(t) = H(t)$.

LEMMA 2. The generalised topological degree $\text{Deg}(f, \Omega, p)$ has the following properties:

(i) $\text{Deg}(f, \Omega, p) = 1$, $\forall p \in \Omega$, where $I$ is an identity operator;

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(iii) If $\text{Deg}(f, \Omega, p) \neq \{0\}$, then the equation $f(x) = p$ has a solution in $\Omega$.

(iii) If $L : [0, 1] \times \Omega \to E$ is continuous and for any fixed $t \in [0, 1]$, $L(t, .) : \Omega \to E$ is an $A$-proper mapping satisfying
\[
\lim_{t \to t_0} \inf_{x \in \Omega} F_{L(t, x)-L(t_0, x)}(\varepsilon) = H(\varepsilon), \quad \forall \varepsilon > 0.
\]

Let $p \notin h_t(\partial \Omega)$, $0 \leq t \leq 1$, where $h_t(x) = L(t, x)$, then we have
\[
\text{Deg}(h_t, \Omega, p) = \text{Deg}(h_0, \Omega, p), \quad \forall 0 \leq t \leq 1;
\]

(iv) If $\Omega_0$ is an open subset of $\Omega$ and $p \notin f(\overline{\Omega} \setminus \Omega_0)$, then we have
\[
\text{Deg}(f, \Omega, p) = \text{Deg}(f, \Omega_0, p);
\]

(v) If $\Omega_1$ and $\Omega_2$ are two disjoint open subsets of $\Omega$ and
\[
p \notin f(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)),
\]
then
\[
\text{Deg}(f, \Omega, p) \subseteq \text{Deg}(f, \Omega_1, p) + \text{Deg}(f, \Omega_2, p).
\]

If either $\text{Deg}(f, \Omega_1, p)$ or $\text{Deg}(f, \Omega_2, p)$ is single-valued, then
\[
\text{Deg}(f, \Omega, p) = \text{Deg}(f, \Omega_1, p) + \text{Deg}(f, \Omega_2, p);
\]

(vi) If $p \notin f(\partial \Omega)$, then $\text{Deg}(f, \Omega, p) = \text{Deg}(f - p, \Omega, \theta)$;

(vii) If $p$ varies on every connected component of $E \setminus f(\partial \Omega)$, then $\text{Deg}(f, \Omega, p)$ is a constant.

2. MAIN RESULTS

**Lemma 3.** Let $(E, F, \Delta)$ be a projected complete Menger PN-space, $\Delta$ is a continuous $t$-norm, and $f : \overline{\Omega} \to E$ is an $A$-proper mapping. Then $\lambda f$ is also an $A$-proper mapping ($\lambda \neq 0$).

**Proof:** For any sequence $\{x_{n_k}\} \subseteq \overline{\Omega}_{n_k}$, we have
\[
\lim_{k \to \infty} F_{Q_{n_k}f(x_{n_k})-Q_{n_k}(y)}(t) = H(t) \quad \forall t > 0.
\]

Because $y \in E$, $E$ is a linear space and $\lambda \neq 0$, then we have $y/\lambda \in E$. Hence the above is equal to
\[
\lim_{k \to \infty} F_{Q_{n_k}f(x_{n_k})-Q_{n_k}(y/\lambda)}(t/\lambda) = H(t) = H(t/\lambda)
\]

Because $f$ is an $A$-proper mapping, then by the definition, there exists a convergent subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_i}} \to x \in \overline{\Omega}$, $f(x) = y/\lambda$. So $\lambda f(x) = y$. Therefore $\lambda f$ is an $A$-proper mapping. \( \square \)
**Lemma 4.** Let \((E, F, \Delta)\) be a projected complete Menger PN-space, \(\Delta\) be a continuous \(t\)-norm, \(f : \overline{\Omega} \to E\) be an \(A\)-proper mapping, \(C : \overline{\Omega} \to E\) be a continuous compact mapping, and \((f + C)(x) = f(x) + C(x)\). Then \(f + C\) is an \(A\)-proper mapping.

**Proof:** If for any sequence \(\{x_{nk}\} \in \overline{\Omega}_{nk}\), we have
\[
\lim_{k \to \infty} F_{Q_{nk}} (f(x_{nk}) + f(x_{nk})) - Q_{nk} (y)(t) = H(t) \quad \forall t > 0
\]
Because \(C\) is a continuous compact mapping and \(x_{nk} \in \overline{\Omega}_{nk} \subset \overline{\Omega}\), then there exists a subsequence (shortly, we assume that it is \(\{x_{nk}\}\) itself) such that \(C(x_{nk}) \to y_0 \in E\). Because \(Q_{nk}\) is continuous and linear, then we have \(Q_{nk} C(x_{nk}) \to Q_{nk} (y_0)\). Because
\[
F_{Q_{nk}} f(x_{nk}) - Q_{nk} (y - y_0)(t) = F_{Q_{nk}} f(x_{nk}) + Q_{nk} C(x_{nk}) - Q_{nk} C(x_{nk}) - Q_{nk} (y - y_0)(t)
\]
\[
\geq \Delta \left( F_{Q_{nk}} f(x_{nk}) + Q_{nk} C(x_{nk}) - Q_{nk} y \left( \frac{t}{2} \right), F_{Q_{nk}} y_0 - Q_{nk} C(x_{nk}) \left( \frac{t}{2} \right) \right),
\]
Taking limit between the two sides, we have
\[
\lim_{k \to \infty} F_{Q_{nk}} f(x_{nk}) - Q_{nk} (y - y_0)(t) = H \left( \frac{t}{2} \right) = H(t).
\]
By the \(A\)-proper properties of \(f\), there must exist a convergent subsequence \(\{x_{nk}\}\) of \(\{x_{nk}\}\) such that \(x_{nk} \to x \in \overline{\Omega}\) and \(f(x) = y - y_0\). By the continuity of \(C\), we have \(C(x_{nk}) \to C(x)\). Therefore \(C(x) = y_0\) and \(f(x) + C(x) = y\). Hence \(f + C\) is an \(A\)-proper mapping. \(\square\)

**Lemma 5.** \(I\) is an \(A\)-proper mapping.

**Proof:** If for any sequence \(\{x_{nk}\} \in \overline{\Omega}_{nk}\), we have \(\lim_{k \to \infty} F_{Q_{nk}} I(x_{nk}) - Q_{nk} (y)(t) = H(t)\), then \(\lim_{k \to \infty} (Q_{nk} (x_{nk}) - Q_{nk} (y)) = 0\). Because \(Q_{nk} x \to x, (n \to \infty)\), then we have \(\lim_{k \to \infty} (x_{nk} - y) = 0\) that is, \(\lim_{k \to \infty} x_{nk} = y\), hence there exists a subsequence \(\{x_{nk}\}\) of \(\{x_{nk}\}\) such that \(x_{nk} \to y\). Because \(I(y) = y\), then \(I\) is an \(A\)-proper mapping. \(\square\)

**Lemma 6.** A and \(B\) are two nonempty number sets. If for any \(x \in A\) and \(y \in B\), we have \(x < y\), then \(\text{Sup} A \leq \text{inf} B\). In particular, when \(A = \{a\}\) the conclusion still holds.

**Proof:** It is obvious. \(\square\)

**Theorem 1.** Let \((E, F, \Delta)\) be a projected complete Menger PN-space, \(\Delta\) be a continuous \(t\)-norm, \(f : \overline{\Omega} \to E\) be an \(A\)-proper mapping, \(C : \overline{\Omega} \to E\) be a continuous compact mapping and \((f - C)(x) = f(x) - C(x), p \notin f(\partial \Omega), p \notin C(\partial \Omega)\). For any \(\varepsilon > 0\), \(\lambda \in R, x \in \partial \Omega, F_{f(x) + \lambda C(x)}(\varepsilon) \geq F_p(\varepsilon)\). Then \(\text{Deg}(f, \Omega, p) = \text{Deg}(C, \Omega, p)\).

**Proof:** Let \(L(t, x) = tf(x) + (1 - t)C(x)\). Because \(f\) and \(C\) are continuous, then \(L(t, x)\) is also continuous. Because \(f\) is an \(A\)-proper mapping, by Lemma 3, \(tf\) is an \(A\)-proper mapping. Because \(C\) is continuous and compact, then \((1 - t)C\) is still continuous.
and compact. By Lemma 4, $tf(x) + (1 - t)C(x)$ is an A-proper mapping. Because $f$ is an A-proper mapping and $C$ is continuous and compact, then $f - C$ is continuous and bounded. Hence when $t \to t_0$, we have $t(f(x) - C(x)) \to t_0(f(x) - C(x))$. Hence

$$\lim_{t \to t_0} F_{t(f(x) - C(x)) - t_0(f(x) - C(x))}(\varepsilon) = H(\varepsilon), \quad \forall \varepsilon > 0, \quad \forall x \in \Omega.$$

Then for any $\lambda > 0$, we have $F_{t(f(x) - C(x)) - t_0(f(x) - C(x))}(\varepsilon) > 1 - \lambda (t \to t_0)$. By Lemma 6, we have $\inf_{x \in \Omega} F_{t(f(x) - C(x)) - t_0(f(x) - C(x))}(\varepsilon) \geq 1 - \lambda$. By the arbitrariness of $\lambda$, we have

$$\inf_{x \in \Omega} F_{t(f(x) - C(x)) - t_0(f(x) - C(x))}(\varepsilon) = 1 (t \to t_0).$$

Therefore

$$\lim_{t \to t_0} \inf_{x \in \Omega} F_{t(f(x) - C(x)) - t_0(f(x) - C(x))}(\varepsilon) = H(\varepsilon).$$

Next, we prove $p \notin h_t(\partial \Omega)$. Using reduction to absurdity, we assume there exist $x_0 \in \partial \Omega$ and $t_0 \in [0,1]$ such that $h_{t_0}(x_0) = p$. Because $p \notin f(\partial \Omega)$, then $t_0 \neq 1$. Because $p \notin C(\partial \Omega)$, then $t_0 \neq 0$. Hence $t_0 \in (0,1)$. By $p = t_0 f(x_0) + (1 - t_0)C(x_0)$, we have $p/t_0 = f(x_0) + (1 - t_0)/(t_0)C(x_0)$. Taking $\lambda = (1 - t_0)/(t_0)$, we have

$$F_{p/t_0 + \lambda C(x_0)}(\varepsilon) = F_{f(x_0) + (1 - t_0)/(t_0)C(x_0)}(\varepsilon) = F_{p/t_0}(\varepsilon) \neq F_p(\varepsilon).$$

It contradicts known conditions. Hence we have $p \notin h_t(\partial \Omega)$. By the Lemma 2 (iii), we have $\text{Deg}(h_1, \Omega, p) = \text{Deg}(h_0, \Omega, p)$. Hence $\text{Deg}(f, \Omega, p) = \text{Deg}(C, \Omega, p)$.

**DEFINITION 2:** Let $(E, F, \Delta)$ be a projected complete Menger PN-space. $\Delta$ is a continuous t-norm. $\Omega$ is the bounded open set of $E$:

(i) If for any $\lambda \geq 0$, $f + \lambda I : \Omega \to E$ is A-proper, then $f$ is called $P_{\lambda}$-compact mapping;

(ii) For given $\gamma > 0$, if for any $\lambda \geq \gamma$, $f - \lambda I : \Omega \to E$ is A-proper, then $f$ is called $P_\gamma$-compact mapping.

**DEFINITION 3:** Let $(E, F, \Delta)$ be a projected complete Menger PN-space. $\Delta$ is a continuous t-norm. $\Omega$ is a bounded open set of $E$, which is symmetric about $0 \in \Omega$. $f : \Omega \to E$ is said to be a convex A-proper mapping, if for $L : [0,1] \times \Omega \to E$, we have $L(t, x) = h_t(x) = (1/1 + t)f(x) - (t/1 + t)f(-x)$ is A-proper.

**THEOREM 2:** Let $(E, F, \Delta)$ be a projected complete Menger PN-space. $\Delta$ is a continuous t-norm. $\Omega$ is a bounded open set of $E$, $f : \Omega \to E$ is a $P_\theta$-compact mapping, $\theta \in \Omega$, we assume that $F_{x - f(x)}(\varepsilon) > F_x(\varepsilon), \forall \varepsilon > 0, \forall x \in \partial \Omega$, then there must exist an $x^* \in \Omega$ such that $f(x^*) = \theta$.

**PROOF:** Let $h_t(x) = L(t, x) = (1 - t)f(x) + tx$, then $L : [0,1] \times \Omega \to E$ is continuous. When $t \neq 1$, we have $h_t = (1 - t)f + tI = (1 - t)(f + (t/1 - t)I)$. Because $(t/1 - t) \geq 0$, by the $P_\theta$-compact property of $f$, $h_t : \Omega \to E$ is A-proper. Because $h_1 = I$ is A-proper,
then for any $t \in [0, 1]$, $h_t$ is A-proper. Because $f$ is a $P_\ast$-compact mapping, then $f(x)$ is bounded. Because $\overline{\Omega}$ is a bounded closed set, $f(x)$ is bounded. Hence $f(x) - x$ is bounded. Thus for any $t, t_0 \in [0, 1], x \in \overline{\Omega}$, when $t \to t_0$, we have $t(f(x) - x) \to t_0(f(x) - x)$. Thus
\[
\lim_{t \to t_0} F_{t(f(x) - x) - t_0(f(x) - x)}(\varepsilon) = H(\varepsilon).
\]
By Lemma 6, we have
\[
\liminf_{t \to t_0, x \in \overline{\Omega}} F_{t(f(x) - x) - t_0(f(x) - x)}(\varepsilon) = H(\varepsilon).
\]

In the following, we prove $\theta \notin h_t(\partial \Omega)$ ($t \in [0, 1]$). Assuming there exist an $x_0 \in \partial \Omega$ and a $t_0 \in [0, 1]$ such that $h_{t_0}(x_0) = \theta$, that is, $(1 - t_0)f(x_0) + t_0x_0 = \theta$. Because $\theta \in \Omega$, then $t_0 \neq 1$. Thus
\[
f(x_0) = (t_0/t_0 - 1)x_0, \quad F_{x_0 - f(x_0)}(\varepsilon) = F_{x_0 - (t_0/t_0 - 1)x_0}(\varepsilon) = F_{(1/t_0 - 1)x_0}(\varepsilon) = F_{x_0}((1 - t_0)\varepsilon).
\]
Because $t_0 \in [0, 1)$, then $((1 - t_0)\varepsilon \leq \varepsilon$. By the properties of distribution function, we have $F_{x_0}((1 - t_0)\varepsilon) \leq F_{x_0}(\varepsilon)$. It contradicts known conditions. Thus $\theta \notin h_t(\partial \Omega)$. By the Lemma 2 (iii), we have
\[
\operatorname{Deg}(f, \Omega, \theta) = \operatorname{Deg}(h_0, \Omega, \theta) = \operatorname{Deg}(h_1, \Omega, \theta) = \operatorname{Deg}(I, \Omega, \theta) = \{1\}.
\]
Therefore, there exists an $x^* \in \Omega$ such that $f(x^*) = \theta$. 

**Theorem 3.** Let $(E, F, \Delta)$ be a projected complete Menger PN-space. $\Delta$ is a continuous $t$-norm. $\Omega$ is a bounded open neighbourhood of $E$, which is symmetric about $\theta \in \Omega$. If $f : \overline{\Omega} \to E$ is an A-proper mapping and
\[
f(-x) = -f(x), \quad f(x) \neq \theta, \quad \forall x \in \partial \Omega,
\]
then there exists an $x_0 \in \Omega$ such that $f(x_0) = \theta$.

**Proof:** Because $f(-x) = -f(x)$, then $f(-x) + f(x) = 0$. Because $Q_{n_\ast}$ is continuous and linear, then $Q_{n_\ast}(f(-x) + f(x)) = 0$ and $Q_{n_\ast}f(-x) = -Q_{n_\ast}f(x)$. Hence $Q_{n_\ast}f$ is an odd mapping. By the properties of topological degree in finite dimensional space, $\deg(Q_{n_\ast}f, \Omega_{1/2}, \theta)$ is an odd integer (see Chang [2]). By the definition $\operatorname{Deg}(f, \Omega, \theta) = \{\gamma \in \mathbb{Z}^+ \mid$ there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $\deg_R(Q_{n_k}f, \Omega_{n_k}, \theta) \to \gamma\}$, then there exists an $x_0 \in \Omega$ such that $f(x_0) = \theta$. 

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