Can. J. Math., Vol. XXXIII, No. 4, 1981, pp. 857-861

CO-ABSOLUTES WITH HOMEOMORPHIC DENSE SUBSPACES

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Recall that the *absolute* $\epsilon(X)$ of a regular space X is the unique (up to a homeomorphism) extremally disconnected space whose image is X under a perfect irreducible map. X and Y are *co-absolute* whenever $\epsilon(X)$ and $\epsilon(Y)$ are homeomorphic. Completely regular spaces X and Y are *weakly co-absolute* whenever βX and βY are co-absolute. For a survey of this area we suggest [**6**] and [**8**].

In this paper we prove

THEOREM 1. *Suppose, for $i \in \{0, 1\}$, $X(i)$ is a compact connected linearly ordered space. Then $X(0)$ and $X(1)$ are co-absolute if, and only if, $X(0)$ and $X(1)$ have homeomorphic dense sets.*

Making use of Theorem 1 and a result from [**7**] we give Theorem 2, a cardinal generalization of

COROLLARY 1. *Suppose for each $i \in \{0, 1\}$, $X(i)$ is a Čech-complete space with a G_δ-diagonal. Then $X(0)$ and $X(1)$ are weakly co-absolute if, and only if, $X(0)$ and $X(1)$ have homeomorphic dense G_δ-sets.*

The latter improves a result of A. Hager (Corollary 2), C. Gates, D. Maharam, and A. Stone (Corollary 3).

For a space X allow $\mathscr{R}(X)$ to denote the Boolean algebra of regular-open sets. It is known [**6**] that X and Y are weakly co-absolute if and only if $\mathscr{R}(X)$ and $\mathscr{R}(Y)$ are isomorphic, and that if D is dense in X, then $\mathscr{R}(D)$ and $\mathscr{R}(X)$ are isomorphic via the map $\mathscr{O} \rightarrow \text{int}(\text{cl}(\mathscr{O}))$, where int and cl denote, respectively the interior and closure operators. Therefore, homeomorphic dense implies weakly co-absolute.

Proof of Theorem 1. From the previous paragraph we need only show the "only if" part. First observe that each proper open interval of $X(i)$, whose closure contains no end-points of $X(i)$, is an element of $\mathscr{R}(X(i))$. Let $\psi_0:\mathscr{R}(X(0)) \rightarrow \mathscr{R}(X(1))$ be an isomorphism with inverse ψ_1. Arbitrarily choose an infinite collection $\mathscr{A}(0, 0)$ of pairwise-disjoint open intervals of $X(0)$ belonging to $\mathscr{R}(X(0))$ such that $\cup \mathscr{A}(0, 0)$ is dense in $X(0)$. As each $\psi_0(A)$, for $A \in \mathscr{A}(0, 0)$, is regular open and non-empty in $X(1)$, we may choose a collection $\mathscr{A}(A)$ of pairwise-disjoint open intervals of $X(1)$ such that $B \in \mathscr{A}(A)$ implies $\text{cl}(B) \subsetneq \psi_0(A)$ and

Received December 14, 1979 and in revised form May 15, 1980.

$B \in \mathscr{R}(X(1))$, and $\bigcup \mathscr{A}(A)$ is dense in $\psi(A)$. Let

$$\mathscr{A}(1, 0) = \{B : B \in \mathscr{A}(A),\ A \in \mathscr{A}(0, 0)\}.$$

Since ψ_0 is an isomorphism $\bigcup \mathscr{A}(1, 0)$ is dense in $X(1)$.

Suppose for a given ordinal α we have, for each $\beta < \alpha$ and $i \in \{0, 1\}$, collections $\mathscr{A}(i, \beta)$ of pairwise-disjoint open intervals of $X(i)$ to satisfy:

(1) There exists a maximal (with respect to inclusion) chain $\mathfrak{M}$ in

$$T(i, \alpha) = \bigcup \{\mathscr{A}(i, \beta) : \beta < \alpha\}$$

such that int $(\bigcap \mathfrak{M}) \neq \emptyset$.

(2) If $\mathfrak{M}$ is a maximal chain in $T(1, \beta)$ and int $(\bigcap \mathfrak{M}) \neq \emptyset$, then

$$\bigcup\{\psi_0(A) : A \in \mathscr{A}(0, \beta),\ \psi_0(A) \subseteq \bigcap \mathfrak{M}\}$$

is dense in $\bigcap \mathfrak{M}$.

(3) If $A \in \mathscr{A}(0, \beta)$, then there exists a maximal chain $\mathfrak{M}$ in $T(1, \beta)$ such that cl $(A) \subsetneq \psi_1(B)$ for all $B \in \mathfrak{M}$.

(4) If $A \in \mathscr{A}(0, \beta)$ then $\bigcup\{\psi_1(B) : B \in \mathscr{A}(1, \beta),\ \psi_1(B) \subseteq A\}$ is dense in A.

(5) If $B \in \mathscr{A}(1, \beta)$, then there exists $A \in \mathscr{A}(0, \beta)$ such that cl $(B) \subsetneq \psi_0(A)$.

We construct $\mathscr{A}(i, \alpha)$ as follows: For each maximal chain $\mathfrak{M}$ in $T(1, \alpha)$ such that int $(\bigcap \mathfrak{M}) \neq \emptyset$, let $\mathscr{A}(\mathfrak{M})$ be a collection of pairwise-disjoint open intervals of $X(1)$ such that $C \in \mathscr{A}(\mathfrak{M})$ implies cl $(C) \subseteq \bigcap \mathfrak{M}$, and $\bigcup \mathscr{A}(\mathfrak{M})$ is dense in $\bigcap \mathfrak{M}$. For each $C \in \mathscr{A}(\mathfrak{M})$ let $\mathscr{A}(C)$ be a collection of pairwise-disjoint open intervals of $X(0)$ such that $A \in \mathscr{A}(C)$ implies $\psi_0(A) \subseteq C$, and $\bigcup \mathscr{A}(I)$ is dense in $\psi_1(C)$. Let

$$\mathscr{A}(0, \alpha) = \{A : A \in \mathscr{A}(C),\ C \in \mathscr{A}(\mathfrak{M}),\ \mathfrak{M} \text{ is a maximal chain in } T(1, \alpha) \text{ with int } (\bigcap \mathfrak{M}) \neq \emptyset\}.$$

$\mathscr{A}(1, \alpha)$ is constructed from $\mathscr{A}(0, \alpha)$ as $\mathscr{A}(1, 0)$ was found from $\mathscr{A}(0, 0)$.

Suppose λ is the first ordinal such that $\alpha < \lambda$ implies $\forall \beta < \alpha$ and $i \in \{0, 1\}$ $\mathscr{A}(i, \beta)$ have been constructed to satisfy (1) through (5), and each maximal chain $\mathfrak{M}$ of $T(i, \lambda) = \bigcup \{\mathscr{A}(i, \alpha) : \alpha < \lambda\}$ has int $(\bigcap \mathfrak{M}) = \emptyset$. Given one of these maximal chains $\mathfrak{M}$ of $T(i, \lambda)$, (3) and (5) imply $|\bigcap \mathfrak{M}| = 1$, since $X(i)$ is compact and connected. So $\mathfrak{M}$ is a local base for $x(i, \mathfrak{M}) \in \bigcap \mathfrak{M}$. Let

$$D(i) = \{x(i, \mathfrak{M}) : \mathfrak{M} \text{ is a maximal chain of } T(i, \lambda)\}.$$

To see that $D(i)$ is dense in $X(i)$ we suppose I is an open interval of $X(i)$ such that $I \cap D(i) = \emptyset$, then

(*) $\quad A \not\subseteq I$ for all $A \in \mathscr{A}(i, \alpha)$ and all $\alpha < \lambda$.

https://doi.org/10.4153/CJM-1981-066-1 Published online by Cambridge University Press

However, there is a first β, $0 < \beta < \lambda$ such that $I \not\subseteq A$ for all $A \in \mathscr{A}(i, \beta)$. So we may find a maximal chain $\mathfrak{N}$ of $T(i, \beta)$ with $I \subseteq \bigcap \mathfrak{N}$. From (*), (2) and (4) I meets at least two elements of $\mathscr{A}(i, \beta)$, say A_1 and A_2. Similarly, each $I \cap A$ intersects two elements of $\mathscr{A}(i, \beta + 1)$. So the open interval I intersects at least four pairwise-disjoint open intervals of $X(i)$. As at least two of these are necessarily subsets of I, (*) is contradicted.

Finally, we observe that (2) through (5) imply that for each i and each maximal chain $\mathfrak{M}(i)$ of $T(i, \lambda)$ there exists a maximal chain $\mathfrak{M}(|1 - i|)$ of $T(|1 - i|, \lambda)$ such that for each limit ordinal $\alpha \leqq \lambda$

$$\bigcap (\mathfrak{M}(|1 - i|) \cap T(|1 - i|, \alpha) = \bigcap \{\psi_i(A) : A \in \mathfrak{M}(i) \cap T(i, \alpha)\}.$$

Since distinct maximal chains of $T(i, \lambda)$ contain distinct elements, and since $\mathfrak{M}$ is a local base for $x(i, \mathfrak{M})$, the map $x(0, \mathfrak{M}(0)) \rightarrow x(1, \mathfrak{M}(1))$ is a homeomorphism from $D(0)$ onto $D(1)$.

Recall that the *Baire number*, $b(X)$, of a space X is the lease cardinal κ such that the intersection of some family of κ many dense open subsets of X fails to be dense. Similarly, we have the *strong Baire number*, $sb(X)$, as the least cardinal κ such that the intersection of some family of κ many dense open subsets of X fails to have dense interior. Finally, we let $\Delta(X)$ denote the least cardinal κ such that the diagonal, $\{(x, x) : x \in X\}$ of X is the intersection of κ many open sets of $X \times X$. In [**7**] a Boolean algebra equivalence of $sb(X)$ in $R(X)$ was given and it was proved that $sb(X)$ is preserved by weak co-absoluteness; and, for any regular space X, $sb(X) \leqq \Delta(X)$.

LEMMA [**7**]. *If a completely regular space Y is the intersection of at most $sb(Y)$ open sets of βY, if $\Delta(Y) = sb(Y) < b(Y)$, and if Y has no isolated points, then Y possesses a dense set G satisfying:*

(1) *G is the intersection of $sb(Y)$ many open sets of $\beta(Y)$.*

(2) *G is linearly orderable via a dense ordering (i.e., G is a dense subset of a compact connected linearly ordered space).*

THEOREM 2. *Suppose, for $i \in \{0, 1\}$, $Y(i)$ is the intersection of at most $sb(Y(i))$ open subsets of $\beta Y(i)$ and $\Delta(Y(i)) < b(Y(i))$. Then $Y(0)$ and $Y(1)$ are weakly co-absolute if, and only if, there is in each $Y(i)$ a dense set $D(i)$, the intersection of $\Delta(Y(i))$ many open sets, such that $D(0)$ and $D(1)$ are homeomorphic.*

Proof. Again we need show just the "only if" part. We let κ be the cardinal $\Delta(Y(0)) = sb(Y(0)) = sb(Y(1)) = \Delta(Y(i))$. We assume without loss of generality that $Y(i)$ has no isolated points. Let $G(i)$ be the subspace of $Y(i)$ given by the lemma and $X(i)$ its Dedekind compactification with end-points. Since $G(i)$ is the intersection of κ many open sets of $\beta Y(i)$, the same is true in $X(i)$ (see [**1**], for example). We write

$$G(i) = \bigcap\{U(i, \alpha):\alpha < \kappa\}, \text{ where } U(i, \alpha) \text{ is open and dense in } X(i),$$

and

$$\{(y, y):y \in Y(i)\} = \bigcap\{\mathcal{O}(i.\alpha):\alpha < \mathfrak{N}\}, \text{ where } \mathcal{O}(i, \alpha) \text{ is open in } Y(i) \times Y(i).$$

Since $Y(0)$ and $Y(1)$ are weakly co-absolute, $X(0)$ and $X(1)$ are co-absolute and, for each i, $\kappa = sb(X(i))$. Thus, we enter the proof of Theorem 1. To the construction of the families $\mathcal{A}(i, \alpha)$ we may easily add

(6) If $\alpha < \kappa$, then $A \subseteq U(i, \alpha)$ for every $A \in \mathcal{A}(i, \alpha)$.

(7) If $\alpha < \kappa$, then $(A \cap G(i)) \times (A \cap G(i)) \subseteq \mathcal{O}(i, \alpha)$.

Suppose $\lambda > \kappa$, then by (1) of the Theorem 1 there exists a maximal chain $\mathfrak{M}$ in $T(i, \kappa + 1)$ such that int $(\bigcap \mathfrak{M}) \neq \emptyset$. From (7)

$$\emptyset \neq (G(i) \cap \text{int } (\bigcap \mathfrak{M})) \times (G(i) \cap \text{int } (\bigcap \mathfrak{M}) \subseteq \mathcal{O}(i, \alpha) \quad \text{for all } \alpha < \kappa,$$

a contradiction. Thus, $\lambda \leqq \kappa$ and by (6) the spaces $D(i) \subseteq G(i)$. If $\lambda < \kappa$, $D(i)$ has interior in $X(i)$ and, hence, contains an interval of $X(i)$ which must be a subset of each member of a maximal chain of $T(i, \lambda)$ which contradicts our choice of λ. So $\lambda = \kappa$. Finally, $D(i) = \bigcap\{U\mathcal{A}(i, \alpha):\alpha < \lambda\}$.

Corollary 1 is now proved, and as well:

COROLLARY 2 [**3**]. *Suppose, for each $i \in \{0, 1\}$, $X(i)$ is a Cech-complete space with a G_δ-diagonal. If $X(0)$ and $X(1)$ are co-absolute, then $X(0)$ and $X(1)$ have homeomorphic dense G_δ-sets.*

COROLLARY 3 [**2, 5**]. *Suppose, for each $i \in \{0, 1\}$, $X(i)$ is a completely metrizable space. If $X(0)$ and $X(1)$ are weakly co-absolute, then $X(0)$ and $X(1)$ have homeomorphic dense G_δ-sets.*

Remarks. (1) I learned of Hager's result through a verbal communication with M. Henriksen and immediately observed Corollary 1 to be true by use of the techniques above; substantially different from those of Hager. After our presentation of these results at the 1979 Annual Spring Topology Conference (Athens, Ohio) we learned of [**2**] and [**5**].

(2) Certainly Corollary 2 implies Corollary 3; however, the lemma reads in the $\Delta(X) = \omega$ case: there exists a dense G_δ-set linearly orderable and having a G_δ-diagonal. From [**4**] this subspace is completely metrizable.

(3) Weakly co-absolute and Cech-complete have generalizations (cardinal in the latter case) to the class of regular spaces. We note that in this case the lemma and Theorem 2 are still true.

(4) Can "connected" be removed from Theorem 1? In [7] it is proved that if X is a space with a dense set of points each having a well-ordered local base, and if X is weakly co-absolute with a linearly ordered space, then X has a dense linearly orderable subspace (densely orderable when X has no isolated points). From this it follows that each separable 1st countable space without isolated points has a dense subspace homeomorphic to the space of rationals, and each compact linearly ordered space without isolated points shares a dense subspace with a compact connected linearly ordered space. In spite of these results I conjecture the answer to the question to be no.

The author wishes to give his appreciation to The Institute for Medicine and Mathematics and Ohio University for their fine hospitality during a period when a substantial part of the results in this paper were discovered.

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