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RESOLVENT MEANS AND INVERTING GENERALIZED FOURIER TRANSFORMS

LOUISE A. RAPHAEL

1. Introduction. Let S-L denote a singular Sturm-Liouville system on the half line with homogeneous boundary conditions, possessing a discrete negative and continuous positive spectrum. Let A be the S-L operator and $S_\alpha(f;\, x)$ the S-L eigenfunction expansion associated with the resolvent operator $(z - A)^{-1}$, z complex. That is, $S_\alpha(f;\, x)$ denotes the resolvent summability means with weight function $z(z - \lambda)^{-1}$ (or $(1 + t\lambda)^{-1}$ where $t = -1/z$).

We first study the problem of determining when

$$(1) \qquad S_\alpha(f;\, x) = \frac{1}{\alpha}\int_0^\infty G\left(x,\, s;\, \frac{1}{\alpha}\right)f(s)ds$$

where $G\left(x,\, s;\, \dfrac{1}{\alpha}\right)$ is the Green's function associated with a certain perturbation of our system.

In proving summability we will use (1) and the fact that the resolvent kernel has properties analogous to those of certain L^1-radially decreasing convolution kernels. We then give an answer to the classical question: given the generalized Fourier transform $F(\lambda)$ of an $L^p(0, \infty)$ $(1 \leqq p < \infty)$ function f, how can we recover f from $F(\lambda)$? Difficulties in answering this stems from the generalized nature of our eigenfunctions, and that $F(\lambda)$ need not be integrable; indeed for $p > 2$, $F(\lambda)$ may not be a function. Our methods depend on the assumption that F is in L^2_{loc}.

A classical solution to the pointwise evaluation of an inverse Fourier transform is to apply summability methods such as those of Cèsaro, Abel [**1**], [**6**], or a method whose kernel is an L^1-dilation of a radially decreasing convolution kernel [**12**], [**3**], [**4**]. We show that our resolvent kernel satisfies the latter condition.

Due to the nature of our problem, eigenfunction expansions have abstract rather than explicit representations. In addition, kernels will be considered on the basis of certain properties rather than computed.

Our answer to pointwise summation questions regarding the inversion of the generalized Fourier transform is that the resolvent summability means of the $L^p(0, \infty)$ $(1 \leqq p < \infty)$ function f converge to f in L^p and

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pointwise on the Lebesgue set of f. Our arguments are based on our results for $p = 2$, proved in [**9**], and simple techniques from harmonic analysis.

Our reason for considering resolvent summability is that in [**10**] we prove that under conditions on the modified resolvent operator, resolvent summability implies general analytic summability methods. This implies the validity of general analytic summability (see [**3**, **4**]) methods.

Historically, for discrete cases, summability means of an eigenfunction expansion are obtainable by an application of the resolvent of the S-L operator [**5**], [**8**]. The resolvent summability method for divergent integrals was introduced in [**9**] under the name Stieltjes summability (because of its relation to the Stieltjes transform). This method is a scaled version of Tikhonov's regularization principle for solving a class of ill-posed problems [**13**].

In the final section of this paper, we consider the ill-posed problem where the coefficients in the expansion of an $L^p(0, \infty)$ $(1 \leqq p < \infty)$ function are perturbed slightly in the $L^2(0, \infty)$ norm. Our last theorem states that the resolvent summability method is stable. We caution the reader that the f_γ, F_γ in the final theorem are not related to the f_n, F_n used for the function and generalized Fourier coefficients in the previous sections.

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2. The S-L **system.** A detailed mathematical formulation of our S-L system is now presented.

Let S-L denote the singular Sturm-Liouville system

$$(2) \qquad -u''(x, \lambda) + q(x)u(x, \lambda) = \lambda u(x, \lambda)$$

with boundary conditions

$$(3) \qquad u(0, \lambda)\cos \beta + u'(0, \lambda)\sin \beta = 0 \quad \text{and}$$
$$u(\infty, \lambda) < \infty, 0 \leqq \beta < \pi$$

where $q(x) \in L^1(0, \infty) \cap L^\infty(0, \infty)$ is continuous and real valued. The functions $u(x, \lambda)$ (for all λ in the spectrum) are normalized by the conditions

$$(4) \qquad u(0, \lambda) = \sin \beta \quad \text{and} \quad u'(0, \lambda) = -\cos \beta.$$

The spectrum of S-L is bounded from below, discrete for $\lambda \leqq 0$ and continuous for $\lambda > 0$ ([**8**], Theorem 3.1, p. 209 and Theorem 3.2, p. 211). The non-positive spectrum is denoted by $\{\lambda_n\}$ and the associated eigenfunctions by $\{u(x, \lambda_n)\}$. In general $u(x, \lambda)$ denotes the eigenfunction associated with the spectral element λ, where $\lambda \in \{\lambda_n\} \cup (0, \infty)$.

For $f(x) \in L(0, \infty)$, the S-L expansion of f is given by

$$\text{(5a)} \qquad f(x) \sim \sum_{\lambda_n} F(\lambda_n) u(x, \lambda_n) d_n + \int_0^\infty F(\lambda) u(x, \lambda) d\rho(\lambda)$$

$$= \int_{-b}^\infty F(\lambda) u(x, \lambda) d\rho(\lambda)$$

where

$$\text{(5b)} \qquad F(\lambda) \sim \int_0^\infty f(x) u(x, \lambda) dx.$$

In (5a), $-b = \inf \lambda_n$, $\rho(\lambda)$ is the spectral function of the system under the normalization (4), and

$$d_n = \rho(\lambda_n^+) - \rho(\lambda_n^-).$$

In (5b), $F(\lambda)$ is the generalized Fourier transform of $f(x)$ and $F \in L_\rho^2(-b, \infty)$ where L_ρ^2 denotes the square norm with respect to the measure $\rho(\lambda)$. The symbol $\sim$ denotes convergence in the L^2-norm as the upper limit of summation or integration becomes infinite.

When $f \in L^2(0, \infty)$ the expansion in (5a) converges to f in the $L^2(0, \infty)$ norm, but the convergence need not be pointwise. For $f \in L^p(0, \infty)$, $p \geqq 1$ and $p \neq 2$, the question of existence of the generalized integrals in (5) must be determined. To illustrate the complexity of this question, we cite the classical Fourier integral. Every $f \in L^p(-\infty, \infty)$ $(1 \leqq p \leqq \infty)$ has a Fourier transform $F(\lambda)$ (defined as a tempered distribution) that coincides with an L^p function if $1 \leqq p \leqq 2$. But for $p > 2$ there exist L^p functions whose Fourier transforms cannot be expressed as a function. So one of our objectives is to determine when (5b) and resolvent summability means of (5a) exist.

All proofs herein are for the class of singular continuous spectrum S-L expansions associated with the system (2) through (4). We remark that these proofs carry over, more simply, to regular S-L expansions on finite intervals.

3. The resolvent kernel. The summator or weight function for the resolvent summability method is $\phi(\lambda) = (1 + \lambda)^{-1}$. The *resolvent summability means*, where they exist, of the S-L expansion (5a) are denoted by

$$\text{(6)} \qquad S_\alpha(f; x) = \int_{-b}^\infty \frac{f(\lambda)}{1 + \alpha\lambda} u(x, \lambda) d\rho(\lambda), \quad 0 < \alpha < \frac{1}{b}$$

where α is the summation parameter and x is fixed. The S-L expansion (5a) is called resolvent-summable at x_0 if

$$\lim_{\alpha \to 0} S_\alpha(f; x_0)$$

exists and called resolvent summable to f at x_0 if

$$\lim_{\alpha\to 0} S_\alpha(f;\, x_0) = f(x_0).$$

Formally $S_\alpha(f;\, x)$ may be rewritten as

$$(7) \qquad S_\alpha(f;\, x) = \int_0^\infty f(x)K_\alpha(x,\, s)ds$$

where the kernel $K_\alpha(x,\, s)$ is formally given by

$$K_\alpha(x,\, s) = \int_0^\infty \frac{u(x,\, \lambda)u(s,\, \lambda)}{1 + \alpha\lambda} d\rho(\lambda).$$

The immediate objectives are to determine when $S_\alpha(f;\, x)$ is defined and to derive a formula for the kernel $K_\alpha(x,\, s)$.

It is a consequence of [**9**] that for $f \in L^2(0,\, \infty)$

$$(8) \qquad S_\alpha(f;\, x) = \frac{1}{\alpha} \int_0^\infty G\left(x,\, s;\, \frac{1}{\alpha}\right) f(s)ds$$

where $G\left(x,\, s;\, \dfrac{1}{\alpha}\right)$ is the Green's function of the distributional equation

$$(9) \qquad -u''(x,\, \lambda) + \left[q(x) + \frac{1}{\alpha}\right]u(x,\, \lambda) = \delta_s(x)$$

with boundary conditions (3), and where δ_s is the Dirac distribution centered at s. From (7) and (8), it is natural to define the resolvent kernel $K_\alpha(x,\, s)$ for $f \in L^p(0,\, \infty)$ to be $\dfrac{1}{\alpha} G\left(x,\, s;\, \dfrac{1}{\alpha}\right)$ where $G\left(x,\, s;\, \dfrac{1}{\alpha}\right)$ is as above.

To determine under what conditions (8) would be true for $L^p(0,\, \infty)$ functions, $(1 \leqq p \leqq \infty)$, we must, due to the nature of the eigenfunctions, consider cases based on whether $\sin \beta = 0$ or not. First we note (8) was proved for $L^2(0,\, \infty)$ functions in [**9**]. In general, our procedure is as follows: For an $L^1(0,\, \infty)$ function $f^{(1)}$, we let $\{f_n\}_n$ be a sequence of functions which belongs to a dense subset of $L^1(0,\, \infty)$, say $L^1 \cap L^2$, and which converges to $f^{(1)}$ with respect to the L^1-norm. We then prove identity (8) for $L^1(0,\, \infty)$ functions by using (8) for $L^2(0,\, \infty)$ functions. An interpolation theorem will prove (8) for $L^p(0,\, \infty)$ $(1 < p < 2)$ functions. Lastly, for $p > 2$ we impose restrictions to insure that the improper integrals exist and thereby proving the identity (8) by standard arguments.

First we need a bound on the Green's function $G\left(x,\, s;\, \dfrac{1}{\alpha}\right)$. Since we are unable to calculate $G\left(x,\, s;\, \dfrac{1}{\alpha}\right)$ directly, we calculate the Green's function $G^*\left(x, s; \dfrac{1}{\alpha}\right)$ for $\alpha > 0$ of

$$(10) \qquad -u''(x) + \frac{1}{\alpha}u(x) = \delta_s(x)$$

where

$$u(0, \lambda)\cos\beta + u'(0, \lambda)\sin\beta = 0,\ u(\infty, \lambda) < \infty, \quad 0 \leqq \beta < \pi$$

$$(11) \qquad G^*\left(x, s; \frac{1}{\alpha}\right) = \begin{cases} \dfrac{\sqrt{\alpha}e^{-x/\sqrt{\alpha}}}{\sqrt{\alpha}\cos\beta - \sin\beta}\left\{\sqrt{\alpha}\cos\beta\sinh\dfrac{s}{\sqrt{\alpha}} - \sin\beta\cosh\dfrac{s}{\sqrt{\alpha}}\right\}, & s < x \\[2ex] \dfrac{\sqrt{\alpha}e^{-s/\sqrt{\alpha}}}{\sqrt{\alpha}\cos\beta - \sin\beta}\left\{\sqrt{\alpha}\cos\beta\sinh\dfrac{x}{\sqrt{\alpha}} - \sin\beta\cosh\dfrac{x}{\sqrt{\alpha}}\right\}, & s > x \end{cases}$$

We recall that the $q(x)$ in our S-L system is uniformly bounded, say by $M > 0$. This was done in order to be able to bound $G\left(x, s; \dfrac{1}{\alpha}\right)$. The import of our next lemma is that the Green's function of our S-L system is bounded between a combination of Green's functions, each of which is bounded by a cusp-shaped convolution kernel which is radially decreasing on **R**.

LEMMA 1. (a)

$$(12a) \qquad G^*\left(x, s; \frac{1}{\alpha}\right) = G^{(1)}\left(x, s; \frac{1}{\alpha}\right) + G^{(2)}\left(x, s; \frac{1}{\alpha}\right)$$

where

$$(12b) \qquad G^{(1)}\left(x, s; \frac{1}{\alpha}\right) = \sqrt{\alpha}\frac{e^{-|x-s|/\sqrt{\alpha}}}{2},$$

$$G^{(2)}\left(x, s; \frac{1}{\alpha}\right) = \frac{\sqrt{\alpha}}{2}\,\frac{\sin\beta + \sqrt{\alpha}\cos\beta}{(\sin\beta - \sqrt{\alpha}\cos\beta)}\,e^{-|x+s|/\sqrt{\alpha}}$$

$$0 \leqq x, s < \infty$$

(b) *For* $|q(x)| \leqq M$, $-M + \dfrac{1}{\alpha} > 0$ *and* $x \in (0, \infty)$

$$(13) \qquad G^{**}\left(x, s; M + \frac{1}{\alpha}\right)$$

$$= G^{(1)}\left(x, s; M + \frac{1}{\alpha}\right) - \left|G^{(2)}\left(x, s; M + \frac{1}{\alpha}\right)\right| \leqq G\left(x, s; \frac{1}{\alpha}\right)$$

$$\leqq G^{(1)}\left(x, s; -M + \frac{1}{\alpha}\right) + \left|G^{(2)}\left(x, s; -M + \frac{1}{\alpha}\right)\right|$$

$$= G^{**}\left(x, s; -M + \frac{1}{\alpha}\right)$$

(c) (14) $$\frac{1}{\alpha}\left|G\left(x, s; \frac{1}{\alpha}\right)\right| \leqq \frac{3}{\sqrt{\alpha}} e^{-|x-s|/\sqrt{\alpha}}.$$

Proof. Part (a) follows from an easy calculation on (11). Part (b) follows by substituting $\pm M + \frac{1}{\alpha}$ for $\frac{1}{\alpha}$ in (12b) and comparing terms. Part (b) implies part (c).

The following approximations of the eigenfunctions and spectral function ρ for large λ are a consequence of [**8**, Equation 3.5, p. 205] and [**8**, Theorem 3.2, p. 211 and p. 206] respectively.

LEMMA 2. *As* $\lambda \to \infty$, $u(x, \lambda)$ *and* $\rho(\lambda)$ *satisfy* (a) *if* $\sin \beta = 0$

(15) $$u(x, \lambda) = -\frac{\cos \beta}{\sqrt{\lambda}} \sin \sqrt{\lambda} x + O\left(\frac{1}{\lambda}\right)$$

$$\rho'(\lambda) = \frac{\sqrt{\lambda}}{\pi \cos^2 \beta} + O(1),$$

and

(b) *if* $\sin \beta \neq 0$

(16) $$u(x, \lambda) = \sin \beta \cos \sqrt{\lambda} x + O\left(\frac{1}{\lambda}\right)$$

$$\rho'(\lambda) = \frac{1}{\pi \sqrt{\lambda} \sin^2 \beta} + O\left(\frac{1}{\lambda}\right).$$

We remark that $u(x, \lambda)$ is bounded in x for fixed λ, and bounded in λ for fixed x, but it is not bounded jointly. Moreover, $u(x, \lambda)$ may get large as λ gets small for x large.

We prove Propositions 4, 6, 7 in the following order: $\sin \beta \neq 0$ and $1 \leqq p \leqq 2$, $\sin \beta = 0$ and $1 \leqq p \leqq 2$; and lastly for arbitrary $\sin \beta$ and $p > 2$. We have noticed that the proofs of these propositions go through identically if we replace the resolvent weight function $(1 + \lambda)^{-1}$ by a summator function $\phi(\lambda)$ which is analytic, bounded and such that

$$\int_{-b}^{\infty} \frac{\phi(\lambda)}{\sqrt{\lambda}} d\lambda < \infty.$$

Moreover, if our summator function $(1 + \lambda)^{-1}$ were replaced by a function $\phi(\lambda)$ which is in $L^1(0, \infty)$ and analytic, then the following propositions can be obtained with fewer assumptions. Of course, the resolvent weight function satisfies

$$\int_{-b}^{\infty} \frac{\phi(\lambda)d\lambda}{\sqrt{\lambda}} < \infty,$$

but is not in $L^1(0, \infty)$.)

In Lemma 3 and Proposition 4 the following notation is used when $\{f_n\}_{n=1}^{\infty}$ is a sequence of $L^1 \cap L^2$ functions. Let

$$F_n(\lambda) = \int_0^{\infty} f_n(x)u(x, \lambda)dx$$

and

$$S_\alpha(f_n, x) = \int_{-b}^{\infty} \frac{F_n(\lambda)}{1 + \alpha\lambda} u(x, \lambda)d\rho(\lambda).$$

LEMMA 3. *For* $\sin \beta \neq 0$, *if* $f \in L^1(0, \infty)$ *and* $\{f_n\}_{n=1}^{\infty}$ *is a net of* $L^1 \cap L^2$ *functions which converges to* f *in the* L^1*-norm, and if* $u(x, \lambda)$ *is uniformly bounded in* λ *for* x *large, then*

(a) $F_n(\lambda) - F(\lambda)$ *converges uniformly in* λ *to zero as* $n \to \infty$;

(b) $S_\alpha(f_n, x) - S_\alpha(f; x)$ *converges pointwise to zero as* $n \to \infty$;

(c) $S_\alpha(f; x)$ *is continuous in* x *and belongs to* $L^1(0, \infty)$;

(d) $$S_\alpha(f; x) = \frac{1}{\alpha} \int_0^{\infty} G\left(x, s; \frac{1}{\alpha}\right) f(s)ds$$

pointwise where $G\left(\cdot; \frac{1}{\alpha}\right)$ *is Green's function for* (9).

Proof. By Lemma 2 part (b), $u(x, \lambda)$ is uniformly bounded in λ for bounded x. The proof of part (a) follows by our assumption on $u(x, \lambda)$ and $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

To prove part (b), we write

$$S_\alpha(f_n, x) - S_\alpha(f; x) = \int_{-b}^{\infty} [F_n(\lambda) - F(\lambda)] \frac{u(x, \lambda)}{1 + \alpha\lambda} d\rho(\lambda).$$

In the case $\sin \beta \neq 0$, Lemma 2 gives

$$u(x, \lambda) = O(\sqrt{\lambda}), \quad d\rho(\lambda) = O\left(\frac{1}{\lambda}\right)$$

and so the product of the last three terms in the integral is $O\left(\frac{1}{\lambda^{3/2}}\right)$.

The proof is completed by using part (a).

To prove part (c) one need only observe that $F(\lambda) \in L^{\infty}(0, \infty)$ (as $f \in L^1(0, \infty)$), $u(x, \lambda)$ is uniformly bounded for bounded x-intervals, and

$$\frac{d\rho(\lambda)}{1 + \alpha\lambda} = O\left(\frac{1}{\lambda^2}\right).$$

Thus the improper integral

$$\int_{-b}^{\infty} \frac{F(\lambda)}{1 + \alpha\lambda} u(x, \lambda) d\rho(\lambda)$$

converges absolutely and uniformly in x. Hence $S_\alpha(f; x)$ is continuous in x and belongs to $L^1(0, \infty)$.

Finally to prove part (d), we recall [**9**] that for $f_n \in L^2(0, \infty)$

$$S_\alpha(f_n; x) = \frac{1}{\alpha} \int_0^{\infty} G\left(x, s; \frac{1}{\alpha}\right) f_n(s) ds$$

pointwise. Next we note that

$$\frac{1}{\alpha} \int_0^{\infty} G\left(x, s; \frac{1}{\alpha}\right) f_n(s) ds$$

converges pointwise to

$$\frac{1}{\alpha} \int_0^{\infty} G\left(x, s; \frac{1}{\alpha}\right) f(s) ds$$

as

$$\left|\frac{1}{\alpha} \int_0^{\infty} G\left(x, s; \frac{1}{\alpha}\right) [f_n(s) - f(x)] ds\right|$$
$$\leqq \frac{1}{\alpha} \left|\left| G\left(\cdot; \frac{1}{\alpha}\right) \right|\right|_{\infty} \|f_n - f\|_1 \to 0$$

pointwise as $n \to \infty$. (For the bound on $G\left(\cdot; \frac{1}{\alpha}\right)$ see Lemma 1.)

So $S_\alpha(f_n; x)$ converges pointwise to both $S_\alpha(f; x)$ (part (b)) and

$$\frac{1}{\alpha} \int_0^{\infty} G_\alpha(x, s) f(s) ds$$

and so the proof of part (d) is complete.

Next we use an interpolation theorem to extend the result of part (d) to $L^p(0, \infty)$ functions for p between 1 and 2.

PROPOSITION 4. *For* $\sin \beta \neq 0, f \in L^p(0, \infty)$, $1 \leqq p \leqq 2$ *if* $\{f_n\}_{n=1}^{\infty}$ *is a* *net of $L^1 \cap L^2$ functions which converges to L^1 part of f in the L^1-norm and $u(x, \lambda)$ is uniformly bounded in λ for x large, then*

(a) $S_\alpha(f; x)$ *is continuous in* x;

(b) $$S_\alpha(f; x) = \frac{1}{\alpha}\int_0^\infty G\left(x, s; \frac{1}{\alpha}\right)f(s)ds$$

pointwise and where $G\left(\cdot; \frac{1}{\alpha}\right)$ *is the Green's function for* (9);

(c) $S_\alpha(f; x)$ *belongs to* $L^p(0, \infty)$, $1 \leqq p \leqq 2$.

Proof. If $f \in L^p(0, \infty)$ where $1 < p < 2$, then f can be expressed as the sum of an $L^1(0, \infty)$ and $L^2(0, \infty)$ function. Namely, if χ denotes the characteristic function, then

$$f(x) = f(x)\chi_{|f|\leqq 1}(x) + f(x)\chi_{|f|\geqq 1}(x) = f_1(x) + f_2(x).$$

We write

$$S_\alpha(f; x) = S_\alpha(f_1; x) + S_\alpha(f_2; x) = \int_0^\infty \frac{F(\lambda)}{1 + \alpha\lambda}u(x, \lambda)d\rho(\lambda)$$

where $F(\lambda)$ is the sum of the $L^\infty(0, \infty)$ Fourier coefficient $F_1(\lambda)$ associated with f_1 and the $L^2(0, \infty)$ Fourier coefficient $F_2(\lambda)$ associated with f_2.

The continuity of $S_\alpha(f; x)$ now follows by separate arguments for the continuity of $S_\alpha(f_1; x)$ and $S_\alpha(f_2; x)$. The first follows by Lemma 3 and the second by Cauchy-Schwarz, [**8**, Corollary 1, p. 116], and [**9**].

Similarly, the proof of part (b) follows by separate arguments on $S_\alpha(f_1; x)$ and $S_\alpha(f_2; x)$ (Lemma 3 and [**9**]).

Finally we show that $S_\alpha(f; x)$ belongs to $L^p(0, \infty)$ by observing that $G\left(x, s; \frac{1}{\alpha}\right)$ is bounded by half of a cusp-shaped convolution kernel (see Lemma 1). As $f \in L^p(0, \infty)$, $G\left(x, s; \frac{1}{\alpha}\right)$ and its bound $\in L^1(0, \infty)$, the convolution is well defined and converges absolutely a.e. in x. So by Young's inequality we have

$$\begin{aligned}
\|S_\alpha(f; s)\|_p &= \left\|\frac{1}{\alpha}\int_0^\infty G\left(x, s; \frac{1}{\alpha}\right)f(s)ds\right\|_p \\
&\leqq C\left\|\int_0^\infty \frac{e^{-|x-s|/\sqrt{\alpha}}}{\sqrt{\alpha}}f(s)ds\right\|_p \\
&\leqq C\left\|\frac{e^{-|x-s|/\sqrt{\alpha}}}{\sqrt{\alpha}}\right\|_1 \|f\|_p,
\end{aligned}$$

where $C = 3$.

The hypotheses of Proposition 4 are satisfied, for example, when $q = 0$ in our S-L system, for then $u(x, \lambda) = \sin \sqrt{\lambda}x$.

We will use the following notation in the next two lemmas. Let $L^2_{\text{loc}}(0, \infty)$ denote the space of functions which are square integrable over every compact subset of $(0, \infty)$. For $f \in L^p(0, \infty)$ we define

$$f_N(x) = \int_{-b}^{N} F(\lambda)u(x, \lambda)d\rho(\lambda)$$

$$S_\alpha(f_N, x) = \int_{-b}^{N} F(\lambda)\frac{u(x, \lambda)}{1 + \alpha\lambda}d\rho(\lambda),$$

where $F(\lambda) \in L^\infty[-b, N]$ for $p = 1$, and $F(\lambda) \in L^2_{\text{loc}}(0, \infty)$ for $p > 2$.

We continue now with the case $\sin \beta = 0$ and $1 \leqq p \leqq 2$.

Due to the scarcity of conditions for equality (8), we include the following pair of sufficient conditions in Lemma 5 and Proposition 6 for (8) to hold.

LEMMA 5. *For* $\sin \beta = 0$ *and* $f \in L^1(0, \infty)$, *if*

$$f_N(x) \to f(x) \quad \textit{in } L^1(0, \infty),$$

and

$$S_\alpha(f_N; x) \to S_\alpha(f; x) \quad \textit{pointwise as } N \to \infty$$

then

(a) $$S_\alpha(f_N; x) = \frac{1}{\alpha}\int_0^\infty G\left(x, s; \frac{1}{\alpha}\right)f_N(s)ds$$

(b) $$S_\alpha(f; x) = \frac{1}{\alpha}\int_0^\infty G\left(x, s; \frac{1}{\alpha}\right)f(s)ds$$

(c) $S_\alpha(f; x) \in L^1(0, \infty)$.

Proof. Define

$$F_N(\lambda) = F(\lambda)\chi_{[-b,N]}(\lambda)$$

$$F_{N,\alpha}(\lambda) = \frac{F(\lambda)}{1 + \alpha\lambda}\chi_{[-b,N]}(\lambda).$$

Clearly F_N and $F_{N,\alpha}$ belong to $L^2 \cap L^\infty$. This in turn implies that f_N and $S_\alpha(f_N; x)$ belong to $L^2(0, \infty)$. And so by [**9**]

$$S_\alpha(f_N; x) = \frac{1}{\alpha}\int_0^\infty G\left(x, s; \frac{1}{\alpha}\right)f_N(s)ds.$$

To prove part (b) we observe that

$$\frac{1}{\alpha}\int_0^\infty G\left(x, s; \frac{1}{\alpha}\right)f_N(s)ds$$

converges to

$$\frac{1}{\alpha}\int_0^\infty G\left(x, s; \frac{1}{\alpha}\right)f(x)ds$$

pointwise as $\|f_N - f\|_1 \to 0$ as $N \to \infty$. By hypothesis, $S_\alpha(f_N; x)$ converges pointwise to $S_\alpha(f; x)$, and so the equality holds.

Part (c) follows from (b), a bound on G and $f \in L^1(0, \infty)$.

Again we use an interpolation theorem to extend Lemma 5 part (b) to $L^p(0, \infty)$ functions, $1 < p < 2$, when $\sin \beta = 0$.

PROPOSITION 6. *For* $\sin \beta = 0$, $f \in L^p(0, \infty)$, $1 \leqq p \leqq 2$ *and* $f_N(x)$ *converge to the* L^1*-part of* f, *in* $L^1(0, \infty)$, $S_\alpha(f_N; x) \to S_\alpha(f; x)$ *pointwise as* $N \to \infty$, *then*

(a) $$S_\alpha(f; x) = \frac{1}{\alpha}\int_0^\infty G\left(x, s; \frac{1}{\alpha}\right)f(s)ds$$

(b) $S_\alpha(f; x) \in L^p(0, \infty)$, $1 \leqq p \leqq 2$.

Proof. If $f \in L^p(0, \infty)$, $1 < p < 2$, then as in the proof of Lemma 4, f can be expressed as the sum of an $L^1(0, \infty)$ and an $L^2(0, \infty)$ function, f_1, f_2 respectively. Thus the generalized Fourier coefficient of f can be expressed as a sum of functions in $L^\infty(0, \infty)$ and $L^2(0, \infty)$. Clearly

$$S_\alpha(f_N; x) = \int_b^N F(\lambda)u(x, \lambda)d\rho(\lambda)$$

exists for finite N. Letting

$$S_\alpha(f; x) = S_\alpha(f_1; x) + S_\alpha(f_2; x)$$

and applying Lemma 5 to $S_\alpha(f_1; x)$ and (8) to $S_\alpha(f_2; x)$ we see part (a) is proved.

The proof of part (b) is similar to that of Proposition 4 part (c).

Lastly we consider the case of $L^p(0, \infty)$ functions for $p > 2$ and $\sin \beta$ arbitrary. The assumptions of the following proposition are motivated by the fact that for $p > 2$, the distribution $F(\lambda)$ may not be a function. We avoid treating this case by assuming F is in L^2_{loc}.

PROPOSITION 7. *For* $f \in L^p(0, \infty)$, $p > 2$ *and* $F(\lambda) \in L^2_{\text{loc}}(0, \infty)$, *if*

(1) $$f_N(x) = \int_{-b}^N F(\lambda)u(x, \lambda)d\rho(\lambda)$$

exists, converges pointwise for each N *and fixed* x.

(2) $f_N(x) \to f(x)$ *in* $L^p(0, \infty)$, $p > 2$;

(3) $S_\alpha(f_N; x)$ *exists for each fixed* N *and* x, *and converges pointwise to* $S_\alpha(f; x)$ *for each fixed* α, x *as* $N \to \infty$, *then*

(a) $S_\alpha(f;\ x) = \frac{1}{\alpha}\int_0^\infty G\left(x,\ s;\ \frac{1}{\alpha}\right)f(s)ds;$

(b) $S_\alpha(f;\ x) \in L^p(0,\ \infty),\ p > 2.$

Proof. As $F(\lambda) \in L^2_{\text{loc}}(0,\ \infty)$ implies $f_N \in L^2(0,\ \infty)$, we have

$$S_\alpha(f_N;\ x) = \int_{-b}^\infty F(\lambda)\chi_{[-b,N]}(\lambda)\frac{u(x,\ \lambda)}{1 + \alpha\lambda}d\rho(\lambda)$$

$$= \frac{1}{\alpha}\int_0^\infty G(x,\ s)f_N(s)ds.$$

Moreover

$$\left|\frac{1}{\alpha}\int_0^\infty G\left(x,\ s;\ \frac{1}{\alpha}\right)f_N(s)ds - \frac{1}{\alpha}\int_0^\infty G\left(x,\ s;\ \frac{1}{\alpha}\right)f(s)ds\right|$$

$$\leqq \left|\left|\frac{1}{\alpha}G\left(\cdot;\ \frac{1}{\alpha}\right)\right|\right|_q \|f_N - f\|_p \to 0$$

as $N \to \infty$ for $p > 0,\ q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. Thus

$$\frac{1}{\alpha}\int_0^\infty G\left(x,\ s;\ \frac{1}{\alpha}\right)f_N(s)ds$$

converges pointwise to

$$\frac{1}{\alpha}\int_0^\infty G\left(x,\ s;\ \frac{1}{\alpha}\right)f(s)ds.$$

The proof of part (a) is completed by using hypothesis (3).

The proof of part (b) follows from Holder's inequality.

The following definition is now motivated by Propositions 4, 6 and 7.

Definition. The *resolvent kernel* $K_\alpha(x,\ s)$ for the resolvent means (6) of an $L^p(0,\ \infty)$, $p \geqq 1$, function f, is defined to be

$$\frac{1}{\alpha}G\left(x,\ s;\ \frac{1}{\alpha}\right)$$

where $G\left(x,\ s;\ \frac{1}{\alpha}\right)$ is the Green's function for the perturbed S-L system (9).

Properties of $K_\alpha(x,\ s)$, the resolvent kernel, analogous to those of the convolution kernels are now proved.

THEOREM 1. *If* $K_\alpha(x, s)$ *is the resolvent kernel, then*

(a) $$\lim_{\alpha \to 0} \int_0^\infty K_\alpha(x, s)ds = 1;$$

(b) *For each* $\epsilon > 0$,
$$\lim_{\alpha \to 0} \int_{|x-s|>\epsilon} K_\alpha(x, s)ds = 0, \quad 0 \leqq x, s < \infty;$$

(c) *For each* $\epsilon > 0$,
$$\lim_{\alpha \to 0} \left[\int_{|x-s|>\epsilon} K_\alpha^q(x, s)ds\right]^{1/q} = 0 \quad \textit{for } q > 1;$$

(d) $K_\alpha(x, s) \to 0$ *uniformly for all* x *and* s *as* $\alpha \to 0$ *for which*
$$|x - s| > \epsilon > 0 \quad \textit{and} \quad 0 \leqq x < s < \infty.$$

Proof. By Lemma 1 it follows that
$$\frac{1}{\alpha} \int_0^\infty G^{**}\left(x, s; M + \frac{1}{\alpha}\right)ds \leqq \frac{1}{\alpha} \int_0^\infty G\left(x, s; \frac{1}{\alpha}\right)ds$$
$$\leqq \frac{1}{\alpha} \int_0^\infty G^{**}\left(x, s; -M + \frac{1}{\alpha}\right)ds.$$

An easy calculation shows that the upper and lower bounds approach one as $\alpha \to 0$. As
$$K_\alpha(x, s) = \frac{1}{\alpha}G_\alpha(x, s),$$
part (a) is proved.

To establish part (b) we use Lemma 1 part (c) and evaluate the bounding function on $|x - s| > \epsilon$ as $\alpha \to 0$.

The proof of part (c) is the same as (b).

Finally to prove part (d) we use Lemma 1 part (c) and note that the bounding function approaches zero as α approaches zero independent of x and s provided $|x - s| > \epsilon > 0$.

We note that the resolvent kernel for $S_\alpha(f_n; x)$ is the same as the resolvent kernel for $S_\alpha(f; x)$.

4. Application. By using two standard theorems of harmonic analysis [**12**, Theorem 1.18, p. 10 and Theorem 1.25, p. 13], we now prove generalized S-L expansions of $L^p(0, \infty)$ $(1 \leqq p < \infty)$ functions are resolvent summable to f with respect to $L^p(0, \infty)$ norm and pointwise. In other words, we are inverting the generalized Fourier transform of an $L^p(0, \infty)$ function $(1 \leqq p < \infty)$.

THEOREM 2. *Let* $f \in L^p(0, \infty)$ $(1 \leqq p < \infty)$ *and* $K_\alpha(x, s)$ *be the resolvent kernel. Then*

(a) $$\frac{1}{\alpha}\int_0^\infty G\left(x, s; \frac{1}{\alpha}\right)f(s)ds = \int_0^\infty K_\alpha(x, s)f(s)ds$$

$$\rightarrow f(x) \text{ as } \alpha \rightarrow 0$$

in $L^p(0, \infty)$ $(1 \leqq p < \infty)$ *and pointwise on the Lebesgue set of* f;

(b) *Under any of the hypotheses of Propositions* 4, 6, 7, *the resolvent summability means*

$$S_\alpha(f; x) = \int_{-b}^\infty \frac{F(\lambda)}{1 + \alpha\lambda}u(x, \lambda)f(s)ds$$

$$\rightarrow f(x) \text{ as } \alpha \rightarrow 0,\ 0 < \alpha < \frac{1}{b}$$

in $L^p(0, \infty)$ $(1 \leqq p < \infty)$ *and pointwise on the Lebesgue set of* f.

Proof. Referring to Lemma 1, we write,

$$\begin{aligned} G\left(x, s; \frac{1}{\alpha}\right) &= G^{**}\left(x, s; M + \frac{1}{\alpha}\right) \\ &\quad + \left[G\left(x, s; \frac{1}{\alpha}\right) - G^{**}\left(x, s; M + \frac{1}{\alpha}\right)\right] \\ &\leqq G^{**}\left(x, s; M + \frac{1}{\alpha}\right) \\ &\quad + \left[G^{**}\left(x, s; -M + \frac{1}{\alpha}\right) - G^{**}\left(x, s; M + \frac{1}{\alpha}\right)\right]. \end{aligned}$$

Similarly,

$$\begin{aligned} &G^{**}\left(x, s; -M + \frac{1}{\alpha}\right) + \left[G^{**}\left(x, s; M + \frac{1}{\alpha}\right)\right. \\ &\quad \left. - G^{**}\left(x, s; -M + \frac{1}{\alpha}\right)\right] \leqq G\left(x, s; \frac{1}{\alpha}\right). \end{aligned}$$

Each of the G^{**} is expressed in terms of $G^{(1)}$ and $G^{(2)}$ (see (12a-b) and (13) of Lemma 1).

Now extend $f(x)$ to be 0 for $x \leqq 0$. We interpret the $G^{(1)}$ to be L^1-dilations of radially decreasing convolution kernels in $\mathbf{R}$. So the conclusion of this theorem holds for all $G^{(1)}$ by well-known theorems in harmonic analysis. That is,

$$\frac{1}{\alpha}\int_0^\infty G^{(1)}\left(x, s; \pm M + \frac{1}{\alpha}\right)f(s)ds \to f(x)$$
$$\text{as } \alpha \to 0 \text{ in } L^p \quad (1 \leqq p < \infty)$$

and pointwise on the Lebesgue set of f.

Considering $G^{(2)}$ on $\mathbf{R}$, replace s by $-s$. This changes

$$G^{(2)}\left(x, s; \pm M + \frac{1}{\alpha}\right)$$

into a convolution kernel on $\mathbf{R}$ and $f(x)$ into a function which is 0 on $\mathbf{R}^+$. And so

$$\frac{1}{\alpha}\int_0^\infty G^{(2)}\left(x, s; \pm M + \frac{1}{\alpha}\right)f(s)ds \to 0$$
$$\text{as } \alpha \to 0 \text{ in } L^p \quad (1 \leqq p < \infty)$$

and on the Lebesgue set of f.

To complete the proof of part (a) we need only observe that

$$\frac{1}{\alpha}\int_0^\infty G^{**}\left(x, s; \pm M + \frac{1}{\alpha}\right)f(s)ds \to f(x) \quad \text{as } \alpha \to 0$$

and

$$\frac{1}{\alpha}\int_0^\infty \left[G^{**}\left(x, s; -M + \frac{1}{\alpha}\right) - G^{**}\left(x, s; M + \frac{1}{\alpha}\right)\right]f(s)ds$$
$$\to f(x) - f(x) = 0$$

as $\alpha \to 0$ in L^p $(1 \leqq p < \infty)$ and on the Lebesgue set of f.

The proof of part (b) is immediate after noting that under the conditions of Propositions 4, 6, and 7,

$$S_\alpha(f; x) = \int_0^\infty K_\alpha(x, s)f(s)ds = \frac{1}{\alpha}\int_0^\infty G_\alpha(x, s)f(s)ds.$$

5. Application to stable summability. In experiments which give the coefficients of eigenfunction expansions [**13**], measuring errors cause small perturbations in the expansion coefficients. Thus stable summability methods which recover from the perturbed expansion a good approximation to the original function f, at points where f is sufficiently regular, are of interest.

For the corollary to Theorem 2, we assume that $f \in L^p(0, \infty)$ for $1 \leqq p < \infty$ satisfies any of the hypotheses of Propositions 4, 6 and 7. This insures that the resolvent summability means $S_\alpha(f; x)$ exist. As usual, $F(\lambda)$ denotes the generalized Fourier transform of f. Let $\{f_n(x)\}_n$ be a sequence of functions in $L^r(0, \infty)$ $1 \leqq r < \infty$ which satisfies one of the Propositions 4, 6 and 7. This corollary insures the pointwise convergence of perturbed summability means to $f(x)$.

COROLLARY. *Let* $f \in L^p(0, \infty)$ $(1 \leqq p < \infty)$ *and* $\{f_n\}_{n=1}^{\infty} \in L^r(0, \infty)$ *satisfy any of the hypotheses of Propositions* 4, 6 *and* 7 *where r is the corresponding value associated with p and* sin β *in these propositions. Then*

$$S_\alpha(f_n; x) \rightarrow f(x)$$

pointwise as $\alpha \rightarrow 0$ *for n sufficiently large.*

Proof. It suffices to show that $S_\alpha(f_n; x) - S_\alpha(f; x)$ approaches zero pointwise for n sufficiently large. This follows immediately from Propositions 4, 6 and 7.

It has been shown in [7] that summability of singular Sturm-Liouville expansions with analytic summator functions is a consequence of resolvent summability. Hence our spectral theory considerations yield conditions when both resolvent summability and analytic multiplier methods, such as Abel ($\phi(z) = e^{-cz}$) and Gauss-Weierstrass ($\phi(z) = e^{-cz^2}$), apply.

Now let the associated generalized Fourier transforms $\{F_\gamma(\lambda)\}$ denote a net of approximations to $F(\lambda)$ in that for each value of the index γ, $F_\gamma(\lambda)$ satisfies

$$\|F_\gamma - F\|_2 = \left[\int_{-b}^{\infty} |F_\gamma(\lambda) - F(\lambda)|^2 d\rho(\lambda)\right]^{1/2} \leqq \gamma.$$

We say a summability method is *pointwise stable* if there exists a non-trivial scaling $\gamma(\alpha)$ such that if $\{F_\gamma(\lambda)\}_\gamma$ satisfies $\|F_\gamma - F\|_2 \leqq \gamma$ and $S_\alpha(f; x) \rightarrow f(x)$ pointwise as $\alpha \rightarrow 0$, then

$$S_\alpha(f_\gamma; x) \rightarrow f(x) \quad \text{pointwise as } \alpha \rightarrow 0.$$

Our final result essentially says that resolvent summability means $S(f_\gamma; x)$ furnish a stable summation method, if the summation parameter α is approximately scaled to go to zero with γ. The proof of this theorem is the same as [**2**, Theorem 1, p. 282]. (We note that the f_γ, F_γ are not related to f_n, F_n of the previous sections or in the above corollary.)

THEOREM 3. *Let* $f \in L^p(0, \infty)$ $(1 \leqq p \leqq \infty)$ *and* $\{f_\gamma\}_{\gamma>0} \in L^r(0, \infty)$ $(1 \leqq r \leqq \infty)$. *Suppose that the following hold*

(1) $\|F_\gamma - F\|_2 \leqq \gamma$

(2) $S_\alpha(f; x) \rightarrow f(x)$ *as* $\alpha \rightarrow 0$ *uniformly on a bounded subset* E *of* $(0, \infty)$

(3) α *is a function of* γ *such that both* $\alpha \rightarrow 0$ *and* $\gamma/\alpha^{1/4} \rightarrow 0$ *as* $\gamma \rightarrow 0$. *Then*

(a) $S_\alpha(f; x) \rightarrow f(x)$ *as* $\alpha \rightarrow 0$ *uniformly in* E, *and*

(b) S_α *is a stable summability method.*

We observe that this theorem holds for general summability methods where the summator function ϕ is real valued and

$$\int_0^\infty \frac{\phi^2(\lambda)d\lambda}{\sqrt{\lambda}} < \infty.$$

We close this paper by noting that all proofs herein were carried out for a class of singular continuous S-L expansions. We emphasize that the hypotheses of the results and their proofs are simplified for regular S-L expansions on finite intervals.

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Howard University,
Washington, D.C.