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A NOTE ON THE TIME-NON-HOMOGENEOUS JOHNSON–MEHL TESSELLATION

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Abstract

The main characteristics of the time-nonhomogeneous Johnson–Mehl tessellation with specified nucleation intensity $\alpha(t)=\alpha t^{\beta}$, $\beta>-1$ and $\alpha>0$ being constants and t time, are investigated as functions of β.

POISSON–VORONOI TESSELLATION; JOHNSON–MEHL TESSELLATION

Miles (1972) presented a review of models concerning the random division of the space $\mathbb{R}^3$ into space-filling and non-overlapping cells. In this note attention is focused on three of these models: (i) the Poisson–Voronoi (PV) tessellation, (ii) the time-homogeneous Johnson–Mehl (THJM) tessellation, (iii) the time-non-homogeneous Johnson–Mehl (TNHJM) tessellation. The PV tessellation (i) has convex cells, whereas the JM tessellations (ii) and (iii) have non-convex cells. The purpose of this note is to extend the results derived by Miles (1972) for the TNHJM tessellation and to show some internal relationships between the PV tessellation and the TNHJM tessellation with nucleation intensity

$$\alpha(t)=\alpha t^{\beta}, \tag{1}$$

$\beta>-1$ and $\alpha>0$ being constants and t time. As in Miles's paper we shall consider the 'ergodic' moments of the volume V, the surface area S, the total edge length L_1 and the number of vertices N_0 of the 'typical cell'.

Inserting (1) into the expressions (69)–(74) of Miles's paper, all integrals involved there convert into gamma functions. Hence for the TNHJM tessellation with intensity (1) we have that

$$E(V)=k_V(\beta)\left\{\frac{v^{3(\beta+1)}}{\alpha^3}\right\}^{1/(\beta+4)}, \tag{2}$$

$$E(S)=k_S(\beta)N_V^{-\frac{2}{3}};\qquad E(L_1)=k_{L_1}(\beta)N_V^{-\frac{1}{3}};\qquad E(N_0)=k_{N_0}(\beta),$$

where

$$k_V(\beta)=\frac{1}{\Gamma\left(\dfrac{\beta+1}{\beta+4}\right)}\left\{\frac{2^{3(\beta+1)}\pi^{\beta+1}(\beta+4)^3}{[(\beta+1)(\beta+2)(\beta+3)]^{\beta+1}}\right\}^{1/(\beta+4)},$$

$$k_S(\beta)=\frac{8}{3}\frac{\Gamma\left(\dfrac{2\beta+7}{\beta+4}\right)}{\left\{\Gamma\left(\dfrac{\beta+1}{\beta+4}\right)\right\}^{\frac{1}{3}}}\frac{\pi^{\frac{1}{3}}(\beta+4)}{\{(\beta+1)(\beta+2)(\beta+3)\}^{\frac{1}{3}}},$$

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$$k_{L_1}(\beta) = \frac{4}{5} \frac{\Gamma\left(\frac{3\beta + 10}{\beta + 4}\right)}{\left\{\Gamma\left(\frac{\beta + 1}{\beta + 4}\right)\right\}^{\frac{2}{3}}} \frac{\pi^{\frac{5}{3}}(\beta + 4)^2}{\{(\beta + 1)(\beta + 2)(\beta + 3)\}^{\frac{2}{3}}},$$

$$k_{N_0}(\beta) = \frac{32}{105} \frac{\Gamma\left(\frac{4\beta + 13}{\beta + 4}\right)}{\Gamma\left(\frac{\beta + 1}{\beta + 4}\right)} \frac{\pi^2(\beta + 4)^3}{(\beta + 1)(\beta + 2)(\beta + 3)},$$

$N_V = 1/E(V)$ is the mean number of cells per unit volume and v constant growth rate, the growth being isotropic.

The numerical values of the coefficients $k_i(\beta)$, $i = V, S, L_1, N_0$, for chosen values of $\beta \in \langle -1; \infty \rangle$ are summarized in Table 1. Of course, for $\beta = 0$ the TNHJM tessellation with intensity (1) converts into the THJM tessellation with constant intensity α and the corresponding coefficients $k_i(\beta)$ in Table 1 agree with those derived in Meijering (1953), Gilbert (1962) and Miles (1972). The limits of $k_s(\beta)$, $k_{L_1}(\beta)$ and $k_{N_0}(\beta)$ for β tending to -1 coincide with the corresponding coefficients of the PV tessellation (see the three papers mentioned above), for which it holds that $E(V) = 1/\alpha$. Note that the term in brackets in (2) converges to $1/\alpha$ as β tends to -1. Furthermore, the coefficients $k_i(\beta)$, $i = S, L_1, N_0$, are decreasing functions in the interval $\langle -1; \infty)$ and have finite positive limits for $\beta \to \infty$. Finally, the coefficient $k_V(\beta)$ is a unimodal function having its maximum 1.2647 at the point $\beta = 0.56$ and it converges to 0 when β tends to -1 and to ∞.

TABLE 1
Coefficients $k_i(\beta)$, $i = V, S, L_1, N_0$

β	$k_V(\beta)$	$k_S(\beta)$	$k_{L_1}(\beta)$	$k_{N_0}(\beta)$
−1.0	0	5.8209	17.4956	27.0709
−0.75	0.3128	5.5808	16.4487	25.2662
−0.50	0.6475	5.4005	15.7075	24.0630
−0.25	0.9245	5.2587	15.1506	23.2036
0	1.1161	5.1433	14.7143	22.5591
0.25	1.2241	5.0473	14.3621	22.0578
0.50	1.2635	4.9659	14.0708	21.6567
0.75	1.2520	4.8959	13.8253	21.3286
1.0	1.2062	4.8348	13.6155	21.0552
2.0	0.8944	4.6526	13.0058	20.3032
∞	0	3.9056	10.7821	18.0473

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