BOREL DIRECTIONS AND ITERATED ORBITS
OF MEROMORPHIC FUNCTIONS

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For transcendental meromorphic functions of finite order, we prove that there exist iterated orbits which tend to the Borel directions. This gives a relation between the value distribution theory and the iteration theory of meromorphic functions.

1. INTRODUCTION

Suppose \( f : \mathbb{C} \to \mathbb{C} \) is a transcendental meromorphic function. If for any \( \varepsilon > 0 \), \( f \) takes every complex value \( a \) infinitely many times on the region: \( |\arg z - \theta_0| < \varepsilon \), with at most two exceptional values \( a \in \overline{\mathbb{C}} \), then the ray \( \arg z = \theta_0 \) is said to be a Julia direction of \( f(z) \). Furthermore, if for any \( \varepsilon > 0 \),

\[
\lim_{r \to \infty} \frac{\log n(r, \theta_0, \varepsilon, f = a)}{\log r} \geq \omega > 0,
\]

with at most two exceptional values of \( a \in \overline{\mathbb{C}} \), where \( n(r, \theta_0, \varepsilon, f = a) \) is the number of roots of \( f(z) = a \) on the region: \( |z| < r \) and \( |\arg z - \theta_0| < \varepsilon \), then the ray \( \arg z = \theta_0 \) is said to be a Borel direction of order at least \( \omega \). These are fundamental concepts in value distribution theory [5].

In this note, we deal with the problem: Can we choose an iterated orbit such that it approximates to the Borel directions? Define

\[
I(f) = \{ z \in \mathbb{C} \mid f^n(z) \neq \infty \text{ for all } n \text{ and } f^n(z) \to \infty \text{ as } n \to \infty \},
\]

where \( f^n \) is the \( n\)-th iterate of \( f \), that is, \( f^0(z) = z \) and \( f^n(z) = f \circ f^{n-1}(z) \) for \( n \geq 1 \). \( f^n(z) \) is defined for all \( z \in \mathbb{C} \) except for a countable set which consists of the poles of \( f, f^2, \ldots, f^{n-1} \). Obviously, the forward orbit \( O^+(a) = \{ f^n(a) \mid n \geq 0 \} \) is an infinite set if \( a \in I(f) \). We want to find a point \( a \in \mathbb{C} \) such that \( a \in I(f) \) and each limiting direction of \( O^+(a) \) (that is, a limit of \( \{ \arg z \mid z \in O^+(a) \} \) is a Borel direction of \( f \). By \( J(f) \) denote the Julia set of \( f \) which is the closure of the set of the repelling periodic points; its complement \( F(f) \) is the Fatou set (see [2]). In this note we shall prove

**Theorem 1.** Let \( f(z) \) be a transcendental meromorphic function, then \( I(f) \cap J(f) \neq \emptyset \).

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REMARK. Eremenko [3] has proved this result for transcendental entire functions.

**THEOREM 2.** Let \( f(z) \) be a transcendental meromorphic function of finite order, the lower order \( \mu > 0 \). Then there exists a point \( a \in I(f) \cap J(f) \) such that each limiting direction of \( O^+(a) \) is a Borel direction of order at least \( \mu \).

**REMARK.** It is well known that there exist transcendental meromorphic functions of lower order zero which don’t have a Julia direction [5].

Since the backward orbit \( O^-(a) = \{ z \mid f^n(z) = a \text{ for some } n \} \) is dense on \( J(f) \) for every point \( a \in J(f) \) with at most one exceptional point [2], we easily have

**COROLLARY.** Let \( f(z) \) be a transcendental meromorphic function of finite order, the lower order \( \mu > 0 \). Then there is a dense subset \( I_B \) of \( J(f) \) such that, for \( a \in I_B \), \( O^+(a) \) tends to infinity and each limiting direction of \( O^+(a) \) is a Borel direction of order at least \( \mu \).

### 2. The Proof of Theorem 1

In order to prove Theorem 1, we need the following lemma:

**LEMMA 1.** [1] Suppose, in a domain \( D \), the analytic functions \( f \) of the family \( G \) omit the values 0, 1, and \( H \) is a compact subset of \( D \) on which the functions all satisfy \( |f(z)| \geq 1 \). Then there exist constants \( k, t \), dependent only on \( H \) and \( D \), such that for any \( z, z' \in H \) and any \( f \in G \) we have \( |f(z')| < k|f(z)|^t \).

**THE PROOF OF THEOREM 1:** We distinguish the following two cases:

**A.** \( f(z) \) has infinitely many poles. Let \( a_0 \) be a pole of \( f(z) \), then there exists a constant \( R > 1 \) such that \( f(V(a_0)) \supset \{ z \mid |z| > R \} \), where \( V(\eta) = \{ z \mid |z - \eta| < 1 \} \). Choose a pole \( a_1 \in \{ z \mid |z| > R + 2 \} \), then \( f(V(a_0)) \supset V(a_1) \). Since \( a_1 \) is also a pole, there exists a constant \( l_1 \geq 2 \) such that \( f(V(a_1)) \supset \{ z \mid |z| > R^{l_1} \} \). By repeating this construction, we obtain a sequence of disks \( V(a_j) \) (\( a_j \) is a pole) such that \( V(a_j) \to \infty \) and

\[
f(V(a_j)) \supset V(a_{j+1}) \quad (j = 0, 1, 2, \ldots).
\]

It is obvious that there exists a sequence of domains \( B_j \subset V(a_0) \) such that \( B_{j+1} \subset B_j \) and \( f^j(B_j) = V(a_j) \). For a point \( a \in \bigcap_{j=1}^{\infty} B_j \), we have \( a \in I(f) \). Since \( a_j \) is a pole, then \( V(a_j) \cap J(f) \neq 0 \), and thus \( B_j \cap J(f) \neq 0 \) for all \( j \) [2]. So we have \( a \in I(f) \cap J(f) \).

**B.** \( f(z) \) has only finitely many poles. By Mittag-Leffler’s theorem,

\[
f(z) = g(z) + \sum_{j=1}^{m} P_j \left( \frac{1}{z - a_j} \right),
\]
where $g(z)$ is a transcendental entire function, $a_j (j = 1, \cdots, m)$ are $m < \infty$ distinct poles of $f(z)$, and $P_j$ is a polynomial with $P_j(0) = 0$. For a transcendental entire function $g(z)$, Eremenko [3] proved: there exist a sequence of positive numbers $r_j \to \infty$, a constant $b > 1$ and a sequence of domains $\sigma_j \subset \{ z \mid r_j/b < |z| < br_j \}$ such that

$$
\begin{equation}
\label{eq:2}
g(\sigma_j) \supset \left\{ z \mid \frac{1}{b_1}r_{j+1} < |z| < b_1r_{j+1} \right\} \quad (j = 1, 2, \cdots),
\end{equation}
$$

where $b_1 > b$ is a constant. For a constant $b_2 \in (b, b_1)$, by (1) and (2) we deduce that there exists $j_0 > 0$ such that

$$
\begin{equation}
\label{eq:3}
f(\sigma_j) \supset \left\{ z \mid \frac{1}{b_2}r_{j+1} < |z| < b_2r_{j+1} \right\} \supset \sigma_{j+1}
\end{equation}
$$

when $j \geq j_0$. So there exists a sequence of domains $B_p \subset \sigma_{j_0}$ such that

$$
\begin{equation}
\label{eq:4}
f^p(B_p) = \sigma_{j_0+p}, \quad B_{p+1} \subset B_p, \quad p = 1, 2, \cdots.
\end{equation}
$$

It follows that $\bigcap_{p=1}^{\infty} B_p \subset I(f)$, thus $I(f) \neq \emptyset$.

If $(\bigcap_{p=1}^{\infty} B_p) \cap J(f) \neq \emptyset$, we have $I(f) \cap J(f) \neq \emptyset$. Below we suppose $(\bigcap_{p=1}^{\infty} B_p) \cap J(f) = \emptyset$, then there exists $p_0 \geq 1$ such that $B_p \subset F(f)$ when $p \geq p_0$. By (3) and (4) we have

$$
\begin{equation}
\label{eq:5}
\left\{ z \mid \frac{1}{b_2}r_{j} < |z| < b_2r_{j} \right\} \subset F(f)
\end{equation}
$$

when $j \geq p_0 + j_0 + 1$.

Now, we prove that $F(f)$ has only bounded components: Assume $D$ is an unbounded component of $F(f)$. By (3) and (5) we know that $f(D) \subset D, f^n(z) \to \infty$ for $z \in D$ and $\sigma_j \subset D$ when $j \geq p_0 + j_0 + 1$. Put

$$
H = \sigma_{p_0+j_0+1} \cup \overline{f(\sigma_{p_0+j_0+1})},
$$

then $H \subset D$. Without loss of generality, we may assume $0, 1 \in J(f)$ and $|f^n(z)| \geq 1$ on $H$ for all $n$. By Lemma 1, for any $z' \in \sigma_{p_0+j_0+1}$ we have

$$
\begin{equation}
\label{eq:6}
|f^{n+1}(z')| < k|f^n(z')|^t, \quad n = 1, 2, \cdots,
\end{equation}
$$

where $k$ and $t$ are two constants. Put $\Omega = \bigcup_{n=0}^{\infty} f^n(\sigma_{p_0+j_0+1})$, then for any $z \in \Omega$, there exists a point $z' \in \sigma_{p_0+j_0+1}$ and a natural number $n$ such that $f^n(z') = z$. By (6) we get

$$
|f(z)| < k|z|^t, \quad z \in \Omega.
$$
Noting $\Omega \supset \{z \mid r_j/b < |z| < br_j \}$ for sufficiently large $j$, we have

$$M(r_j, g) = M(r_j, f) + O(1) = O(r_j^\delta) \quad (r_j \to \infty).$$

This contradicts the transcendency of $g(z)$. Therefore $F(f)$ has only bounded components.

Denote the component of $F(f)$ containing $B_{P_0}$ by $D_{Q_0}$. Since $B_{P_0} \cap I(f) \neq \emptyset$, so $f^n(z) \to \infty$ for $z \in D_0$. It follows from (5) and the boundedness of $D_0$ that $f^n(\partial D_0) \to \infty$, and thus $\partial D_0 \subset I(f) \cap J(f)$. The proof of Theorem 1 is complete. \(\square\)

3. The Proof of Theorem 2

Denote the Nevanlinna characteristic function of $f(z)$ by $T(r, f)$ [5]. Since $f$ is of positive lower order and finite order, there exists a constant $\alpha > 1$ such that $T(2r, f) < T^\alpha(r, f)$ for sufficiently large $r$. Therefore, Theorem 2 is the corollary of the following result:

**Theorem 3.** Let $f(z)$ be a transcendental meromorphic function of lower order $\mu \in (0, \infty)$. If

$$\lim_{r \to \infty} \frac{\log T(2r, f)}{\log T(r, f)} < \infty,$$

then there exists a point $a \in I(f) \cap J(f)$ such that each limiting direction of $O^+(a)$ is a Borel direction of order at least $\mu$.

In order to prove Theorem 3, we need the following lemmas:

**Lemma 2.** [5] Let $f$ be a transcendental meromorphic function. If $R$ is sufficiently large to satisfy

$$T(R, f) \geq \max \left\{ 240, \frac{240 \log (2R)}{\log k}, 12T(r, f), \frac{12T(k, f)}{\log k} \log \frac{2R}{r} \right\},$$

then there exists a point $z_j$ lying in $r < |z| < 2R$ such that in the domain

$$\Gamma : \quad |z - z_j| < \frac{4\pi}{q} |z_j|,$$

$f$ takes every complex value at least

$$n = c^* \frac{T(R, f)}{q^2 \left( \log \frac{r}{R} \right)^2}$$
times except for those complex values which can be contained in two spherical disks each with radius $e^{-n}$, where $k > 1$ is a constant, $q$ is a sufficiently large integer, and $c^* > 0$ is an absolute constant. The disk $T$ is called a filling disk of $f(z)$.

**Lemma 3.** [4] Let $T(r)$ be a positive, increasing and continuous function, and $T(r) \to +\infty$ ($r \to +\infty$). If

$$\lim_{r \to \infty} \frac{\log T(r)}{\log r} \leq \nu < +\infty,$$

then for any two numbers $\tau_1 > 1$, $\tau_2 > 1$, the lower logarithmic density of the set \{$r \mid T(\tau_1 r) \leq \tau_2 T(r)$\} is not less than $1 - (\nu \log \tau_1)/(\log \tau_2)$.

**Lemma 4.** Let $T(r)$ be a positive, increasing and continuous function, and $T(r) \to +\infty$ ($r \to +\infty$). If

$$\lim_{r \to \infty} \frac{\log T(r)}{\log r} \geq \omega > 0,$$

where $\tau_1 > 1$, $\tau_2 > 1$ are two constants satisfying $\tau_2 < \tau_1^\nu$, then for any constant $m > 1/(1 - (\log \tau_2)/(\omega \log \tau_1))$, there exists a constant $R_0 > 0$ such that

$$\{ t \mid \tau_2 T(t) \leq T(\tau_1 t) \} \cap [r, T^{-1}(T^m(r))] \neq \emptyset$$

when $r > R_0$.

**The proof of Lemma 4:** Put $s = T(r)$, $T_0(s) = T^{-1}(s)$. Then $T_0(s)$ is a positive, increasing and continuous function, and $T_0(s) \to +\infty$ ($s \to +\infty$). Obviously,

$$\lim_{s \to +\infty} \frac{\log T_0(s)}{\log s} \leq \frac{1}{\omega} < +\infty.$$

By Lemma 3,

$$\text{lower-logdens}\{ s \mid T_0(\tau_2 s) \leq \tau_1 T_0(s) \} \geq 1 - \frac{\log \tau_2}{\omega \log \tau_1}.$$

Therefore, there exists $s_0 \in [s, s^m]$ such that $T_0(\tau_2 s_0) \leq \tau_1 T_0(s_0)$ for sufficiently large $s$. Put $r_0 = T^{-1}(s_0)$, then $r_0 = T_0(s_0)$, $T(r_0) = s_0$. Thus $\tau_2 T(r_0) \leq T(\tau_1 r_0)$. Since $r_0 \geq r$, $T(r_0) = s_0 \leq s^m = T^m(r)$, we deduce $r_0 \in [r, T^{-1}(T^m(r))]$. The proof of Lemma 4 is complete.

**The proof of Theorem 3:** Choose two constants $k > 1$ and $\tau_1 > 1$ such that

$$\frac{12}{\log k} \log (2k \tau_1) < \tau_1^\nu.$$
Put
\[ \tau_2 = \frac{12}{\log k} \log (2k \tau_1) \quad \text{and} \quad \alpha = \lim_{r \to \infty} \frac{\log T(2r, f)}{\log T(r, f)}. \]

Choose a natural number \( m \) such that
\[ m > \max \left( \frac{1}{1 - (\log \tau_2)/(\mu \log \tau_1)}, 2\alpha \right). \]

For convenience, we put \( T(r, f) = T(r) \). It is obvious that there exists a constant \( M_0 > 0 \) such that
\[ M_0 > \max \{ R_0, e^{2\tau} \}, \]
\[ T(r) > \max \left\{ \frac{1}{K} (\log r)^{2m^{2p+1}+2}, \frac{240 \log (2r)}{\log k} \right\}, \]
\[ T(2r, f) < T^{2\alpha}(r, f), \]
\[ c \cdot c^* \frac{\tau_1^{\mu/2} r^\mu/4}{(\log (k \tau_1) \log r)^2} > 1 \]
when \( r \geq M_0 \), where \( R_0 > 0 \) is the constant stated in Lemma 4, \( c^* > 0 \) is the constant stated in Lemma 2, and
\[
\frac{1}{1 + q^{\tau_1}}, \quad K = \frac{c^{\mu/2}(\mu/2)^{m^{2p+1}+1}}{(m^{4p+1} + 1)!} \frac{(c^*)^{m^{2p+1}+1}}{(\log (k \tau_1))^{2m^{2p+1}+2}}, \quad P = \left[ \frac{\log (6k \tau_1)}{\log 2} \right] + 2,
\]
(where \( [\cdot] \) denotes the integral part). Put \( r^* = \max\{M_0, M_0^{4/\mu}\} \). From (11) we deduce that
\[ c \cdot c^* \frac{(\tau_1 r)^{\mu/2}}{(\log (k \tau_1) \log r)^2} > r^\mu/4 \geq M_0 \]
for \( r \geq r^* \).

By Lemma 4, there exists \( r_0 \in \left[ r^*, T^{-1}(T^m(r^*)) \right] \) such that
\[ \tau_2 T(r_0) \leq T(\tau_1 r_0). \]

Put \( r_1 = r_0/k, \quad R_1 = \tau_1 r_0 \), then
\[ \frac{12}{\log k} \log \frac{2R_1}{r_1} T(k \tau_1) \leq T(R_1), \]
and
\[ 12T(r_1) \leq \frac{12}{\log k} \log \frac{2R_1}{r_1} T(k \tau_1) \leq T(R_1). \]
By (8), (9), (14), (15) and Lemma 2, there exists $z_0$ lying in $r_1 < |z| < 2R_1$ such that in the disk

$$\Gamma_0 : |z - z_0| < \frac{4\pi}{\log r^* |z_0|}$$

$f$ takes every complex value $a$ at least

$$n_0 = c^* \frac{T(R_1)}{(\log r^*)^2 (\log (k_1))^2}$$

times except for those complex values which can be contained in two spherical disks $\gamma_0$ and $\gamma_0'$ with radius $e^{-n_0}$, that is, $\Gamma_0$ is a filling disk of $f(z)$. Obviously,

$$n_0 \geq \frac{c^* T(1/2|z_0|)}{(\log k_1)^2 (\log (k|z_0|))^2} \geq (|z_0|)^{n-\varepsilon(|z_0|)},$$

where $c(r) > 0$, and $\varepsilon(r) \to 0$ as $r \to \infty$. It can be easily verified from (8) that

$$\Gamma_0 \subset \left\{ z \mid \frac{1}{2^k} r^* < |z| < 3\tau_1 T^{-1}(T^m(r^*)) \right\}.$$

Put $t_j = T^{-1}(T^m(r^*))$. It is obvious that $t_0 = r^*$, $\{t_j\}$ is an increasing sequence and $t_j \to \infty$. So the sequence of annuli $A_j = \{z \mid t_j/2^k < |z| < 3\tau_1 t_{j+1}\}$ tends to infinity as $j \to \infty$ and $\Gamma_0 \subset A_0$. By $T(t_{j+1}) = T^m(t_j)$, (7) and (10) we get

$$T(t_{j+2}) = T^m(t_{j+1}) > T^{2\alpha}(t_{j+1}) > T(2t_{j+1}).$$

It follows that $t_{j+2} > 2t_{j+1}$, so $t_{j+p} > 2^p t_{j+1}$, and thus $t_{j+p} > 6k\tau_1 t_{j+1}$. Therefore,

$$A_j \cap A_{j+p} = \emptyset \quad (j = 0, 1, 2, \cdots).$$

Next we prove that there is at least one in five annuli $A_p, A_{2p}, A_{3p}, A_{4p}, A_{5p}$ which does not meet $\gamma_0 \cup \gamma_0'$. Assume $\gamma_0'$ (or $\gamma_0''$) meet both $A_{jp}$ and $A_{(j+2)p}$ $(j \in \{1, 2, 3\})$. Then we have

$$e^{-n_0} \geq \frac{3\tau_1 t(j+1)p+1 - \frac{1}{2^k} t(j+1)p}{\sqrt{1 + 9\tau_1^2 t(j+1)p+1} \sqrt{1 + \frac{1}{4k^2}} t(j+1)p} \geq \frac{c}{t(j+1)p+1},$$

where $c > 0$ is the constant in (12). This means

$$T^{m(j+1)p+1}(r^*) \geq T(ce^{n_0}).$$
On the other hand, by (10) and (13) we have

\[ cn_0 > c \cdot c^* \frac{(\tau_1 r^*)^{\mu/2}}{(\log (k\tau_1) \log r^*)^2} \geq M_0, \]

and hence

(18)

\[ T(cn_0) > cn_0 > c^{\mu/2} \cdot (\mu/2)^{n_0 + 1} \frac{(m^{(j+1)p+1} + 1)!}{(m^{(j+1)p+1} + 1)!} n_0^{m^{(j+1)p+1} + 1} > K \frac{T^{m^{(j+1)p+1} + 1}(r^*)}{(\log r^*)^{2m^{4p+1} + 2}}, \]

where \( K > 0 \) is the constant in (12). By (17) and (18) we have

\[ T(r^*) < \frac{1}{K} (\log r^*)^{2m^{4p+1} + 2}. \]

This contradicts (9). Therefore, \( \gamma'_0 \) (or \( \gamma''_0 \)) can not meet both \( A_{jp} \) and \( A_{(j+2)p} \) \( (j \in \{1, 2, 3\}) \). It follows immediately that there exists at least one in five annuli \( A_p, A_{2p}, A_{3p}, A_{4p}, A_{5p} \) which does not meet \( \gamma'_0 \) or \( \gamma''_0 \). Denote this annulus by \( A^0_0 \). So \( f(\Gamma_0) \supset A^0_0 \).

By the same discussion, we can deduce that there exists a filling disk \( \Gamma_1 \subset A^0_0 \) and an annulus \( A^0_2 \in \{A_j:j \in \mathbb{N}\} \) such that \( f(\Gamma_1) \supset A^0_2 \). Repeating this construction, we obtain a sequence of filling disks \( \Gamma_j \) such that

(19)

\[ f(\Gamma_j) \supset \overline{\Gamma_{j+1}}, \Gamma_j \to \infty \quad (j \to \infty). \]

Denote the centre of \( \Gamma_j \) by \( z_j \). From (16) we know that each limiting point of \( \{\arg z_j \mid j = 1,2, \cdots \} \) is a Borel direction of order at least \( \mu \) (see [5]). It follows (19) that there is a sequence of domains \( B_j \subset A_0 \) such that \( f^{-1}(B_j) = \Gamma_j \) and \( \Gamma_0 \supset B_j \supset \overline{B_{j+1}} \).

Now, we prove \( \left( \bigcap_{j=1}^{\infty} B_j \right) \cap J(f) \neq \emptyset \): Otherwise, there exists a natural number \( j_0 \) such that \( B_j \subset F(f) \) when \( j \geq j_0 \). Since \( \Gamma_0 \) is a filling disk, we have \( f^j(B_j) = f(\Gamma_j) \supset \overline{C \setminus (\gamma'_j \cup \gamma''_j)} \) (where \( \gamma'_j \) and \( \gamma''_j \) are two spherical disks each with radius \( e^{-n_j} \) and \( n_j \to \infty \) as \( j \to \infty \)), so \( J(f) \subset \gamma'_j \cup \gamma''_j \) when \( j \geq j_0 \). This implies \( J(f) \) contains at most two points. This is a contradiction [2].

For a point \( a \in \left( \bigcap_{j=1}^{\infty} B_j \right) \cap J(f) \), we have \( a \in I(f) \) and each limiting direction of \( O^+(a) \) is a Borel direction of order at least \( \mu \). The proof of Theorem 3 is complete. \( \square \)

REFERENCES

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