Borel Directions and Iterated Orbits of Meromorphic Functions

Jianyong Qiao

For transcendental meromorphic functions of finite order, we prove that there exist iterated orbits which tend to the Borel directions. This gives a relation between the value distribution theory and the iteration theory of meromorphic functions.

1. Introduction

Suppose \( f : \mathbb{C} \to \mathbb{C} \) is a transcendental meromorphic function. If for any \( \varepsilon > 0 \), \( f \) takes every complex value \( a \) infinitely many times on the region: \( |\arg z - \theta_0| < \varepsilon \), with at most two exceptional values \( a \in \overline{\mathbb{C}} \), then the ray \( \arg z = \theta_0 \) is said to be a Julia direction of \( f(z) \). Furthermore, if for any \( \varepsilon > 0 \),

\[
\lim_{r \to \infty} \frac{\log n(r, \theta_0, \varepsilon, f = a)}{\log r} \geq \omega > 0,
\]

with at most two exceptional values of \( a \in \overline{\mathbb{C}} \), where \( n(r, \theta_0, \varepsilon, f = a) \) is the number of roots of \( f(z) = a \) on the region: \( |z| < r \) and \( |\arg z - \theta_0| < \varepsilon \), then the ray \( \arg z = \theta_0 \) is said to be a Borel direction of order at least \( \omega \). These are fundamental concepts in value distribution theory [5].

In this note, we deal with the problem: Can we choose an iterated orbit such that it approximates to the Borel directions? Define

\[
I(f) = \{ z \in \mathbb{C} \mid f^n(z) \neq \infty \text{ for all } n \text{ and } f^n(z) \to \infty \text{ as } n \to \infty \},
\]

where \( f^n \) is the \( n \)-th iterate of \( f \), that is, \( f^0(z) = z \) and \( f^n(z) = f \circ f^{n-1}(z) \) for \( n \geq 1 \). \( f^n(z) \) is defined for all \( z \in \mathbb{C} \) except for a countable set which consists of the poles of \( f, f^2, \ldots, f^{n-1} \). Obviously, the forward orbit \( O^+(a) = \{ f^n(a) \mid n \geq 0 \} \) is an infinite set if \( a \in I(f) \). We want to find a point \( a \in \mathbb{C} \) such that \( a \in I(f) \) and each limiting direction of \( O^+(a) \) (that is, a limit of \( \{ \arg z \mid z \in O^+(a) \} \)) is a Borel direction of \( f \). By \( J(f) \) denote the Julia set of \( f \) which is the closure of the set of the repelling periodic points; its complement \( F(f) \) is the Fatou set (see [2]). In this note we shall prove

Theorem 1. Let \( f(z) \) be a transcendental meromorphic function, then \( I(f) \cap J(f) \neq \emptyset \).
2. THE PROOF OF THEOREM 1

In order to prove Theorem 1, we need the following lemma:

**LEMMA 1.** [1] Suppose, in a domain $D$, the analytic functions $f$ of the family $G$ omit the values $0,1,$ and $H$ is a compact subset of $D$ on which the functions all satisfy $|f(z)| \geq 1$. Then there exist constants $k, t$, dependent only on $H$ and $D$, such that for any $z, z' \in H$ and any $f \in G$ we have $|f(z')| < k |f(z)|^t$.

**THE PROOF OF THEOREM 1:** We distinguish the following two cases:

A. $f(z)$ has infinitely many poles. Let $a_0$ be a pole of $f(z)$, then there exists a constant $R > 1$ such that $f(V(a_0)) \supset \{z \mid |z| > R\}$, where $V(\eta) = \{z \mid |z - \eta| < 1\}$. Choose a pole $a_1 \in \{z \mid |z| > R + 2\}$, then $f(V(a_0)) \supset \overline{V(a_1)}$. Since $a_1$ is also a pole, there exists a constant $t \geq 2$ such that $f(V(a_1)) \supset \{z \mid |z| > R^t\}$. By repeating this construction, we obtain a sequence of disks $V(a_j)$ ($a_j$ is a pole) such that $V(a_j) \to \infty$ and

$$f(V(a_j)) \supset \overline{V(a_{j+1})} \quad (j = 0, 1, 2, \ldots).$$

It is obvious that there exists a sequence of domains $B_j \subset V(a_0)$ such that $\overline{B_{j+1}} \subset B_j$ and $f^j(B_j) = V(a_j)$. For a point $a \in \bigcap_{j=1}^{\infty} B_j$, we have $a \in I(f)$. Since $a_j$ is a pole, then $V(a_j) \cap J(f) \neq \emptyset$, and thus $B_j \cap J(f) \neq \emptyset$ for all $j \geq 2$. So we have $a \in I(f) \cap J(f)$.

B. $f(z)$ has only finitely many poles. By Mittag-Leffler's theorem,

$$f(z) = g(z) + \sum_{j=1}^{m} P_j \left( \frac{1}{z - a_j} \right),$$

(1)

REMARK. Eremenko [3] has proved this result for transcendental entire functions.

**THEOREM 2.** Let $f(z)$ be a transcendental meromorphic function of finite order, the lower order $\mu > 0$. Then there exists a point $a \in I(f) \cap J(f)$ such that each limiting direction of $O^+(a)$ is a Borel direction of order at least $\mu$.

**REMARK.** It is well known that there exist transcendental meromorphic functions of lower order zero which don't have a Julia direction [5].

Since the backward orbit $O^-(a) = \{z \mid f^n(z) = a \text{ for some } n\}$ is dense on $J(f)$ for every point $a \in J(f)$ with at most one exceptional point [2], we easily have

**COROLLARY.** Let $f(z)$ be a transcendental meromorphic function of finite order, the lower order $\mu > 0$. Then there is a dense subset $I_B$ of $J(f)$ such that, for $a \in I_B$, $O^+(a)$ tends to infinity and each limiting direction of $O^+(a)$ is a Borel direction of order at least $\mu$. 

2. THE PROOF OF THEOREM 1

In order to prove Theorem 1, we need the following lemma:
where \( g(z) \) is a transcendental entire function, \( a_j (j = 1, \ldots, m) \) are \( m < \infty \) distinct poles of \( f(z) \), and \( P_j \) is a polynomial with \( P_j(0) = 0 \). For a transcendental entire function \( g(z) \), Eremenko [3] proved: there exist a sequence of positive numbers \( r_j \to \infty \), a constant \( b > 1 \) and a sequence of domains \( \sigma_j \subset \{ z \mid r_j/b < |z| < br_j \} \) such that

\[
(2) \quad g(\sigma_j) \supset \left\{ z \mid \frac{1}{b_1} r_{j+1} < |z| < b_1 r_{j+1} \right\} \quad (j = 1, 2, \ldots),
\]

where \( b_1 > b \) is a constant. For a constant \( b_2 \in (b, b_1) \), by (1) and (2) we deduce that there exists \( j_0 > 0 \) such that

\[
(3) \quad f(\sigma_j) \supset \left\{ z \mid \frac{1}{b_2} r_{j+1} < |z| < b_2 r_{j+1} \right\} \supset \sigma_{j+1}
\]

when \( j \geq j_0 \). So there exists a sequence of domains \( B_p \subset \sigma_{j_0} \) such that

\[
(4) \quad f^p(B_p) = \sigma_{j_0+p}, \quad \overline{B_{p+1}} \subset B_p, \quad p = 1, 2, \ldots.
\]

It follows that \( \bigcap_{p=1}^{\infty} \overline{B_p} \subset I(f) \), thus \( I(f) \neq \emptyset \).

If \( \bigcap_{p=1}^{\infty} \overline{B_p} \cap J(f) \neq \emptyset \), we have \( I(f) \cap J(f) \neq \emptyset \). Below we suppose \( \bigcap_{p=1}^{\infty} \overline{B_p} \cap J(f) = \emptyset \), then there exists \( p_0 \geq 1 \) such that \( B_p \subset F(f) \) when \( p \geq p_0 \). By (3) and (4) we have

\[
(5) \quad \left\{ z \mid \frac{1}{b_2} r_j < |z| < b_2 r_j \right\} \subset F(f)
\]

when \( j \geq p_0 + j_0 + 1 \).

Now, we prove that \( F(f) \) has only bounded components: Assume \( D \) is an unbounded component of \( F(f) \). By (3) and (5) we know that \( f(D) \subset D, f^n(z) \to \infty \) for \( z \in D \) and \( \sigma_j \subset D \) when \( j \geq p_0 + j_0 + 1 \). Put

\[
H = \sigma_{p_0+j_0+1} \cup f(\sigma_{p_0+j_0+1}),
\]

then \( H \subset D \). Without loss of generality, we may assume \( 0, 1 \in J(f) \) and \( |f^n(z)| \geq 1 \) on \( H \) for all \( n \). By Lemma 1, for any \( z' \in \sigma_{p_0+j_0+1} \) we have

\[
(6) \quad |f^{n+1}(z')| < k|f^n(z')|^t, \quad n = 1, 2, \ldots,
\]

where \( k \) and \( t \) are two constants. Put \( \Omega = \bigcup_{n=0}^{\infty} f^n(\sigma_{p_0+j_0+1}) \), then for any \( z \in \Omega \), there exist a point \( z' \in \sigma_{p_0+j_0+1} \) and a natural number \( n \) such that \( f^n(z') = z \). By (6) we get

\[
|f(z)| < k|z|^t, \quad z \in \Omega.
\]
Noting $\Omega \supset \{z \mid r_j/b < |z| < br_j\}$ for sufficiently large $j$, we have

$$M(r_j, g) = M(r_j, f) + o(1) = O(r_j^\epsilon) \quad (r_j \to \infty).$$

This contradicts the transcendence of $g(z)$. Therefore $F(f)$ has only bounded components.

Denote the component of $F(f)$ containing $B_{P_0}$ by $D_Q$. Since $B_{P_0} \cap I(f) \neq \emptyset$, so $f^n(z) \to \infty$ for $z \in D_0$. It follows from (5) and the boundedness of $D_0$ that $f^n(\partial D_0) \to \infty$, and thus $\partial D_0 \subset I(f) \cap J(f)$. The proof of Theorem 1 is complete.

3. THE PROOF OF THEOREM 2

Denote the Nevanlinna characteristic function of $f(z)$ by $T(r, f)$ [5]. Since $f$ is of positive lower order and finite order, there exists a constant $\alpha > 1$ such that $T(2r, f) < T^\alpha(r, f)$ for sufficiently large $r$. Therefore, Theorem 2 is the corollary of the following result:

**THEOREM 3.** Let $f(z)$ be a transcendental meromorphic function of lower order $\mu \in (0, \infty)$. If

$$\lim_{r \to \infty} \frac{\log T(2r, f)}{\log T(r, f)} < \infty,$$

then there exists a point $a \in I(f) \cap J(f)$ such that each limiting direction of $O^+(a)$ is a Borel direction of order at least $\mu$.

In order to prove Theorem 3, we need the following lemmas:

**Lemma 2.** [5] Let $f$ be a transcendental meromorphic function. If $R$ is sufficiently large to satisfy

$$T(R, f) \geq \max \left\{ 240, \frac{240 \log (2R)}{\log k}, \frac{12T(r, f)}{\log k}, \frac{12T(kr, f)}{\log k} \right\},$$

then there exists a point $z_j$ lying in $r < |z| < 2R$ such that in the domain

$$\Gamma : \quad |z - z_j| < \frac{4\pi}{q} |z_j|,$$

$f$ takes every complex value at least

$$n = c \left( \frac{T(R, f)}{q^2 (\log \frac{r}{R})^2} \right).$$
times except for those complex values which can be contained in two spherical disks each with radius $e^{-n}$, where $k > 1$ is a constant, $q$ is a sufficiently large integer, and $c^*>0$ is an absolute constant. The disk $\Gamma$ is called a filling disk of $f(z)$.

**Lemma 3.** [4] Let $T(r)$ be a positive, increasing and continuous function, and $T(r) \to +\infty (r \to +\infty)$. If

$$\lim_{r \to \infty} \frac{\log T(r)}{\log r} \leq \nu < +\infty,$$

then for any two numbers $\tau_1 > 1, \tau_2 > 1$, the lower logarithmic density of the set \{ $r \mid T(\tau_1 r) \leq \tau_2 T(r)$ \} is not less than $1 - (\nu \log \tau_1) / (\log \tau_2)$.

**Lemma 4.** Let $T(r)$ be a positive, increasing and continuous function, and $T(r) \to +\infty (r \to +\infty)$. If

$$\lim_{r \to \infty} \frac{\log T(r)}{\log r} \geq \omega > 0,$$

where $\tau_1 > 1, \tau_2 > 1$ are two constants satisfying $\tau_2 < \tau_1^\omega$, then for any constant $m > 1 / (1 - (\log \tau_2) / (\omega \log \tau_1))$, there exists a constant $R_0 > 0$ such that

\{ $t \mid \tau_2 T(t) \leq T(\tau_1 t)$ \} $\cup$ $[r, T^{-1}(T^m(r))] \neq \emptyset$

when $r > R_0$.

**The proof of Lemma 4:** Put $s = T(r)$, $T_0(s) = T^{-1}(s)$. Then $T_0(s)$ is a positive, increasing and continuous function, and $T_0(s) \to +\infty (s \to +\infty)$. Obviously,

$$\lim_{s \to \infty} \frac{\log T_0(s)}{\log s} \leq \frac{1}{\omega} < +\infty.$$

By Lemma 3,

lower-logdens \{ $s \mid T_0(\tau_2 s) \leq \tau_1 T_0(s)$ \} $\geq 1 - \frac{\log \tau_2}{\omega \log \tau_1}$.

Therefore, there exists $s_0 \in [s, s^m]$ such that $T_0(\tau_2 s_0) \leq \tau_1 T_0(s_0)$ for sufficiently large $s$. Put $r_0 = T^{-1}(s_0)$, then $r_0 = T_0(s_0)$, $T(r_0) = s_0$. Thus $\tau_2 T(r_0) \leq T(\tau_1 r_0)$. Since $r_0 > r$, $T(r_0) = s_0 \leq s^m = T^m(r)$, we deduce $r_0 \in [r, T^{-1}(T^m(r))]$. The proof of Lemma 4 is complete.

**The proof of Theorem 3:** Choose two constants $k > 1$ and $\tau_1 > 1$ such that

$$\frac{12}{\log k} \log (2k\tau_1) < \tau_1^\omega.$$
Put
\[ \tau_2 = \frac{12}{\log k} \log (2k\tau_1) \] and \[ \alpha = \lim_{r \to \infty} \frac{\log T(2r, f)}{\log T(r, f)}. \]

Choose a natural number \( m \) such that
\[ m > \max \left( \frac{1}{1 - (\log \tau_2)/(\mu \log \tau_1)}, 2\alpha \right). \]  

For convenience, we put \( T(r, f) = T(r) \). It is obvious that there exists a constant \( M_0 > 0 \) such that
\[ M_0 > \max\{R_0, e^{8r}\}, \]
\[ T(r) > \max \left\{ \frac{1}{K}, \frac{240 \log (2r)}{\log k} \right\}, \]
\[ T(2r, f) < T^{2\alpha}(r, f), \]
\[ c \cdot c^* \frac{\tau_2^{\mu/2}r^{\mu/4}}{(\log (k\tau_1) \log r)^2} > 1 \]
when \( r \geq M_0 \), where \( R_0 > 0 \) is the constant stated in Lemma 4, \( c^* > 0 \) is the constant stated in Lemma 2, and
\[ c = \frac{1}{1 + 9\tau_1^2}, \quad K = \frac{c^{\mu/2}((\mu/2))^{m2p+1}+1}{(m^{4p+1} + 1)!}, \quad p = \left[ \log \left( \frac{\log (6k\tau_1)}{\log 2} \right) \right] + 2, \]
(\( \lfloor . \rfloor \) denotes the integral part). Put \( r^* = \max\{M_0, 4^{4/\mu}\} \). From (11) we deduce that
\[ c \cdot c^* \frac{(\tau_1 r)^{\mu/2}}{(\log (k\tau_1) \log r)^2} > r^{\mu/4} \geq M_0 \]
for \( r \geq r^* \).

By Lemma 4, there exists \( r_0 \in \left[ r^*, T^{-1}(T^{m}(r^*)) \right] \) such that
\[ \tau_2 T(r_0) \leq T(\tau_1 r_0). \]

Put \( r_1 = r_0/k, \quad R_1 = \tau_1 r_0 \), then
\[ \frac{12}{\log k} \log \frac{2R_1}{r_1} T(kr_1) \leq T(R_1), \]
and
\[ 12T(r_1) \leq \frac{12}{\log k} \log \frac{2R_1}{r_1} T(kr_1) \leq T(R_1). \]
By (8), (9), (14), (15) and Lemma 2, there exists $z_0$ lying in $r_1 < |z| < 2R_1$ such that in the disk

$$\Gamma_0 : |z - z_0| < \frac{4\pi}{\log r^*|z_0|}$$

$f$ takes every complex value $a$ at least $n_0 = c^* \frac{T(R_1)}{(\log r^*)^2 (\log (k\tau_1)^2)}$
times except for those complex values which can be contained in two spherical disks $\gamma_0$ and $\gamma_0'$ with radius $e^{-n_0}$, that is, $\Gamma_0$ is a filling disk of $f(z)$. Obviously,

$$n_0 \geq \frac{c^* \frac{T(1/2|z_0|)}{(\log k\tau_1)^2 (\log |z_0|)^2}}{\log (k|z_0|)} \geq (|z_0|)^1 - \varepsilon(|z_0|),$$

where $c(r) > 0$, and $\varepsilon(r) \to 0$ as $r \to \infty$. It can be easily verified from (8) that

$$\Gamma_0 \subset \left\{ z \mid \frac{1}{2k} r^* < |z| < 3\tau_1 T^{-1}(T^m(r^*)) \right\}.$$
On the other hand, by (10) and (13) we have
\[ c e^{n_0} > c n_0 > c \cdot c^* \frac{(\tau_1 r^*)^{\mu/2}}{(\log (k \tau_1) \log r^*)^2} \geq M_0, \]
and hence
\[ (18) \]
\[ T(e^{n_0}) > c^{\mu/2} e(\mu/2)n_0 > c^{\mu/2} \frac{(\mu/2)^p+1}{(m(j+1)p+1)!} n_0^{m(j+1)p+1+1} > K \frac{T^m(j+1)p+1+1(r^*)}{(\log r^*)^{2m(j+1)p+1+2}}, \]
where \( K > 0 \) is the constant in (12). By (17) and (18) we have
\[ T(r^*) < \frac{1}{K} (\log r^*)^{2m(j+1)p+1+2}. \]
This contradicts (9). Therefore, \( \gamma_0' \) (or \( \gamma_0'' \)) can not meet both \( A_{j+2p} \) and \( A_{j+2p} \) \((j \in \{1,2,3\})\). It follows immediately that there exists at least one in five annuli \( A_p, A_{2p}, A_{3p}, A_{4p}, A_{5p} \) which does not meet \( \gamma_0 \) or \( \gamma_0'' \). Denote this annulus by \( A_0^j \). So \( f(\Gamma_0) \supseteq A_0^j \).

By the same discussion, we can deduce that there exists a filling disk \( \Gamma_1 \subset A_0^j \) and an annulus \( A_0^j \in \{A_j \mid j \in \mathbb{N}\} \) such that \( f(\Gamma_1) \supseteq A_0^j \). Repeating this construction, we obtain a sequence of filling disks \( \Gamma_j \) such that
\[ (19) \]
\[ f(\Gamma_j) \supset \overline{B}_{j+1}, \quad \Gamma_j \to \infty \ (j \to \infty). \]
Denote the centre of \( \Gamma_j \) by \( z_j \). From (16) we know that each limiting point of \( \{\arg z_j \mid j = 1,2,\cdots\} \) is a Borel direction of order at least \( \mu \) (see [5]). It follows (19) that there is a sequence of domains \( B_j \subset \Omega_0 \) such that \( f^{-1}(B_j) = \Gamma_j \) and \( \Gamma_0 \supset B_j \supset \overline{B}_{j+1} \).

Now, we prove \( \bigcap_{j=1}^{\infty} B_j \cap J(f) \neq \emptyset \): Otherwise, there exists a natural number \( j_0 \) such that \( B_j \subset F(f) \) when \( j \geq j_0 \). Since \( \Gamma_0 \) is a filling disk, we have \( f^j(B_j) = f(\Gamma_j) \supset \overline{C \setminus (\gamma_j' \cup \gamma_j'')} \) (where \( \gamma_j' \) and \( \gamma_j'' \) are two spherical disks each with radius \( e^{-n_j} \) and \( n_j \to \infty \) as \( j \to \infty \)), so \( J(f) \subset \gamma_j' \cup \gamma_j'' \) when \( j \geq j_0 \). This implies \( J(f) \) contains at most two points. This is a contradiction [2].

For a point \( a \in \bigcap_{j=1}^{\infty} B_j \cap J(f) \), we have \( a \in I(f) \) and each limiting direction of \( O^+(a) \) is a Borel direction of order at least \( \mu \). The proof of Theorem 3 is complete. \( \square \)

REFERENCES


