1. Introduction and preliminaries. Let $X$ be an infinite-dimensional complex Banach space and denote the set of bounded (compact) linear operators on $X$ by $B(X)$ ($K(X)$). Let $\sigma(A)$ and $\sigma_a(A)$ denote, respectively, the spectrum and approximate point spectrum of an element $A$ of $B(X)$. Set

$$
\sigma_{em}(A) = \bigcap_{K \in K(X)} \sigma(A + K),
$$

$$
\sigma_{ea}(A) = \bigcap_{K \in K(X)} \sigma_a(A + K),
$$

$$
\sigma_{eb}(A) = \bigcap_{AK = KA \in K(X)} \sigma(A + K),
$$

$$
\sigma_{ab}(A) = \bigcap_{AK = KA \in K(X)} \sigma_a(A + K).
$$

$\sigma_{em}(A)$ and $\sigma_{eb}(A)$ are respectively Schechter’s and Browder’s essential spectrum of $A$ ([16], [9]). $\sigma_{ea}(A)$ is a non-empty compact subset of the set of complex numbers $\mathbb{C}$ and it is called the essential approximate point spectrum of $A$ ([13], [14]). In this note we characterize $\sigma_{ab}(A)$ and show that if $f$ is a function analytic in a neighborhood of $\sigma(A)$, then $\sigma_{ab}(f(A)) = f(\sigma_{ab}(A))$. The relation between $\sigma_a(A)$ and $\sigma_{ab}(A)$, that is exhibited in this paper, resembles the relation between the $\sigma(A)$ and the $\sigma_{eh}(A)$, and it is reasonable to call $\sigma_{ab}(A)$ Browder’s essential approximate point spectrum of $A$.

Throughout this paper $N(A)$ and $R(A)$ will denote respectively the null space and the range space of $A$. Set $\alpha(A) = \dim N(A)$ and $\beta(A) = \dim X/R(A)$. An operator $A \in B(X)$ is called semi-Fredholm if $R(A)$ is closed and at least one of $\alpha(A)$ and $\beta(A)$ is finite. For such an operator $A$ we define an index $i(A)$ by $i(A) = \alpha(A) - \beta(A)$. Let $\Phi_+(X)$ denote the set of semi-Fredholm operators with $\alpha(A) < \infty$, and $\Phi_+(X) = \{ A \in \Phi_+(X) : i(A) \leq 0 \}$. Then $\sigma_{ea}(A) = \{ \lambda \in \mathbb{C} : A - \lambda \notin \Phi_+(X) \}$ ([14, Theorem 3.1]). Let $(G_n)$ be a sequence of compact subsets of $\mathbb{C}$. The limit superior, $\limsup G_n$, is the set of all $A$ in $\mathbb{C}$ such that every neighbourhood of $A$ intersects infinitely many $G_n$.

A mapping $\tau$ defined on $B(X)$ whose values are compact subsets of $\mathbb{C}$ is said to be upper semi-continuous at $A$ when if $A_n \to A$ then $\limsup \tau(A_n) \subseteq \tau(A)$ ([11]). The polynomial hull $\hat{E}$ of a compact subset $E$ of the complex plane $\mathbb{C}$ is the complement of the unbounded component of $\mathbb{C}\setminus E$. Given a compact subset $E$ of the plane, a hole of $E$ is a component of $\hat{E}\setminus E$. If $F$ is another compact set such that $\partial E \subset F \subset E$, it follows that $\partial E \subset \partial F$, $\hat{E} = \hat{F}$ and $E$ can be obtained from $F$ by filling in some holes of $F$. (Here and in what follows $\partial E$ denotes the boundary of the set $E$ [15].) Finally $a(A)$, the ascent of $A$, is the smallest non-negative integer $n$ such that $N(A^n) = N(A^{n+1})$. If no such $n$ exists, then $a(A) = \infty$. 

2. Characterization of $\sigma_{ab}(A)$.

**Theorem 2.1.** $\lambda \notin \sigma_{ab}(A)$ if and only if $A - \lambda \in \Phi_{+}^{-}(X)$ and $a(A - \lambda) < \infty$.

**Proof.** If $\lambda \notin \sigma_{ab}(A)$, there is a $K \in K(X)$ such that $AK = KA$ and $\lambda \notin \sigma_{k}(A + K)$. In particular, $A + K - \lambda \in \Phi_{+}^{-}(X)$ and $a(A + K - \lambda) = 0$. Adding the operator $-K$ to $A + K - \lambda$, we see that $A - \lambda \in \Phi_{-}^{-}(X)$ ([7, Theorem 5.26 of Chapter IV]) and $a(A - \lambda) < \infty$ ([3, Theorem 2]). To prove the converse suppose that $A - \lambda_0 \in \Phi_{-}^{-}(X)$ and that $a(A - \lambda_0) < \infty$. If $\lambda_0 \notin \sigma(A)$, then $\lambda_0 \notin \sigma_{ab}(A)$ and the proof is complete. Suppose that $\lambda_0 \in \sigma(A)$. Then $\lambda_0$ is an isolated point of $\sigma_{a}(A)$ ([10, Lemma 2.5]). Now $0 < a(A - \lambda_0) < \infty$ implies that Kato's number $v(A - \lambda_0 : I)$ is finite ([6, Theorem 3]). Following Zemánek's method of removing jumping points ([18, Theorem 7.1]), while applying Kato's reduction theorem ([6, Theorem 4]), we conclude that the space $X$ decomposes into a direct sum of two closed subspaces $X_0$ and $X_1$. These subspaces are $(A - \lambda_0)$-invariant, hence $A$-invariant, and have the following properties: (we quote only those relevant to our problem). The space $X_1$ is finite dimensional (and $A - \lambda_0$ is nilpotent on it). If $A_0$ is the restriction of $A$ to $X_0$ considered as an operator from $X_0$ into itself then $a(A_0 - \lambda_0)$ is constant on a neighbourhood of $\lambda_0$, and hence it is 0. Let $F$ be the finite rank operator defined by $F = I$ on $X_1$, $F = 0$ on $X_0$. Hence, $AF = FA$, $a(A + F - \lambda_0) = 0$ and $A + F - \lambda_0 \in \Phi_{+}(X)$ ([7, Theorem 5.26 of Chapter IV]). Thus, $\lambda_0 \notin \sigma_{a}(A + F)$ and the proof is complete.

**Corollary 2.2.** $\lambda \notin \sigma_{a}(A) \setminus \sigma_{ab}(A)$ if and only if $\lambda$ is an isolated point of $\sigma_{a}(A)$, an eigenvalue of $A$ of finite multiplicity, $a(A - \lambda) < \infty$ and $R(A - \lambda)$ is closed.

**Corollary 2.3.** Let $\lambda \notin \sigma_{a}(A)$ be an isolated point of $\sigma_{a}(A)$ and let $a(A - \lambda) = \infty$. Then $\lambda \in \sigma_{es}(A)$.

**Proof.** Let $\lambda$ be an isolated point of $\sigma_{a}(A)$, $a(A - \lambda) = \infty$ and $\lambda \notin \sigma_{a}(A)$. Then $0 < a(A - \lambda) < \infty$, $R(A - \lambda)$ is closed and Kato's number $v(A - \lambda : I)$ is finite ([6, Theorem 3]). Let us apply the operator $F$ from the proof of Theorem 2.1. Then, by [3, Theorem 2], $a(A - \lambda) < \infty$, which provides a contradiction. This completes the proof.

**Corollary 2.4.** $\sigma_{ab}(A) = \sigma_{es}(A) \cup \{\text{limit points of } \sigma_{a}(A)\}$.

**Corollary 2.5.** Let $A \in B(X)$. Then

(i) $\sigma_{es}(A) \subset \sigma_{ab}(A) \subset \sigma_{eb}(A)$,

(ii) $\partial \sigma_{eb}(A) \subset \partial \sigma_{ab}(A) \subset \partial \sigma_{es}(A)$,

(iii) $\sigma_{es}(A) = \sigma_{eb}(A) = \sigma_{eb}(A)$,

(iv) $\sigma_{ab}(A)$ (or $\sigma_{eb}(A)$) can be obtained from $\sigma_{es}(A)$ (or $\sigma_{eb}(A)$) by filling in some holes of $\sigma_{es}(A)$ (or $\sigma_{eb}(A)$).

(v) If $\sigma_{es}(A)$ is connected, $\sigma_{ab}(A)$ is connected, and if $\sigma_{eb}(A)$ is connected, $\sigma_{eb}(A)$ is connected.

**Proof.** It is sufficient to prove (ii). Since $\partial \sigma_{eb}(A) \subset \partial \sigma_{em}(A)$ ([15, Theorem 1(b)]) and $\partial \sigma_{em}(A) \subset \partial \sigma_{ea}(A)$ ([13, Theorem 1]), then $\partial \sigma_{eb}(A) \subset \partial \sigma_{ab}(A)$. Suppose $\lambda_0 \in \partial \sigma_{ab}(A)$ and $\lambda_0 \notin \sigma_{ea}(A)$. Hence, $0 < a(A - \lambda_0) < \infty$ and $R(A - \lambda_0)$ is closed. Then there
exists an \( \varepsilon > 0 \) such that \( 0 < |\lambda_0 - \lambda| < \varepsilon \) implies that \( R(A - \lambda) \) is closed and \( \alpha(A - \lambda) \) is constant ([7, p. 243]). Since \( \lambda_0 \in \partial \sigma_{ab}(A) \), we have, by Theorem 2.1, that the constant is 0. Thus, \( \lambda_0 \) is an isolated point of \( \sigma_{ab}(A) \), and again by Theorem 2.1 we have that \( a(A - \lambda_0) = \infty \). Hence, \( \lambda_0 \in \sigma_{ea}(A) \) (Corollary 2.3). This is a contradiction, and the proof is complete.

The following example was used by Salinas ([15]) in another context. We use it to show that in general \( \sigma_{ea}(A) \neq \sigma_{ab}(A) \).

**Example 2.6.** Let \( H \) be a separable Hilbert space, and let \( V \) be a unilateral shift of multiplicity one on \( H \); also let \( N \in B(H) \) be any quasinilpotent operator. Set \( A = K \circ V^* \otimes N \). If we denote by \( D \) the closed unit disc in \( \mathbb{C} \) we have \( \sigma_{ab}(A) = D \), while \( \sigma_{ea}(A) = \partial D \cup \{0\} \).

**Proof.** Salinas showed that \( \sigma_{em}(A) = \partial D \cup \{0\} \) and \( \sigma_{ab}(A) = D \). Hence, by ([13, Theorem 2.1]) we have that \( \sigma_{ea}(A) = \partial D \cup \{0\} \). Suppose \( 0 < |\lambda| < 1 \) and \( \lambda \notin \sigma_{ab}(A) \). Then \( a(A - \lambda) < \infty \) (Theorem 2.1). Now \( \lambda \notin \sigma_{ea}(A) \) ([16], [9, Theorem 1(4)]). This is a contradiction, and the proof is complete.

From the proof of Theorem 2.1 and Corollary 2.3 we have the following of a T. J. Laffey and T. T. West theorem ([8, Proposition 2]).

**Corollary 2.7.** Let \( A \in \Phi_\tau(X) \). Then the following statements are equivalent:

(i) \( A = V + F \), where \( \alpha(V) = 0 \), \( F \) is finite rank and \( VF = FV \);

(ii) there exists a finite rank projection \( P \) commuting with \( A \) such that \( \alpha(A | N(P)) = 0 \);

(iii) there exists \( \varepsilon > 0 \) such that \( \alpha(A + \lambda) = 0 \) for \( 0 < |\lambda| < \varepsilon \);

(iv) \( a(A) < \infty \).

3. **Spectral mapping theorem for \( \sigma_{ab}(A) \).**

**Theorem 3.1.** If \( A \) is any operator and \( p \) is any polynomial, then

\[
\sigma_{ab}(p(A)) = p(\sigma_{ab}(A)).
\]

**Proof.** Let \( \lambda \notin \sigma(ab)(A) \) and \( p(t) = c(t - \lambda_1) \ldots (t - \lambda_n), \ c \neq 0 \). Thus, \( p(A) - \lambda = c(A - \lambda_1) \ldots (A - \lambda_n) \), where \( A - \lambda_i \in \Phi_\tau(X) \) and \( a(A - \lambda_i) < \infty \) for \( i = 1, \ldots, n \) (Theorem 2.1). Then \( p(A) - \lambda \in \Phi_\tau(X) \) ([17, Theorem 6.6, Theorem 3.5, Theorem 2.3 of Chapter V]). Let us show that \( a(p(A) - \lambda) < \infty \). By ([5, Proposition 38.7]) it is sufficient to prove that \( p(A) - \lambda \) is injective on the subspace \( U = \bigcap_{n=1}^{\infty} (p(A) - \lambda)^n(X) \). We shall use the method of mathematical induction. This is true for \( n = 1 \). Suppose that this is true for all polynomials of degree \( n - 1 \). Set \( p(A) - \lambda = q(A)(A - \lambda_i) \) where \( q(t) \) is polynomial of degree \( n - 1 \). Let \( x \in U \) and \( (p(A) - \lambda)x = 0 \). Then \( (A - \lambda_i)(q(A)x) = 0 \) and \( q(A)x \in \bigcap_{n=1}^{\infty} (A - \lambda_i)^n(X) \). Hence, by ([5, Proposition 38.7]) \( q(A)x = 0 \). Since \( x \in \bigcap_{n=1}^{\infty} (q(A))^n(X) \) and \( q(t) \) is a polynomial of degree \( n - 1 \), we have that \( x = 0 \). Thus, we see by Theorem 2.1 that \( \lambda \notin \sigma_{ab}(p(A)) \). This shows that \( \sigma(ab)(p(A)) \subset p(\sigma_{ab}(A)) \). We now turn to the proof of the opposite inclusion. Suppose that \( \lambda \notin p(\sigma_{ab}(A)) \) and \( \lambda \notin \sigma_{ab}(p(A)) \).
Then \( p(A) - \lambda \in \Phi_+(X) \) and \( a(p(A) - \lambda) < \infty \) (Theorem 2.1). By ([4, p. 20]) \( A - \lambda_i \in \Phi_+(X) \) for \( i = 1, \ldots, n \). Let \( \lambda_i \in \sigma_{ab}(A) \) and \( \lambda = p(\lambda_i) \). Now, \( \lambda \) is an isolated point of \( \sigma_p(p(A)) \) (Corollary 2.2) and by ([4]) \( \lambda_i \) is an isolated point of \( \sigma_{ab}(A) \). Thus, \( A - \lambda_i \in \Phi_+(X) \) ([7, Theorem 5.22 of Chapter IV]), and by Corollary 2.3 we have \( a(A - \lambda_i) < \infty \). Again, by Theorem 2.1, we have that \( \lambda_i \notin \sigma_{ab}(A) \), which provides a contradiction. This completes the proof.

To show that if \( f \) is an analytic function defined on a neighbourhood of \( \sigma(A) \), then \( f(\sigma_{ab}(A)) = \sigma_{ab}(f(A)) \) we shall apply K. Oberai's method ([12]). First we shall prove two following statements which are of particular interest.

**Theorem 3.2.** Let \( A \in B(X) \). Then the mapping \( A \mapsto \sigma_{ab}(A) \) is upper semi-continuous.

**Proof.** Let \( A_n \to A \). We have to show that \( \limsup \sigma_{ab}(A_n) \subseteq \sigma_{ab}(A) \). It is enough to show that if \( 0 \notin \sigma_{ab}(A) \), then \( 0 \notin \limsup \sigma_{ab}(A_n) \). Let \( 0 \notin \sigma_{ab}(A) \). Then 0 is an isolated point of \( \sigma_p(A) \) and \( A \in \Phi_+(X) \) (Theorem 2.1). By ([7, Theorem 5.22 of Chapter IV]) there exists an \( \varepsilon > 0 \) and an integer \( n_1 \) such that \( A_n - \lambda \in \Phi_+(X) \) for \( |\lambda| < \varepsilon \) and for \( n \geq n_1 \). We may assume that \( \alpha(A - \lambda) = 0 \) for \( 0 < |\lambda| < \varepsilon \). Let \( n \geq n_1 \). Hence, by ([7, p. 243]) \( \alpha(A_n - \lambda) \) are constant for all \( |\lambda| < \varepsilon \) except for an isolated set. Let \( 0 < |\lambda_0| < \varepsilon \). Set \( S_0 = \{ \lambda \in \mathbb{C} : |\lambda| = |\lambda_0| \} \) and \( m(A) = \inf\{\|Ax\| : \|x\| = 1\} \). Now, \( m(A - \lambda) > 0 \) for \( \lambda \in S_0 \). Since \( m(A - \lambda) \) is a continuous function of \( \lambda \) ([2, p. 19]) and \( S_0 \) is compact, there exists a \( \mu_0 \in S_0 \) such that \( m(A - \mu_0) = \inf\{m(A - \lambda) : \lambda \in S_0\} \). Let \( n_2 \) be an integer such that for \( n \geq n_2 \) we have \( \|A_n - A\| < m(A - \mu_0) \). Hence, for \( n \geq n_0 = \max(n_1, n_2) \) and \( \lambda \in S_0 \) we have \( m(A - \lambda) - \|A_n - A\| < m(A_n - \lambda) \) ([2, Lemma 2.2]). Thus, \( m(A_n - \lambda) > 0 \) and \( \alpha(A_n - \lambda) = 0 \) for all \( |\lambda| < \varepsilon \) except for an isolated set and for \( n \geq n_0 \). Therefore, for \( n \geq n_0 \) we see that \( \sigma_{ab}(A_n) \cap \{ \lambda \in \mathbb{C} : |\lambda| < \varepsilon \} \) is empty (Corollary 2.3 and Theorem 2.1). Thus we have \( 0 \notin \limsup \sigma_{ab}(A_n) \), and the proof is complete.

**Theorem 3.3.** Let \( A \in B(X) \) and let \( f \) be an analytic function defined on a neighbourhood of \( \sigma(A) \). Then \( \sigma_{ae}(f(A)) = f(\sigma_{ae}(A)) \).

**Proof.** \( \Phi_+(X) \) is an F-semigroup with index \( i \) ([4, p. 20]). Also

\[
\sigma_{ae}(A) = \sigma_{ae}(A) \cup \left( \bigcup_{n \geq 1} F_n \right),
\]

(3.1)

where \( \sigma_{ae}(A) = \{ \lambda \in \mathbb{C} : A - \lambda \notin \Phi_+(X) \} \), \( F_n = \{ \lambda \in \sigma(A) : A - \lambda \in \Phi_+(X) \} \), \( i(A - \lambda) = n \). Now suppose that \( \mu \notin f(\sigma_{ae}(A)) \). Then \( \mu - f(\lambda) \) has no zeros on \( \sigma_{ae}(A) \) and in particular has no zeros on \( \sigma_{ae}(A) \). Applying ([4, Theorem 1]) we conclude that \( \mu - f(A) \in \Phi_+(X) \) and

\[
i(\mu - f(A)) = \sum_n n \alpha_n,
\]

where \( \alpha_n \) is the number of isolated zeros of \( \mu - f(\lambda) \) on \( F_n \) counted according to their multiplicities. From (3.1) it follows that \( \alpha_n = 0 \) for \( n \geq 1 \). Thus \( i(\mu - f(A)) \leq 0 \), which implies that \( \mu \notin \sigma_{ae}(f(A)) \). This completes the proof.
THEOREM 3.4. Let \( A \in B(X) \) and let \( f \) be an analytic function defined on a neighbourhood of \( \sigma(A) \). Then
\[
f(\sigma_{ab}(A)) = \sigma_{ab}(f(A)).
\]

Proof. Let \((p_n(t))\) be a sequence of polynomials converging uniformly to \( f(t) \) on a neighbourhood of \( \sigma(A) \). We have
\[
f(\sigma_{ab}(A)) = \lim p_n(\sigma_{ab}(A)) \quad \text{(by ([12, p. 370])})
\]
\[
= \lim \sigma_{ab}(p_n(A)) \quad \text{(by Theorem 3.1)}
\]
\[
\leq \sigma_{ab}(f(A)) \quad \text{(by Theorem 3.2)}.
\]

To prove the converse suppose that \( \lambda \in \sigma_{ab}(f(A)) \).

Case I. \( \lambda \in \sigma_{ea}(f(A)) \). By Theorem 3.3 \( \lambda \in f(\sigma_{ea}(A)) \). Thus, we have \( \lambda \in f(\sigma_{ab}(A)) \).

Case II. \( \lambda \notin \sigma_{ea}(f(A)) \). In this case \( \lambda \) is a limit point of \( \sigma_{a}(f(A)) \) (Corollary 2.4), and there exists a sequence \((\lambda_n) \subset \sigma_{a}(f(A))\) such that \( \lambda_n \to \lambda \). Now, there exists a sequence \((\mu_n) \subset \sigma_{a}(A)\) such that \( f(\mu_n) = \lambda_n \to \lambda \) ([4]). Then \((\mu_n)\) contains a convergent subsequence and we may assume that \( \lim \mu_n = \mu \in \sigma_{a}(A) \). Then \( \lambda = f(\mu) \in f(\sigma_{ab}(A)) \). This completes the proof of the theorem.

4. A perturbation theorem.

THEOREM 4.1. Let \( A \in B(X) \) and let \( N \in B(X) \) be a quasinilpotent operator commuting with \( A \). Then \( \sigma_{ab}(A + N) = \sigma_{ab}(A) \).

Proof. It is enough to show that if \( 0 \notin \sigma_{ab}(A) \), then \( 0 \notin \sigma_{ab}(A + N) \). Let \( 0 \notin \sigma_{ab}(A) \). If \( 0 \notin \sigma_{q}(A) \), then \( 0 \notin \sigma_{q}(A + N) \) ([1, p. 320]). Hence, we have that \( 0 \notin \sigma_{ab}(A + N) \). If \( 0 \in \sigma_{q}(A) \), then \( 0 \) is an isolated point of \( \sigma_{q}(A) \) (Corollary 2.2), and therefore \( 0 \) is an isolated point of \( \sigma_{q}(A + N) \) ([1, p. 320]). Since \( 0 \notin \sigma_{ab}(A) \), we see that \( 0 \notin \sigma_{ea}(A + N) \) ([19, Theorem 7]). This implies that \( 0 \notin \sigma_{ab}(A + N) \) (Corollary 2.3), and the proof is complete.

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