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ALGEBRAIC NUMBERS WITH BOUNDED DEGREE AND WEIL HEIGHT

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Abstract

For a positive integer d and a nonnegative number ξ, let $N(d,\xi)$ be the number of $\alpha \in \overline{\mathbb{Q}}$ of degree at most d and Weil height at most ξ. We prove upper and lower bounds on $N(d,\xi)$. For each fixed $\xi > 0$, these imply the asymptotic formula $\log N(d,\xi) \sim \xi d^2$ as $d \to \infty$, which was conjectured in a question at Mathoverflow [https://mathoverflow.net/questions/177206/].

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1. Introduction

For an algebraic number α of degree d over $\mathbb{Q}$ with conjugates $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ and minimal polynomial

$$a_d(x-\alpha_1)\cdots(x-\alpha_d) = a_d x^d + \cdots + a_1 x + a_0 \in \mathbb{Z}[x],$$

where $a_d > 0$, we denote by $H(\alpha) := \max_{0\le j\le d} |a_j|$ the *height* of α and by

$$M(\alpha) := a_d \prod_{i=1}^{d} \max\{1, |\alpha_i|\}$$

the *Mahler measure* of α. For $\alpha \in \overline{\mathbb{Q}}$, these quantities are related by the inequalities

$$H(\alpha)2^{-d} \le M(\alpha) \le H(\alpha)\sqrt{d+1} \tag{1.1}$$

(see, for instance, [15] and [16, Lemma 3.11]).

Set

$$M(d,T) := \#\{\alpha \in \overline{\mathbb{Q}} \ :\ \deg \alpha = d,\ M(\alpha) \le T\},$$

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where $\#A$ stands for the cardinality of the set A. For $d \geq 2$ and

$$V_d := 2^{d+1}(d+1)^{\lfloor (d-1)/2 \rfloor} \prod_{i=1}^{\lfloor (d-1)/2 \rfloor} \frac{(2i)^{d-2i}}{(2i+1)^{d+1-2i}},$$

the asymptotic formula

$$M(d,T) = \frac{dV_d}{2\zeta(d+1)} T^{d+1} + O(T^d (\log T)^{\lfloor 2/d \rfloor}) \quad \text{as } T \to \infty$$

has been established in [2] and [10]. (Throughout, $\zeta(s)$ is the Riemann zeta-function.) See also [1, 11, 17] and the references therein for some generalisations. In [9], this formula is given with an explicit error term: for any $d \geq 3$ and any $T \geq 1$,

$$\left| M(d,T) - \frac{dV_d}{2\zeta(d+1)} T^{d+1} \right| \leq 3.37 \cdot 15.01^{d^2} \cdot T^d.$$

This inequality gives the asymptotic formula for $M(d,T)$ as $d \to \infty$ in the range $\log T \gg d^2$. (Here and below, the notation $v \gg w$ means that the inequality $v \geq cw$ holds with some positive constant c.) By [2, Theorem 4], this asymptotic formula holds in a wider range $\log T \gg d \log d$, but with slightly larger error term in T. However, for small T, for example, T fixed at $T = 2$, the problem of finding the correct order of $M(d,T)$ is wide open. See, for instance, the papers [3, 5, 13]. More precisely, from the main result of [5] one can derive $M(d,2) > cd^5$ with some absolute constant $c > 0$, whereas the best known upper bound is only $M(d,2) < 2^{(1+\varepsilon)d}$ for any $\varepsilon > 0$ and $d \geq d(\varepsilon)$ [6]. Another interesting case, $T = 1$, corresponds to the constant

$$C := \limsup_{d \to \infty} \frac{\log M(d,1)}{\log d}, \tag{1.2}$$

which has been studied by Erdős [7] and Pomerance [14]. This constant can be expressed as the number of solutions of the equation $\phi(n) = d$ for $n \in \mathbb{N}$ (when d is fixed), where ϕ is Euler's totient function, and bounds can be found using tools from analytic number theory. Erdős and Pomerance showed that $0.55 < C \leq 1$ and Erdős conjectured that $C = 1$ [8].

In the upper bound direction, for d sufficiently large and any $T \geq 1$, we showed in [6] that the number of integer polynomials of degree d and with positive leading coefficient, nonzero constant coefficient and Mahler measure at most T is bounded above by $T^{d(1+16\log\log d/\log d)} e^{3.58\sqrt{d}}$ for d large enough. Furthermore, the factor $e^{3.58\sqrt{d}}$ can be removed for $T \geq 1.32$. The roots of any such polynomial, irreducible over $\mathbb{Q}$ and whose coefficients are relatively prime, give d algebraic numbers of degree d and Mahler measure at most T. Hence, the main result of [6] yields the inequality

$$M(d,T) < dT^{d(1+16\log\log d/\log d)} \tag{1.3}$$

for each $T \geq 1.32$ and each sufficiently large integer d, say $d \geq d_0$.

In this paper, we consider the related quantity

$$N(d,\xi) := \#\{\alpha \in \overline{\mathbb{Q}} \ : \ \deg \alpha \leq d, \ h(\alpha) \leq \xi\},$$

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where

$$h(\alpha) := \frac{\log M(\alpha)}{\deg \alpha}$$

is the *Weil height* of α. Using [2, Theorem 4] and following the approach of [10], for $\xi \gg \log d$, one can derive the asymptotic formula

$$N(d,\xi) \sim \frac{dV_d e^{\xi d(d+1)}}{2\zeta(d+1)} \quad \text{as } d \to \infty. \tag{1.4}$$

In [12], the problem of finding the asymptotic formula for $N(d,1)$ (noting that $\xi = 1$ is much less than $\log d$), or, less ambitiously, for $\log N(d,1)$ as $d \to \infty$, has been raised. From the discussion in [12] and also from (1.4), one can conjecture that the expected formula is

$$\log N(d,1) \sim d^2 \quad \text{as } d \to \infty. \tag{1.5}$$

In this note, we prove the following theorem, which implies (1.5).

Theorem 1.1. *For each $\xi \geq 2d^{-1}\log d$ and each sufficiently large d,*

$$-\frac{d\log d}{2} < \log N(d,\xi) - \xi d^2 < \frac{17\xi d^2 \log\log d}{\log d}.$$

It is clear that Theorem 1.1 yields the asymptotic formula

$$\log N(d,\xi) \sim \xi d^2 \quad \text{as } d \to \infty \quad \text{and} \quad \frac{\xi d}{\log d} \to \infty.$$

Of course, equation (1.4) immediately implies this asymptotic formula, but only in the range $\xi \gg \log d$. We also remark that, for $0 \leq \xi \leq d^{-1}(\log d)^{-3}$, by combining a Dobrowolski-type bound with the above mentioned results [7, 8, 14], one gets

$$\log N(d_k,\xi) \sim C\log d_k \quad \text{as } d_k \to \infty,$$

where C is the constant defined in (1.2) and $(d_k)_{k=1}^{\infty}$ is some increasing sequence of positive integers.

In fact, the lower bound on $\log N(d,\xi) - \xi d^2$ as claimed in Theorem 1.1 will be proved for $d \geq 1.784 \cdot 10^8$. In principle, some explicit constant D_0 such that the upper bound of Theorem 1.1 for $\log N(d,\xi) - \xi d^2$ is true for each $d \geq D_0$ can also be given. However, it depends on the corresponding bound $d \geq d_0$ in (1.3), which was not calculated in [6], so we will not give it here.

For $\log M(d,T)$, by applying the same arguments, we get the following bounds.

Theorem 1.2. *For each $T \geq 38d^{3/2}(\log d)^2$ and each sufficiently large d,*

$$-\frac{d\log d}{2} < \log M(d,T) - d\log T < \frac{17d\log T\log\log d}{\log d}.$$

We will prove the lower bound on $\log M(d,T) - d\log T$ for each $d \geq 6$. Note that Theorem 1.2 implies the asymptotic formula

$$\log M(d,T) \sim d\log T \quad \text{as } d \to \infty \quad \text{and} \quad \frac{\log T}{\log d} \to \infty.$$

In the next section, we give some auxiliary results and combine them into Lemma 2.3. Then, in Section 3, we give the proofs of the theorems.

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2. Auxiliary results

Let $d, H \geq 2$ be two integers. Consider the set $P(d, H)$ of integer polynomials defined by

$$P(d,H) := \Big\{a_d x^d + \cdots + a_1 x + a_0 \in \mathbb{Z}[x] \;:\; a_d > 0, a_0 \neq 0, \max_{0 \leq j \leq d} |a_j| \leq H\Big\}.$$

In [4, Theorem 1], we showed that the number of integer polynomials reducible over $\mathbb{Q}$ and of degree d and height at most H is less than

$$2H(2H+1)^{d-1} + 2dH(2H+1)^{d-1}(\log(2H))^2.$$

Here, the first term corresponds to the polynomials whose free term is zero. Since the polynomials with $a_d < 0$ are also counted in the above formula, we can remove the factor 2 from the second term and restate this result as shown in the following lemma.

Lemma 2.1. *For any integers $d, H \geq 2$, the number of polynomials in $P(d, H)$ reducible over $\mathbb{Q}$ is less than*

$$dH(2H+1)^{d-1}(\log(2H))^2.$$

Of course, the coefficients of a polynomial irreducible over $\mathbb{Q}$ are not necessarily coprime (for instance, the coefficients of $2x^2 - 6x + 2$ are all divisible by 2). For this reason, we also need the following result.

Lemma 2.2. *For any integers $d \geq 6$ and $H \geq 6d$, the set $P(d, H)$ contains at least*

$$\frac{2^d H^{d+1}}{\zeta(d+1)} - d2^{d+2}H^d$$

polynomials $a_d x^d + \cdots + a_1 x + a_0$ satisfying $\gcd(a_d, \ldots, a_1, a_0) = 1$.

Proof. Let g be an integer in the range $1 \leq g \leq H$. Suppose there are $N_g(H)$ polynomials in $P(d, H)$ satisfying $\gcd(a_d, \ldots, a_1, a_0) = g$. Our aim is to estimate $N_1(H)$ from below. Clearly,

$$\#P(d,H) = 2H^2(2H+1)^{d-1},$$

since there are H possibilities for a_d, $2H$ possibilities for a_0, and $2H + 1$ possibilities for each a_j, where $j = 1, \ldots, d-1$. Consequently,

$$N_1(H) + N_2(H) + \cdots + N_H(H) = 2H^2(2H+1)^{d-1}.$$

Observe that $N_g(H) = N_1(\lfloor H/g \rfloor)$ for $g = 1, \ldots, H$. Hence,

$$\sum_{g=1}^{H} N_1(\lfloor H/g \rfloor) = 2H^2(2H+1)^{d-1}.$$

Now, by the Möbius inversion formula,

$$N_1(H) = \sum_{g=1}^{H} \mu(g) 2\lfloor H/g \rfloor^2 (2\lfloor H/g \rfloor + 1)^{d-1}. \tag{2.1}$$

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Split the sum on the right-hand side of (2.1) into two sums $N_1(H) = S_1 + S_2$, where S_1 is taken over g in the interval $1 \le g \le \lfloor H/d \rfloor$ and S_2 is over $\lfloor H/d \rfloor + 1 \le g \le H$. Since $H/g \le d$, we find that

$$|S_2| \le (H - \lfloor H/d \rfloor) 2 (H/g)^2 (2H/g + 1)^{d-1} < 2d^2 (2d+1)^{d-1} H.$$

So, in view of

$$2d^2(2d+1)^{d-1} < 2d^2(13d/6)^{d-1} \le 2d^2(13H/36)^{d-1} < 0.5H^{d-1},$$

we conclude that

$$|S_2| < 0.5H^d.$$

To evaluate the sum

$$S_1 := \sum_{g=1}^{\lfloor H/d \rfloor} \mu(g) 2 \lfloor H/g \rfloor^2 (2\lfloor H/g \rfloor + 1)^{d-1}, \tag{2.2}$$

we first show that the difference between $2\lfloor H/g \rfloor^2 (2\lfloor H/g \rfloor + 1)^{d-1}$ and $2^d (H/g)^{d+1}$ is small, and then investigate

$$S_0 := \sum_{g=1}^{\lfloor H/d \rfloor} \mu(g) 2^d (H/g)^{d+1}. \tag{2.3}$$

Indeed, both numbers, $2\lfloor H/g \rfloor^2 (2\lfloor H/g \rfloor + 1)^{d-1}$ and $2^d (H/g)^{d+1}$, belong to the interval

$$(2(y-1)^2(2y-1)^{d-1}, 2y^2(2y+1)^{d-1}],$$

where $y := H/g \ge 2$. Thus, the difference between them does not exceed the length of the interval, namely,

$$2y^2(2y+1)^{d-1} - 2(y-1)^2(2y-1)^{d-1} < \frac{(2y+1)^{d+1} - (2y-2)^{d+1}}{2}.$$

By the mean value theorem, the latter difference equals $1.5(d+1)y_0^d$ for some y_0 in the interval $[2y-2, 2y+1]$. Consequently,

$$|2\lfloor H/g \rfloor^2 (2\lfloor H/g \rfloor + 1)^{d-1} - 2^d (H/g)^{d+1}| < 1.5(d+1)(2H/g + 1)^d.$$

Combining this with (2.2) and (2.3), we derive

$$|S_1 - S_0| \le 1.5(d+1) \sum_{g=1}^{\lfloor H/d \rfloor} (2H/g + 1)^d.$$

The first term in the above sum is $(2H+1)^d$. The quotient of the gth term and the first term is

$$\frac{(2H/g+1)^d}{(2H+1)^d} = \frac{(2H+g)^d}{(2H+1)^d} \cdot \frac{1}{g^d} \le \frac{(2H+H/d)^d}{(2H+1)^d} \cdot \frac{1}{g^d}$$
$$< \left(1 + \frac{1}{2d}\right)^d \cdot \frac{1}{g^d} < \frac{1.65}{g^d}.$$

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It follows that

$$|S_1 - S_0| < 1.5(d+1)\frac{1.65}{\zeta(d)}(2H+1)^d < \frac{2.5(d+1)}{\zeta(d)}(2H+1)^d.$$

Therefore, applying the inequality

$$\Big(1 + \frac{1}{2H}\Big)^d \le \Big(1 + \frac{1}{12d}\Big)^d < 1.09, \tag{2.4}$$

we conclude that

$$|S_1 - S_0| < \frac{(3d+3)(2H)^d}{\zeta(d)} < 3.5d2^d H^d. \tag{2.5}$$

Next, since the Dirichlet series that generates the Möbius function is the inverse of the Riemann zeta function, from (2.3) we find that

$$\frac{S_0}{2^d H^{d+1}} = \sum_{g=1}^{\lfloor H/d \rfloor} \frac{\mu(g)}{g^{d+1}} = \frac{1}{\zeta(d+1)} - \sum_{g=\lfloor H/d \rfloor + 1}^{\infty} \frac{\mu(g)}{g^{d+1}}.$$

This leads to

$$\begin{aligned}\Big|S_0 - \frac{2^d H^{d+1}}{\zeta(d+1)}\Big| &\le 2^d H^{d+1} \sum_{g=\lfloor H/d \rfloor + 1}^{\infty} \frac{1}{g^{d+1}} < \frac{2^d H^{d+1}}{d\lfloor H/d \rfloor^d} \\ &< \frac{2^d H^{d+1}}{d(H/d-1)^d} \le \frac{2^d H^{d+1}}{d(5H/6d)^d} = 2.4^d d^{d-1} H \\ &\le 2.4^d (H/6)^{d-1} H < 0.1H^d.\end{aligned}$$

Combining this with (2.1)–(2.3) and (2.5), we deduce that

$$\begin{aligned}\Big|N_1(H) - \frac{2^d H^{d+1}}{\zeta(d+1)}\Big| &= \Big|S_2 + S_1 - S_0 + S_0 - \frac{2^d H^{d+1}}{\zeta(d+1)}\Big| \\ &\le |S_2| + |S_1 - S_0| + \Big|S_0 - \frac{2^d H^{d+1}}{\zeta(d+1)}\Big| \\ &< 0.5H^d + 3.5d2^d H^d + 0.1H^d < d2^{d+2} H^d.\end{aligned}$$

This yields the required lower bound on $N_1(H)$ and proves the lemma. □

From Lemmas 2.1 and 2.2 we will derive the following lemma.

Lemma 2.3. *For any $d \ge 6$ and any $H \ge 37d(\log d)^2$ there are at least*

$$d2^{d-1}H^{d+1} \tag{2.6}$$

algebraic numbers of degree d and height at most H.

Proof. Lemmas 2.1 and 2.2 imply that, for $d \ge 6$ and $H \ge 6d$,

$$I(d,H) > \frac{2^d H^{d+1}}{\zeta(d+1)} - d2^{d+2}H^d - dH(2H+1)^{d-1}(\log(2H))^2,$$

where $I(d,H)$ is the number of irreducible polynomials in $\mathbb{Z}[x]$ lying in the set $P(d,H)$.

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By (2.4), we have $(2H+1)^d < 1.09 \cdot 2^d H^d$. It follows that

$$dH(2H+1)^{d-1} < \frac{d}{2}(2H+1)^d < d2^d H^d,$$

and hence

$$d2^{d+2}H^d + dH(2H+1)^{d-1}(\log(2H))^2 < d2^d H^d(4 + (\log(2H))^2).$$

Therefore,

$$\begin{aligned} I(d,H) &> 2^d H^d (H\zeta(d+1)^{-1} - 4d - d(\log(2H))^2) \\ &> 2^d H^d (0.98H - 4d - d(\log(2H))^2). \end{aligned}$$

Note that the function

$$u(x) := \frac{0.24x}{4 + (\log x)^2} - d$$

is increasing in $x > 0$. Furthermore, one can easily verify that, for each $d \geq 6$,

$$u(74d(\log d)^2) = d\Big(\frac{17.76(\log d)^2}{4 + (\log(74d(\log d)^2))^2} - 1\Big) > 0.$$

Hence, $u(x) > 0$ for $x \geq 74d(\log d)^2$. Now, assuming that

$$H \geq 37d(\log d)^2$$

and $d \geq 6$, from $u(2H) > 0$ we deduce that

$$0.98H - 4d - d(\log(2H))^2 > 0.5H.$$

Therefore,

$$I(d,H) > 2^d H^d \cdot 0.5H = 2^{d-1}H^{d+1}.$$

This implies (2.6), since each of these polynomials (with positive leading coefficients) gives d algebraic numbers of degree d and height at most H. □

3. Proofs of the theorems

Proof of Theorem 1.1. We will apply Lemma 2.3 with

$$H := \lfloor e^{\xi d}(d+1)^{-1/2} \rfloor$$

and d so large that $H \geq 37d(\log d)^2$. (Recall that $\xi \geq 2d^{-1}\log d$, so the inequality $H \geq 37d(\log d)^2$ holds for $d \geq 1.784 \cdot 10^8$.) Then, by (1.1) and (2.6), each of those $\geq d2^{d-1}H^{d+1}$ algebraic numbers α has degree d and Weil height

$$h(\alpha) = \frac{\log M(\alpha)}{d} \leq \frac{\log(H(\alpha)\sqrt{d+1})}{d} \leq \frac{\log e^{\xi d}}{d} = \frac{\xi d}{d} = \xi.$$

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Hence, for all $d \geq 1.784 \cdot 10^8$ and $\xi \geq 2d^{-1}\log d$,

$$N(d,\xi) \geq d2^{d-1}\lfloor e^{\xi d}(d+1)^{-1/2}\rfloor^{d+1} > d2^{d-1}\Big(\frac{e^{\xi d} - \sqrt{d+1}}{\sqrt{d+1}}\Big)^{d+1}$$
$$> \frac{d2^{d-1}(e^{\xi d}/2)^{d+1}}{\sqrt{d+1}(d+1)^{d/2}} = \frac{d/4}{\sqrt{d+1}(1+1/d)^{d/2}} \cdot \frac{e^{\xi d(d+1)}}{d^{d/2}} > \frac{e^{\xi d^2}}{d^{d/2}}.$$

This implies the required lower bound on $\log N(d,\xi)$.

For the upper bound, we first observe that, by (1.1), each $\alpha \in \overline{\mathbb{Q}}$ of degree d whose Mahler measure is bounded by T, satisfies

$$H(\alpha) \leq 2^d M(\alpha) \leq 2^d T.$$

Thus,

$$M(d,T) \leq (2^{d+1}T+1)^{d+1} < (2^{d+2}T)^{d+1} = 2^{(d+1)(d+2)}T^{d+1}. \tag{3.1}$$

Next, observe that each α of degree at most d and Weil height at most ξ satisfies $M(\alpha) \leq e^{\xi \deg \alpha} \leq e^{\xi d}$. Now, using (1.3) with $T = e^{\xi d}$ for j in the range $d_0 \leq j \leq d$, where d_0 is so large that (1.3) is true for $d \geq d_0$, and (3.1) for $j < d_0$, we deduce that

$$N(d,\xi) \leq \sum_{j=0}^{d} M(j, e^{\xi d}) = \sum_{j=0}^{d_0-1} M(j, e^{\xi d}) + \sum_{j=d_0}^{d} M(j, e^{\xi d})$$
$$\leq \sum_{j=0}^{d_0-1} 2^{(j+1)(j+2)}e^{\xi d(j+1)} + \sum_{j=d_0}^{d} je^{\xi dj(1+16\log\log j/\log j)}$$
$$< d_0 2^{(d_0+1)(d_0+2)}e^{\xi d d_0} + d^2 e^{\xi d^2(1+16\log\log d/\log d)}$$
$$< e^{\xi d^2(1+17\log\log d/\log d)}$$

for d large enough. This proves the required upper bound.

Proof of Theorem 1.2. By (1.3), we find that

$$M(d,T) < T^{d(1+17\log\log d/\log d)}$$

for $T \geq 1.32$ and d large enough. This implies the claimed upper bound.

To prove the lower bound, apply Lemma 2.3 with

$$H := \lfloor T(d+1)^{-1/2}\rfloor,$$

where $T \geq 38d^{3/2}(\log d)^2$ and $d \geq 6$. Then, by (1.1) and (2.6), each of those $\geq d2^{d-1}H^{d+1}$ algebraic numbers has degree d and Mahler measure at most T. Consequently, using the bounds $T - \sqrt{d+1} > T/2$ and $d \geq 6$, we deduce that

$$M(d,T) \geq d2^{d-1}\lfloor T(d+1)^{-1/2}\rfloor^{d+1} > d2^{d-1}\Big(\frac{T - \sqrt{d+1}}{\sqrt{d+1}}\Big)^{d+1}$$
$$> d2^{d-1}(d+1)^{-(d+1)/2}\Big(\frac{T}{2}\Big)^{d+1} = \frac{dT^{d+1}}{4\sqrt{d+1}(d+1)^{d/2}}$$
$$> \frac{2dT^d\sqrt{d+1}}{4\sqrt{d+1}d^{d/2}(1+1/d)^{d/2}} = \frac{d/2}{(1+1/d)^{d/2}} \cdot \frac{T^d}{d^{d/2}} > \frac{T^d}{d^{d/2}},$$

which gives the claimed lower bound.

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