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BOUNDS FOR THE MULTIPLICITIES OF THE ROOTS OF A COMPLEX POLYNOMIAL

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Abstract We refine a result of Dubickas on the maximal multiplicity of the roots of a complex polynomial, and obtain several separability criteria for complex polynomials with large leading coefficient. We also give p-adic analogous results for polynomials with integer coefficients.

Keywords: separable polynomial; multiplicity of a root; non-Archimedean absolute value

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1. Introduction

In the study of the canonical factorization of a given polynomial f it is often useful to find information on the multiplicities of its roots or of its irreducible factors. For instance, if all the roots of f have multiplicity 1, then f has no repeated irreducible factors, and the polynomial is separable, or square free. One of the important uses of derivatives in the study of polynomials consists of the investigation of their multiple factors. A fundamental result in this direction is that, given two non-constant polynomials f and g with coefficients in a unique factorization domain of characteristic 0, g irreducible, g^2 divides f if and only if g divides both f and f' [**12**]. An equivalent result is that a polynomial over a unique factorization domain of characteristic 0 has a repeated non-constant factor if and only if its discriminant is zero. It is also well known that if K is a field and F an extension of K, then a polynomial $f \in K[X]$ has $c \in F$ for a multiple root if and only if $f(c) = f'(c) = 0$ [**7**]. We note here that there exist some square-free criteria for primitive polynomials over unique factorization domains, which do not make use of derivatives or discriminants [**1**]. By considering the resultant of two different derivatives of a given polynomial, and then making use of some of Ostrowski's conditions for non-vanishing of determinants, in [**5**] we have provided explicit upper bounds for the multiplicities of the roots for some classes of complex polynomials. We have also derived bounds for the multiplicities of the roots for some classes of integral polynomials in terms of the prime factorization of their coefficients.

Using some elegant techniques, Dubickas [**6**] sharpened some of the results in [**5**], by replacing some conditions on the coefficients of a given polynomial f that force the multiplicities of its roots to all be bounded by the same positive integer k, with conditions that are symmetric with respect to f and its reciprocal $\bar{f}(x) = x^{\deg f} f(1/x)$. For instance, he obtained the following nice result.

Theorem A (Dubickas). *Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{C}[x]$, where $a_0 a_n \neq 0$, and let $k < n$ be a positive integer. If*

$$\binom{n}{k}|a_0| > \sum_{i=k}^{n-1}\binom{i}{k}|a_{n-i}| \quad \text{and} \quad \binom{n}{k}|a_n| > \sum_{i=k}^{n-1}\binom{i}{k}|a_i|, \tag{1.1}$$

then the multiplicity of each root of f is less than or equal to k. Moreover, one of the inequalities '$>$' in (1.1) (but, for $k \in \{1, n-1\}$, not both!) can be replaced by '$\geqslant$'.

The first aim of this paper is to refine Theorem A by introducing a parameter into the expressions appearing in (1.1), this approach considerably enlarging the class of polynomials to which results of this type apply. Our second aim is to replace the two conditions in (1.1) by a single inequality either on the size of $|a_n|$ or on the size of $|a_0|$. We will achieve this by combining the techniques in [**6**] with some ideas in [**3**, **4**]. We use the following notation. For a non-constant complex polynomial $f(X) = a_0 + a_1 x + \cdots + a_n x^n$, which factors as

$$f(x) = a_n \prod_{i=1}^{s}(x - \theta_i)^{n_i}$$

with $a_n, \theta_1, \ldots, \theta_s \in \mathbb{C}$, θ_i pairwise distinct and n_i positive integers, we set

$$e(f) = \max\{n_1, \ldots, n_s\}.$$

The first result is a variant of Theorem A above.

Theorem 1.1. *Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{C}[x]$, $a_0 a_n \neq 0$, and let $k < n$ be a positive integer. If for a positive real δ we have*

$$\binom{n}{k}|a_0| > \sum_{i=k}^{n-1}\binom{i}{k}|a_{n-i}|\delta^{i-n} \quad \text{and} \quad \binom{n}{k}|a_n| > \sum_{i=k}^{n-1}\binom{i}{k}|a_i|\delta^{n-i}, \tag{1.2}$$

then $e(f) \leqslant k$. Moreover, one of the inequalities '$>$' in (1.2) can be replaced by '$\geqslant$'.

For the sake of completeness we will include a detailed proof of this result. The proof will require the following lemma.

Lemma 1.2. *Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{C}[x]$, $a_0 a_n \neq 0$, and let $k < n$ be a positive integer. If $a_1 = a_2 = \cdots = a_{n-k} = 0$ or $a_k = a_{k+1} = \cdots = a_{n-1} = 0$, then $e(f) \leqslant k$.*

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Theorem 1.1 has many consequences that are not immediately revealed by Theorem A, and this is mainly due to the flexibility given by the parameter δ above. The first such consequence, which is a result complementary to Lemma 1.2, is as follows.

Corollary 1.3. *Let $f(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{C}[x]$, where $a_0a_n \neq 0$, and assume that $(|a_1| + |a_2| + \cdots + |a_{n-k}|) \cdot (|a_k| + |a_{k+1}| + \cdots + |a_{n-1}|) \neq 0$ for a positive integer $k < n$. Let θ and ψ be the unique positive roots of the polynomials*

$$f_1(x) = \binom{n}{k}|a_0|x^{n-k} - \sum_{i=k}^{n-1}\binom{i}{k}|a_{n-i}|x^{i-k}$$

and

$$f_2(x) = \binom{n}{k}|a_n|x^{n-k} - \sum_{i=k}^{n-1}\binom{i}{k}|a_i|x^{i-k},$$

respectively. If $\theta\psi < 1$, then $e(f) \leqslant k$.

Recall that, according to Descartes's Rule of Signs, a polynomial with real coefficients and a single sign difference between its consecutive non-zero coefficients has a unique positive root.

In practice we need to replace the condition $\theta\psi < 1$ by an effective one involving the absolute values of the coefficients of f, by using suitable estimates for the moduli of the roots of f_1 and f_2. We will only give a couple of examples in this respect, and refer the interested reader to the excellent account of polynomial root estimates provided by Marden [**9**].

Corollary 1.4. *Let $f(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{C}[x]$, where $a_0a_n \neq 0$, and let $k < n$ be a positive integer. If*

$$\binom{n}{k}|a_n| > \sum_{i=k}^{n-1}\binom{i}{k}|a_i| \cdot \max\left\{1, \frac{\sum_{i=k}^{n-1}\binom{i}{k}|a_{n-i}|}{\binom{n}{k}|a_0|}\right\}^{n-i},$$

then $e(f) \leqslant k$.

Corollary 1.5. *Let $f(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{C}[x]$, where $a_0a_n \neq 0$, and let $k < n$ be a positive integer. If*

$$\binom{n}{k}|a_n| > \sum_{i=k}^{n-1}\binom{i}{k}|a_i| \cdot \max\left\{\frac{\binom{k}{k}|a_{n-k}|}{\binom{n}{k}|a_0|}, 1 + \frac{\binom{k+1}{k}|a_{n-k-1}|}{\binom{n}{k}|a_0|}, \ldots, 1 + \frac{\binom{n-1}{k}|a_1|}{\binom{n}{k}|a_0|}\right\}^{n-i},$$

then $e(f) \leqslant k$.

In particular, Theorem 1.1 and Corollaries 1.4 and 1.5 provide the following separability criteria, respectively.

Corollary 1.6. *Let $f(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{C}[x]$, $a_0a_n \neq 0$. If for a positive real δ we have*

$$n|a_0| > \sum_{i=1}^{n-1} i|a_{n-i}|\delta^{i-n} \quad \text{and} \quad n|a_n| > \sum_{i=1}^{n-1} i|a_i|\delta^{n-i}, \tag{1.3}$$

then the polynomial f is separable. The same conclusion holds if we replace one of the inequalities '$>$' in (1.3) by '$\geqslant$'.

Corollary 1.7. *Let $f(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{C}[x]$, where $a_0a_n \neq 0$. If*

$$n|a_n| > \sum_{i=1}^{n-1} i|a_i| \cdot \max\left\{1, \frac{|a_{n-1}| + 2|a_{n-2}| + \cdots + (n-1)|a_1|}{n|a_0|}\right\}^{n-i},$$

then f is a separable polynomial.

Corollary 1.8. *Let $f(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{C}[x]$, where $a_0a_n \neq 0$. If*

$$n|a_n| > \sum_{i=1}^{n-1} i|a_i| \cdot \max\left\{\frac{|a_{n-1}|}{n|a_0|}, 1 + \frac{2|a_{n-2}|}{n|a_0|}, \ldots, 1 + \frac{(n-1)|a_1|}{n|a_0|}\right\}^{n-i},$$

then f is a separable polynomial.

We will also prove a p-adic version of Theorem 1.1. For a rational prime p and an arbitrary non-zero rational number r, we will denote by $\nu_p(r)$ the exponent of p in the prime decomposition of r ($\nu_p(0) = \infty$). With this notation we have the following result.

Theorem 1.9. *Let $f(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{Z}[x]$, $a_0a_n \neq 0$, and let $k < n$ be a positive integer. If for a real δ we have*

$$\left.\begin{aligned} \nu_p\left(\binom{n}{k}a_n\right) - n\delta &< \min_{k \leqslant i \leqslant n-1}\left[\nu_p\left(\binom{i}{k}a_i\right) - \mathrm{i}\delta\right], \\ \nu_p\left(\binom{n}{k}a_0\right) + n\delta &< \min_{k \leqslant i \leqslant n-1}\left[\nu_p\left(\binom{i}{k}a_{n-i}\right) + \mathrm{i}\delta\right], \end{aligned}\right\} \tag{1.4}$$

then $e(f) \leqslant k$. Moreover, one of the inequalities '$<$' in (1.4) can be replaced by '$\leqslant$'.

We note that by letting $\delta = 0$ in Theorem 1.9, one obtains Theorem 3 in [**6**], which is the p-adic version of Theorem A. Before proceeding with the proof of the results, we also note that, like Theorem A, Theorem 1.1 is in some cases best possible, in the sense that the inequality signs '$>$' in (1.2) cannot both be replaced by '$\geqslant$'. To see this, we need to show that there exist polynomials f for which we have equalities in (1.2), and at least one root with multiplicity greater than or equal to $k+1$. We give two examples of such polynomials, obtained from the corresponding examples in [**6**]. First let $n \geqslant 2$, $k = 1$ and take $f(x) = (1 - \delta x)(1 - \delta^{n-1}x^{n-1})$. It is easy to check that in this case we have equalities in (1.2), while f has $1/\delta$ for a double root. Next, let $n \geqslant 2$, $k = n-1$ and take $f(x) = (1 - \delta x)^n$. Here the inequalities (1.2) to be satisfied are $n|a_0| > |a_1|/\delta$ and $n|a_n| > |a_{n-1}|\delta$. In this case we have $n|a_0| = |a_1|/\delta = n$ and $n|a_n| = |a_{n-1}|\delta = n\delta^n$, while $1/\delta$ is a root of f with multiplicity $n = k+1$.

Similarly, Theorem 1.9 is in some cases the best possible, since there exist polynomials f for which we have equalities in (1.4), and at least one root with multiplicity greater than or equal to $k+1$. Let first $n \geqslant 2$, $k=1$, let p be a prime number such that $p \nmid n(n-1)$ and take $f(x) = (1-px)(1-p^{n-1}x^{n-1})$. It is easy to check that in this case we have equalities in (1.4) for $\delta = 1$, while f has a double root at $x = 1/p$. Next, let $n \geqslant 2$, $k = n-1$, let p be a prime number such that $p \nmid n$ and take $f(x) = (1-px)^n$. Here the inequalities (1.4) to be satisfied are $\nu_p(na_n) - n\delta < \nu_p(a_{n-1}) - (n-1)\delta$ and $\nu_p(na_0) + n\delta < \nu_p(a_1) + (n-1)\delta$. In our case we have $\nu_p(na_n) - n\delta = n - n\delta$, $\nu_p(a_{n-1}) - (n-1)\delta = (n-1) - (n-1)\delta$, $\nu_p(na_0) + n\delta = n\delta$ and $\nu_p(a_1) + (n-1)\delta = 1 + (n-1)\delta$, so we have equalities in (1.4) for $\delta = 1$, while $1/p$ is a root of f with multiplicity $n = k+1$.

2. Proof of the results

Proof of Lemma 1.2. Let us assume to the contrary that f has a root θ with multiplicity greater than or equal to $k+1$. Then $f^{(k)}(\theta) = 0$ and, since $a_0 \neq 0$, we must have $\theta \neq 0$. If we assume that $a_i = 0$ for $i = k, \dots, n-1$, then since

$$\frac{f^{(k)}(\theta)}{k!} = \sum_{i=k}^{n} \binom{i}{k} a_i \theta^{i-k},$$

we will obtain $\binom{n}{k} a_n \theta^{n-k} = 0$: a contradiction.

Let us assume now that $a_{n-i} = 0$ for $i = k, \dots, n-1$. Since $1/\theta$ is a root of $\bar{f}(x) = x^n f(1/x)$, the reciprocal of f, and moreover, the multiplicity of $1/\theta$ in the canonical factorization of $\bar{f}$ and the multiplicity of θ in the canonical factorization of f are the same, we must have $\bar{f}^{(k)}(1/\theta) = 0$. Notice now that the coefficients of $\bar{f}$ are precisely the coefficients of f written in reverse order. Therefore, we have

$$\frac{\bar{f}^{(k)}(1/\theta)}{k!} = \sum_{i=k}^{n} \binom{i}{k} \frac{a_{n-i}}{\theta^{i-k}} = 0,$$

which yields $\binom{n}{k} a_0 \theta^{-(n-k)} = 0$, again a contradiction. This completes the proof of the lemma. □

For the proof of Theorem 1.1 we will also need the following lemma.

Lemma 2.1. *Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{C}[x]$, $a_0 a_n \neq 0$, $k < n$ a positive integer, and suppose there exists $\theta \in \mathbb{C}$ such that $(x-\theta)^{k+1} \mid f(x)$. If for a positive real δ we have*

$$\binom{n}{k} |a_n| > \sum_{i=k}^{n-1} \binom{i}{k} |a_i| \delta^{n-i},$$

then $|\theta| < 1/\delta$. Similarly, if

$$\binom{n}{k} |a_n| \geqslant \sum_{i=k}^{n-1} \binom{i}{k} |a_i| \delta^{n-i},$$

then $|\theta| \leqslant 1/\delta$.

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Proof. Let us first note that $\theta \neq 0$ and that, by Lemma 1.2, at least one of the coefficients $a_k, \dots, a_{n-1}$ must be non-zero, since $e(f) \geqslant k+1$. Now, since

$$\frac{f^{(k)}(x)}{k!} = \sum_{i=k}^{n} \binom{i}{k} a_i x^{i-k} \quad \text{and} \quad f^{(k)}(\theta) = 0,$$

we deduce that

$$\begin{aligned} 0 = \left| \sum_{i=k}^{n} \binom{i}{k} a_i \theta^{i-k} \right| &\geqslant \binom{n}{k} |a_n| \cdot |\theta^{n-k}| - \sum_{i=k}^{n-1} \binom{i}{k} |a_i| \cdot |\theta^{i-k}| \\ &= |\theta^{n-k}| \cdot \left[\binom{n}{k} |a_n| - \sum_{i=k}^{n-1} \binom{i}{k} |a_i| \cdot |\theta^{i-n}| \right]. \end{aligned}$$

If we assume now that $|\theta| \geqslant 1/\delta$, we deduce by the above inequalities that

$$0 \geqslant |\theta^{n-k}| \cdot \left[\binom{n}{k} |a_n| - \sum_{i=k}^{n-1} \binom{i}{k} |a_i| \cdot \delta^{n-i} \right],$$

which contradicts our hypothesis that

$$\binom{n}{k} |a_n| > \sum_{i=k}^{n-1} \binom{i}{k} |a_i| \delta^{n-i}.$$

Similarly, if

$$\binom{n}{k} |a_n| \geqslant \sum_{i=k}^{n-1} \binom{i}{k} |a_i| \delta^{n-i}$$

and we assume that $|\theta| > 1/\delta$, we obtain

$$\begin{aligned} 0 &\geqslant |\theta^{n-k}| \cdot \left[\binom{n}{k} |a_n| - \sum_{i=k}^{n-1} \binom{i}{k} |a_i| \cdot |\theta^{i-n}| \right] \\ &> |\theta^{n-k}| \cdot \left[\binom{n}{k} |a_n| - \sum_{i=k}^{n-1} \binom{i}{k} |a_i| \cdot \delta^{n-i} \right], \end{aligned}$$

again a contradiction, and this completes the proof of the lemma. □

Proof of Theorem 1.1. We will first assume that the first inequality sign '>' in (1.2) is replaced by '⩾'. Now let us assume to the contrary that $e(f) \geqslant k+1$, which means that there exists $\theta \in \mathbb{C}$ such that $(x-\theta)^{k+1} \mid f(x)$. Since

$$\binom{n}{k} |a_n| > \sum_{i=k}^{n-1} \binom{i}{k} |a_i| \delta^{n-i},$$

we deduce by Lemma 2.1 that

$$|\theta| < \frac{1}{\delta}. \tag{2.1}$$

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Notice now that $\theta \neq 0$ and that $1/\theta$ is a root of the polynomial $\bar{f}(x) = x^{\deg f} f(1/x)$. Moreover, since $(x-\theta)^{k+1} \mid f(x)$, we deduce that $(x-1/\theta)^{k+1} \mid \bar{f}(x)$.

Next, we write $\bar{f}(x) = b_0 + b_1 x + \cdots + b_n x^n$, say, with $b_i = a_{n-i}$, $i = 0, \ldots, n$, so our condition

$$\binom{n}{k}|a_0| \geqslant \sum_{i=k}^{n-1} \binom{i}{k}|a_{n-i}|\delta^{i-n}$$

may be written as

$$\binom{n}{k}|b_n| \geqslant \sum_{i=k}^{n-1} \binom{i}{k}|b_i|(1/\delta)^{n-i}.$$

By Lemma 2.1 we then obtain $|1/\theta| \leqslant \delta$, which contradicts (2.1).

Similarly, if we replace '$>$' by '$\geqslant$' in the second inequality in (1.2), we will first obtain by Lemma 2.1 that $|\theta| \leqslant 1/\delta$, while the same lemma applied to $\bar{f}$ will yield $|1/\theta| < \delta$, again a contradiction. This completes the proof of the theorem. □

Proof of Corollary 1.3. According to our assumption, for a positive integer $k < n$ we have $|a_1| + |a_2| + \cdots + |a_{n-k}| \neq 0$ and $|a_k| + |a_{k+1}| + \cdots + |a_{n-1}| \neq 0$, so by Descartes's Rule of Signs either of the polynomials

$$f_1(x) = \binom{n}{k}|a_0|x^{n-k} - \sum_{i=k}^{n-1} \binom{i}{k}|a_{n-i}|x^{i-k}$$

and

$$f_2(x) = \binom{n}{k}|a_n|x^{n-k} - \sum_{i=k}^{n-1} \binom{i}{k}|a_i|x^{i-k}$$

has a unique positive root that we denote by θ and ψ, respectively. Recall that we may replace the first inequality sign '$>$' in (1.2) by '$\geqslant$', so according to our notation we need a δ that satisfies the inequalities $f_1(\delta) \geqslant 0$ and $f_2(1/\delta) > 0$. A suitable candidate for δ is therefore θ, since $1/\theta > \psi$, and this completes the proof. □

Proof of Corollary 1.4. We will actually prove a more general result, that depends on a suitable set of parameters. For the proof we will need the following classical result given in [**10**].

Theorem 2.2 (Perron). *If $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ is an arbitrary set of positive real numbers, then all the characteristic roots of the $n \times n$ complex matrix $\mathcal{M} = (a_{ij})$ lie in the disc*

$$|z| \leqslant A_\mu = \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} \frac{\mu_j}{\mu_i}|a_{ij}|. \tag{2.2}$$

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Proof of Theorem 2.2. Indeed, for any characteristic root λ of $\mathcal{M}$ the system of equations

$$\sum_{j=1}^{n} a_{ij}x_j = \lambda x_i, \quad i = 1, 2, \dots, n, \tag{2.3}$$

has a non-trivial solution $(x_1, x_2, \dots, x_n)$. Let us set $x_j = \mu_j y_j$ and denote by y_m the y_j of maximum modulus. By the mth equation of (2.3) we then infer that

$$|\lambda \mu_m y_m| \leqslant \sum_{j=1}^{n} |a_{mj}|\mu_j|y_j| \leqslant \left(\sum_{j=1}^{n} |a_{mj}|\mu_j\right)|y_m|.$$

Hence, $|\lambda| \leqslant A_\mu$, as claimed. □

Let now $F(x) = b_0 + b_1 x + \dots + b_n x^n \in \mathbb{C}[x]$, $b_0 b_n \neq 0$. If we apply the previous result to the companion matrix of the polynomial $\tilde{F}(x) = F(x)/b_n$,

$$\mathcal{M}_{\tilde{F}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -\dfrac{b_0}{b_n} & -\dfrac{b_1}{b_n} & -\dfrac{b_2}{b_n} & \cdots & -\dfrac{b_{n-2}}{b_n} & -\dfrac{b_{n-1}}{b_n} \end{bmatrix},$$

we find that all the roots of F lie in the disc

$$|z| \leqslant \max\left\{\frac{\mu_2}{\mu_1}, \frac{\mu_3}{\mu_2}, \dots, \frac{\mu_n}{\mu_{n-1}}, \sum_{j=1}^{n} \frac{\mu_j}{\mu_n} \cdot \frac{|b_{j-1}|}{|b_n|}\right\}. \tag{2.4}$$

We return now to the proof of Corollary 1.4. Let $\mu_1, \mu_2, \dots, \mu_{n-k}$ be arbitrary positive constants. By (2.4), all the roots of f_1 lie in the disc

$$|z| \leqslant A := \max\left\{\frac{\mu_2}{\mu_1}, \frac{\mu_3}{\mu_2}, \dots, \frac{\mu_{n-k}}{\mu_{n-k-1}}, \sum_{j=1}^{n-k} \frac{\mu_j}{\mu_{n-k}} \cdot \frac{\binom{k+j-1}{k}|a_{n-k-j+1}|}{\binom{n}{k}|a_0|}\right\}.$$

Therefore, if $f_2(1/A) > 0$, we may apply Corollary 1.3 to conclude that $e(f) \leqslant k$. Corollary 1.4 follows now by choosing $\mu_1 = \mu_2 = \dots = \mu_{n-k} = 1$ and observing that

$$\sum_{j=1}^{n-k} \binom{k+j-1}{k} |a_{n-k-j+1}| = \sum_{i=k}^{n-1} \binom{i}{k} |a_{n-i}|.$$

□

Proof of Corollary 1.5. Here too we will prove a more general result that depends on a suitable set of parameters. For the proof we will use the following result of Ballieu [**2**,**9**] on the location of the roots of a complex polynomial.

Theorem 2.3. *Let $F(x) = b_0 + b_1x + \cdots + b_nx^n \in \mathbb{C}[x]$ with $b_0b_n \neq 0$ and let $\mu_0 = 0$ and $\mu_1, \ldots, \mu_n$ be arbitrary positive constants. Then all the roots of F lie in the disc*

$$|z| \leqslant \max_{0\leqslant j\leqslant n-1} \left\{ \frac{\mu_j}{\mu_{j+1}} + \frac{\mu_n}{\mu_{j+1}} \cdot \frac{|b_j|}{|b_n|} \right\}. \tag{2.5}$$

Proof of Theorem 2.3. This result follows immediately by using (2.2) for the transpose of $\mathcal{M}_{\tilde{F}}$. □

Let now $\mu_0 = 0$ and let $\mu_1, \mu_2, \ldots, \mu_{n-k}$ be arbitrary positive constants. By (2.5), all the roots of f_1 lie in the disc

$$|z| \leqslant A := \max_{0\leqslant j\leqslant n-k-1} \left\{ \frac{\mu_j}{\mu_{j+1}} + \frac{\mu_{n-k}}{\mu_{j+1}} \cdot \frac{\binom{k+j}{k}|a_{n-k-j}|}{\binom{n}{k}|a_0|} \right\}.$$

Therefore, if $f_2(1/A) > 0$, we deduce by Corollary 1.3 that $e(f) \leqslant k$. Corollary 1.5 follows now by choosing $\mu_1 = \mu_2 = \cdots = \mu_{n-k} = 1$.

Proof of Theorem 1.9. We will first introduce a non-Archimedean absolute value $|\cdot|_p$ on $\mathbb{Q}$, as follows. For an arbitrary rational number r we define $|r|_p = p^{-\nu_p(r)}$. This absolute value satisfies $|ab|_p = |a|_p|b|_p$ and $|a+b|_p \leqslant \max\{|a|_p, |b|_p\}$, and it can be extended to a number field (see, for example, [**8**, **11**]). Let K be a number field containing all the roots of f and let us fix an extension of our absolute value $|\cdot|_p$ to K, which we will also denote by $|\cdot|_p$.

By the definition of $|\cdot|_p$ we see that the first inequality in (1.4) is equivalent to

$$\left|\binom{n}{k}a_n\right|_p > \max_{k\leqslant i\leqslant n-1} \left|\binom{i}{k}a_i\right|_p \cdot p^{(i-n)\delta}, \tag{2.6}$$

while the second one is equivalent to

$$\left|\binom{n}{k}a_0\right|_p > \max_{k\leqslant i\leqslant n-1} \left|\binom{i}{k}a_{n-i}\right|_p \cdot p^{(n-i)\delta}. \tag{2.7}$$

Let us assume to the contrary that f has a root θ with multiplicity greater than or equal to $k+1$. Since $\theta \neq 0$, $f^{(k)}(\theta) = 0$, and our absolute value also satisfies the triangle inequality, we deduce that

$$\begin{aligned}
0 &= \left|\frac{1}{k!}f^{(k)}(\theta)\right|_p \\
&\geqslant \left|\binom{n}{k}a_n\theta^{n-k}\right|_p - \left|\sum_{i=k}^{n-1}\binom{i}{k}a_i\theta^{i-k}\right|_p \\
&\geqslant \left|\binom{n}{k}a_n\right|_p \cdot |\theta|_p^{n-k} - \max_{k\leqslant i\leqslant n-1}\left|\binom{i}{k}a_i\right|_p \cdot |\theta|_p^{i-k} \\
&= |\theta|_p^{n-k} \cdot \left[\left|\binom{n}{k}a_n\right|_p - \max_{k\leqslant i\leqslant n-1}\left|\binom{i}{k}a_i\right|_p \cdot |\theta|_p^{i-n}\right].
\end{aligned}$$

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These inequalities show that

$$|\theta|_p < p^{\delta}, \tag{2.8}$$

for otherwise, if $|\theta|_p \geqslant p^{\delta}$, we would obtain

$$0 \geqslant |\theta|_p^{n-k} \cdot \left[\left| \binom{n}{k} a_n \right|_p - \max_{k \leqslant i \leqslant n-1} \left| \binom{i}{k} a_i \right|_p \cdot p^{(i-n)\delta} \right],$$

which contradicts (2.6).

On the other hand, $\theta \neq 0$, and hence $1/\theta$ is a root of $\bar{f}(x) = x^n \cdot f(1/x)$ and, moreover, since $f^{(k)}(\theta) = 0$, we must have $\bar{f}^{(k)}(1/\theta) = 0$. Therefore, if we write $\bar{f}(x) = b_0 + b_1 x + \cdots + b_n x^n$, say, with $b_i = a_{n-i}$, $i = 0, \dots, n$, we obtain

$$\begin{aligned} 0 &= \left| \frac{1}{k!} \bar{f}(\theta^{-1}) \right|_p \\ &\geqslant \left| \binom{n}{k} b_n \theta^{k-n} \right|_p - \left| \sum_{i=k}^{n-1} \binom{i}{k} b_i \theta^{k-i} \right|_p \\ &\geqslant \left| \binom{n}{k} b_n \right|_p \cdot |\theta|_p^{k-n} - \max_{k \leqslant i \leqslant n-1} \left| \binom{i}{k} b_i \right|_p \cdot |\theta|_p^{k-i} \\ &= |\theta|_p^{k-n} \cdot \left[\left| \binom{n}{k} a_0 \right|_p - \max_{k \leqslant i \leqslant n-1} \left| \binom{i}{k} a_{n-i} \right|_p \cdot |\theta|_p^{n-i} \right]. \end{aligned}$$

The latter inequalities show us that we must have

$$|\theta|_p > p^{\delta}, \tag{2.9}$$

for otherwise we would obtain

$$0 \geqslant |\theta|_p^{k-n} \cdot \left[\left| \binom{n}{k} a_0 \right|_p - \max_{k \leqslant i \leqslant n-1} \left| \binom{i}{k} a_{n-i} \right|_p \cdot p^{(n-i)\delta} \right],$$

which contradicts (2.7). Finally, since obviously one of the inequalities (2.8) and (2.9) must fail, we conclude that $e(F) \leqslant k$, and this proves the first statement of the theorem.

Let us replace now '$<$' by '$\leqslant$' in the first inequality in (1.4), so instead of (2.6) we have

$$\left| \binom{n}{k} a_n \right|_p \geqslant \max_{k \leqslant i \leqslant n-1} \left| \binom{i}{k} a_i \right|_p \cdot p^{(i-n)\delta}. \tag{2.10}$$

We may assume that at least one of the coefficients $a_k, a_{k+1}, \dots, a_{n-1}$ is non-zero, for otherwise the conclusion would follow by Lemma 1.2. Then we deduce successively

$$\begin{aligned} 0 &\geqslant |\theta|_p^{n-k} \cdot \left[\left| \binom{n}{k} a_n \right|_p - \max_{k \leqslant i \leqslant n-1} \left| \binom{i}{k} a_i \right|_p \cdot |\theta|_p^{i-n} \right] \\ &> |\theta|_p^{n-k} \cdot \left[\left| \binom{n}{k} a_n \right|_p - \max_{k \leqslant i \leqslant n-1} \left| \binom{i}{k} a_i \right|_p \cdot p^{(i-n)\delta} \right] \quad \text{(by (2.9))} \\ &\geqslant 0 \quad \text{(by (2.10))}, \end{aligned}$$

which is a contradiction.

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Similarly, if we replace '$<$' by '$\leqslant$' in the second inequality in (1.4), then instead of (2.7) we have

$$\left|\binom{n}{k}a_0\right|_p \geqslant \max_{k\leqslant i\leqslant n-1}\left|\binom{i}{k}a_{n-i}\right|_p \cdot p^{(n-i)\delta}. \tag{2.11}$$

Assuming this time that at least one of the coefficients $a_1, a_2, \dots, a_{n-k}$ is non-zero, we obtain

$$\begin{aligned}
0 &\geqslant |\theta|_p^{k-n} \cdot \left[\left|\binom{n}{k}a_0\right|_p - \max_{k\leqslant i\leqslant n-1}\left|\binom{i}{k}a_{n-i}\right|_p \cdot |\theta|_p^{n-i}\right] \\
&> |\theta|_p^{k-n} \cdot \left[\left|\binom{n}{k}a_0\right|_p - \max_{k\leqslant i\leqslant n-1}\left|\binom{i}{k}a_{n-i}\right|_p \cdot p^{(n-i)\delta}\right] \quad (\text{by } (2.8)) \\
&\geqslant 0 \quad (\text{by } (2.11)),
\end{aligned}$$

again a contradiction. In either case we deduce that $e(F) \leqslant k$, and this completes the proof of the theorem. □

Remark 2.4. Let $f(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{Z}[x]$, $a_0a_n \neq 0$, and let $k < n$ be a positive integer. Let

$$\gamma = \min_{k\leqslant i\leqslant n-1} \nu_p\left(\binom{i}{k}a_{n-i}\right) - \nu_p\left(\binom{n}{k}a_0\right).$$

If

$$\nu_p\left(\binom{n}{k}a_n\right) - n\cdot\min\{0,\gamma\} < \min_{k\leqslant i\leqslant n-1}\left[\nu_p\left(\binom{i}{k}a_i\right) - i\cdot\min\{0,\gamma\}\right], \tag{2.12}$$

then $e(f) \leqslant k$.

Indeed, if $\gamma \geqslant 0$, the conclusion follows by Theorem 1.9 with $\delta = 0$ and the inequality sign '$<$' replaced by '$\leqslant$' in the second inequality in (1.4).

If $\gamma < 0$, then in view of (2.12) we may take $\delta = \gamma$, which satisfies the first inequality in (1.4). It remains to prove that $\delta = \gamma$ also satisfies the second inequality in (1.4), with the inequality sign '$<$' replaced by '$\leqslant$'. Indeed, since $\gamma < 0$, we have

$$n\gamma - \min_{k\leqslant i\leqslant n-1} i\gamma = \gamma = \min_{k\leqslant i\leqslant n-1}\nu_p\left(\binom{i}{k}a_{n-i}\right) - \nu_p\left(\binom{n}{k}a_0\right),$$

so we deduce that

$$\begin{aligned}
\nu_p\left(\binom{n}{k}a_0\right) + n\gamma &= \min_{k\leqslant i\leqslant n-1}\nu_p\left(\binom{i}{k}a_{n-i}\right) + \min_{k\leqslant i\leqslant n-1} i\gamma \\
&\leqslant \min_{k\leqslant i\leqslant n-1}\left[\nu_p\left(\binom{i}{k}a_{n-i}\right) + i\gamma\right],
\end{aligned}$$

which completes the proof. □

We end by noting that one may obtain similar results by using other estimates for the moduli of the roots of f_1 and f_2 in Corollary 1.3, or by using different values of the parameter δ in the p-adic case.

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