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A REMARK ON BASES IN HARDY SPACES

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ABSTRACT. The Franklin spline system in $[0, 1]$ has been generalized by Strömberg to a system in $\mathbb{R}^n$ which is an unconditional basis in $H^p(\mathbb{R}^n)$ for $p > n/(n+m+1)$. Here the natural number m is the order of the system. For some of these values of p, it was known that the H^p quasi-norm is equivalent to a certain expression containing the coefficients of the function with respect to this basis. We prove this equivalence for all $p > n/(n+m+1)$.

1. **Introduction.** In [4] J.-O. Strömberg constructs on orthonormal basis $(f_\nu^\omega)_{\nu,\omega}$ in $L^2(\mathbb{R}^n)$ which generalizes the Franklin system in $L^2[0, 1]$. Here $\nu = (j, k) = (j, k_1, \ldots, k_n)$ ranges over $\mathbb{Z}^{n+1}$ and ω over a finite set. Each f_ν^ω is a tensor product of one-dimensional spline functions of order m and takes its largest values near the cube

$$Q_\nu = \{x : 2^{-j}k_i \le x_i \le 2^{-j}(k_i + 1), \qquad i = 1, \ldots, n\},$$

whose characteristic function is denoted by χ_ν.

Strömberg proves that this system is an unconditional basis in the Hardy space $H^p(\mathbb{R}^n)$, $p > n/(n+m+1)$, and compares the quasinorm of f in H^p to its coefficients $c_\nu^\omega = (f, f_\nu^\omega)$. For all $p > n/(n+m+1)$ he shows that

$$\left\| \left(\sum_{\nu,\omega} |c_\nu^\omega|^2 \, 2^{nj} \chi_\nu \right)^{1/2} \right\|_{L^p} \le C \, \|f\|_{H^p}. \tag{1.1}$$

Here and in the sequel, C denotes various constants. At least in one dimension, (1.1) holds only for these values of p, see [3]. The converse inequality

$$\|f\|_{H^p} \le C \left\| \left(\sum_{\nu,\omega} |c_\nu^\omega|^2 \, 2^{nj} \chi_\nu \right)^{1/2} \right\|_{L^p} \tag{1.2}$$

is proved only for $p > n/(n/2 + m + 2)$. We shall prove (1.2) for all $p > n/(n+m+1)$, by modifying Strömberg's proof. This answers a question asked at the end of [4]. It is enough to prove the following.

THEOREM. *Let $n/(n+m+1) < p \le 1$. There is a constant $C = C(n, m, p)$ such that for any finite set of numbers c_ν^ω*

$$\left\| \sum c_\nu^\omega f_\nu^\omega \right\|_{H^p} \le C \left\| \left(\sum_{\nu,\omega} |c_\nu^\omega|^2 \, 2^{nj} \chi_\nu \right)^{1/2} \right\|_{L^p}. \tag{1.3}$$

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We shall need some properties of f_ν^ω. Strömberg's definition reads

$$f_\nu^\omega(x) = 2^{nj/2}\tau^\omega(2^j x - k) \tag{1.4}$$

with $\nu = (j, k) \in \mathbb{Z} \times \mathbb{Z}^n$, where τ^ω is a tensor product

$$\tau^\omega(x) = \prod_{i=1}^n \omega_i(x_i).$$

Here each ω_i is either of two fixed spline functions τ and ρ in $\mathbb{R}$, and at least one is τ. These two functions are in $C^m(\mathbb{R})$ and they are polynomials in the complementary intervals of a countable discrete set. Moreover,

$$|D^k\tau(t)| \le Cr^{|t|}, \quad k = 0, \dots, m+1, \tag{1.5}$$

and similarly for ρ, for some $r < 1$. The moments $\int \tau(t)t^\alpha\, dt$ vanish for $\alpha = 0, \dots, m+1$.

2. **Proof of the theorem.** Choose a radial nonzero $\psi \in C^\infty(\mathbb{R}^n)$ with support in $|x| \le 1$ and vanishing moments up to order m. Writing $\psi_s(x) = s^{-n}\psi(x/s)$ and $G(x, s) = \psi_s * f(x)$ for $f \in \mathcal{S}'$, we define a Lusin function

$$A_\gamma G(z) = \left(\iint_{|x-z|<\gamma s} |G(x, s)|^2 \frac{dx\, ds}{s^{n+1}} \right)^{1/2}.$$

Here $z, x \in \mathbb{R}^n$ and $s > 0$. Then

$$\|A_\gamma G\|_{L^p} \sim \|f\|_{H^p}, \tag{2.1}$$

in the sense that the quotient between these two quantities stays away from 0 and ∞, see [1, Theorems 6.6 and 6.9].

Let a be an atom in H^p with vanishing moments up to order m. The arguments on p. 492 of [4] show that $\sum \pm(a, f_\nu^\omega)f_\nu^\omega$ belongs to H^p with quasi-norm at most C, for any sign combination. Here molecules could also be used. Hence, the operator which maps $f \in H^p$ onto $\sum \pm(f, f_\nu^\omega)f_\nu^\omega$ is bounded in H^p, uniformly over all sign choices, and the H^p quasi-norms of all these sums are comparable.

We consider the probability measure on the set of all sign choices making the signs independent and each sign of probability 1/2. Let E denote the corresponding expectation. We conclude from (2.1) with $\gamma = 1$ that

$$\left\|\sum c_\nu^\omega f_\nu^\omega\right\|_{H^p}^p \sim E\left\|\sum \pm c_\nu^\omega f_\nu^\omega\right\|_{H^p}^p \sim E\int dz \left(\iint_{|x-z|<s} \left|\psi_s * \sum \pm c_\nu^\omega f_\nu^\omega(x)\right|^2 \frac{dx\, ds}{s^{n+1}} \right)^{p/2}$$

$$\le \int dz \left(\iint_{|x-z|<s} E\left|\sum \pm c_\nu^\omega \psi_s * f_\nu^\omega(x)\right|^2 \frac{dx\, ds}{s^{n+1}} \right)^{p/2},$$

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by Hölder's inequality. But $E\,|\sum \pm a_j|^2 = \sum |a_j|^2$, so

$$\left\|\sum c_\nu^\omega f_\nu^\omega\right\|_{H^p}^p \le C \int dz\Bigl(\iint_{|x-z|<s} \sum |c_\nu^\omega|^2\, |\psi_s * f_\nu^\omega(x)|^2 \frac{dx\,ds}{s^{n+1}}\Bigr)^{p/2}. \tag{2.2}$$

We must thus estimate $\psi_s * f_\nu^\omega$. Our inequalities are similar to those of [4, p. 488–9].

LEMMA. *For each $N>0$ there exists $C=C(N)$ such that for $x\in\mathbb{R}^n$*

$$|\psi_s * f_\nu^\omega(x)| \le C2^{nj/2}(2^js)^{m+1}(1+2^j\,|x-2^{-j}k|)^{-N},\ s\le 2^{-j}$$
$$\le C2^{nj/2}(2^js)^{-n-m-2}(1+s^{-1}\,|x-2^{-j}k|)^{-N},\ s>2^{-j}.$$

Proof. We let $t>0$ and estimate $\psi_s * \tau_t^\omega$. Denote by P_x the Taylor polynomial of τ^ω at x of degree m. For $s\le t$ we use the vanishing moments of ψ to get

$$|\psi_s * \tau_t^\omega(x)| = \left|\int_{|y|\le s} \psi_s(y)t^{-n}\Bigl(\tau^\omega\Bigl(\frac{x-y}{t}\Bigr) - P_{x/t}\Bigl(-\frac{y}{t}\Bigr)\Bigr)\,dy\right|.$$

Because of (1.5), the parenthesis here is dominated by $C(|y|/t)^{m+1}(1+t^{-1}\,|x|)^{-N}$. Thus,

$$|\psi_s * \tau_t^\omega(x)| \le Ct^{-n}\Bigl(\frac{s}{t}\Bigr)^{m+1}\Bigl(1+\frac{|x|}{t}\Bigr)^{-N}.$$

For $s>t$, we use instead the Taylor polynomial Q_x of ψ at x of degree $m+1$. Since $\tau^\omega(x)$ contains at least one factor $\omega_i(x_i)=\tau(x_i)$, its moments of order up to $m+1$ vanish. Hence,

$$\psi_s * \tau_t(x) = s^{-n}\int \Bigl(\psi\Bigl(\frac{x-y}{s}\Bigr) - Q_{x/s}\Bigl(-\frac{y}{s}\Bigr)\Bigr)\tau_t^\omega(y)\,dy.$$

If $|x|\le 2s$, this is easily estimated by $Cs^{-n}(t/s)^{m+2}$. If $|x|>2s$, then $Q_{x/s}=0$ and we may assume $|y|>|x|/2$ since otherwise $|x-y|>s$ and the integrand vanishes. Then (1.5) implies

$$|\psi_s * \tau_t^\omega(x)| \le Cs^{-n}\int_{|y|>|x|/2} |\tau_t^\omega(y)|\,dy \le Cs^{-n}\Bigl(\frac{|x|}{t}\Bigr)^{-N}.$$

So for $s>t$ and $N>m+2$,

$$|\psi_s * \tau_t^\omega(x)| \le Cs^{-n}\Bigl(\frac{t}{s}\Bigr)^{m+2}\Bigl(1+\frac{|x|}{s}\Bigr)^{-N}.$$

The estimates obtained imply the lemma, as seen from (1.4).

Because of this lemma, (2.2) implies

$$\left\|\sum c_\nu^\omega f_\nu^\omega\right\|_{H^p}^p \le C\int dz\Bigl(\sum |c_\nu^\omega|^2\, 2^{nj}(I_1+I_2)\Bigr)^{p/2},$$

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where

$$I_1 = \iint_{|x-z|<s\le 2^{-j}} (2^j s)^{2(m+1)}\left(1+\frac{|x-2^{-j}k|}{2^{-j}}\right)^{-2N}\frac{dx\,ds}{s^{n+1}}$$

and

$$I_2 = \iint_{\substack{|x-z|<s\\ s>2^{-j}}} (2^j s)^{-2(n+m+2)}\left(1+\frac{|x-2^{-j}k|}{s}\right)^{-2N}\frac{dx\,ds}{s^{n+1}}.$$

It is easily seen that

$$I_1 \le C\int_0^{2^{-j}} (2^j s)^{2(m+1)}\frac{ds}{s}\left(1+\frac{|z-2^{-j}k|}{2^{-j}}\right)^{-2N} \le C\left(1+\frac{|z-2^{-j}k|}{2^{-j}}\right)^{-2N}$$

and

$$I_2 \le C\int_{2^{-j}}^{\infty} (2^j s)^{-2(n+m+2)}\left(1+\frac{|z-2^{-j}k|}{s}\right)^{-2N}\frac{ds}{s} \le C\left(1+\frac{|z-2^{-j}k|}{2^{-j}}\right)^{-2(n+m+2)},$$

if N is large. Thus,

$$\left\|\sum c_\nu^\omega f_\nu^\omega\right\|_{H^p}^p \le C\int dz\left(\sum |c_\nu^\omega|^2\, 2^{nj}\left(1+\frac{|z-2^{-j}k|}{2^{-j}}\right)^{-2(n+m+2)}\right)^{p/2}. \tag{2.3}$$

Now define a function F in $\mathbb{R}_+^{n+1}$ as in [2, p. 118], setting $F = c_\nu^\omega 2^{nj/2}$ on $\frac{1}{2}Q_\nu \times [2^{-j}, 2^{-j+1}]$ for each ν and $F=0$ elsewhere. Here $\frac{1}{2}$ of course means concentric scaling of the cube in $\mathbb{R}^{n+1}$. Then (2.3) implies

$$\left\|\sum c_\nu^\omega f_\nu^\omega\right\|_{H^p} \le C\,\|g_\lambda^*(F)\|_{L^p}, \qquad \lambda = n+m+2,$$

where

$$g_\lambda^*(F)(z) = \left(\iint\left(1+\frac{|x-z|}{s}\right)^{-2N}|F(x,s)|^2\frac{dx\,ds}{s^{n+1}}\right)^{1/2}$$

is a Littlewood-Paley function. Further,

$$\|g_\lambda^*(F)\|_{L^p} \le C\,\|A_\gamma F\|_{L^p}, \qquad \gamma>0,$$

by [1, Theorem 3.5], since $\lambda > n/p$ here. But $A_\gamma F$ is dominated by $C(\sum |c_\nu^\omega|^2\, 2^{nj}\chi_\nu)^{1/2}$ if γ is small enough, and (1.3) and the theorem follow.

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