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Generalizing Hopf's Boundary Point Lemma

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Abstract. We give a Hopf boundary point lemma for weak solutions of linear divergence form uniformly elliptic equations, with Hölder continuous top-order coefficients and lower-order coefficients in a Morrey space.

1 Introduction

We illustrate how the Hopf boundary point lemma can be proved for divergence form equations, given sufficient regularity of the coefficients. Here, we show the case when the top-order coefficients are Hölder continuous, while the lower-order terms are in a Morrey space (see Definition 2.1).

The Hopf boundary point lemma states that if $u \in C(B_1(0)) \cap C^2(\overline{B_1(0)})$ satisfies a second-order linear equation

$$\sum_{i,j=1}^{n} a^{ij} D_i D_j u + \sum_{i=1}^{n} c^i(x) D_i u + du = 0$$

over $B_1(0)$, for functions $a^{ij} = a^{ji}, c^i, d \in L^\infty(B_1(0))$ for $i,j \in \{1,\ldots,n\}$ with $\{a^{ij}\}_{i,j=1}^n$ uniformly elliptic over $B_1(0)$ with respect to some $\lambda \in (0,\infty)$ (see Definition 2.4), and if $u(x) > u(-e_n) = 0$ for all $x \in B_1(0)$, then

$$\liminf_{h \searrow 0} \frac{u((h-1)e_n)}{h} > 0. \tag{1.1}$$

See the proof given by Hopf in [10] as well as [8, Lemma 3.4].

It is useful to have the Hopf boundary point lemma for divergence form equations. We consider $u \in C(\overline{B_1(0)}) \cap W^{1,2}(B_1(0))$ a weak solution over $B_1(0)$ of the equation

$$\sum_{i,j=1}^{n} D_i(a^{ij} D_j u + b^i u) + \sum_{i=1}^{n} c^i D_i u + du = 0 \tag{1.2}$$

for functions $a^{ij}, c^i \in L^2(B_1(0))$ and $b^i, d \in L^1(B_1(0))$ for each $i,j \in \{1,\ldots,n\}$ (see Definition 2.3). Assuming again that $u(x) > u(-e_n) = 0$ for all $x \in B_1(0)$, the aim is to show that (1.1) holds.

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The most recent result is given by [20, Theorem 1.1], which shows that (1.1) holds if

$$a^{ij} = a^{ji} \in C^{0,\alpha}(\overline{B_1(0)})$$

for some $\alpha \in (0,1)$ for $i, j \in \{1,\ldots,n\}$ with $\{a^{ij}\}_{i,j=1}^n$ uniformly elliptic with respect to some $\lambda \in (0,\infty)$ (see Definition 2.4), and $b^i = 0$ while $c^i, d \in L^\infty(B_1(0))$ for each $i \in \{1,\ldots,n\}$. We also refer the reader to [20], which discusses previous generalizations of the Hopf boundary point lemma, and gives examples showing the assumption $a^{ij} \in C^{0,\alpha}(\overline{B_1(0)})$ for each $i, j \in \{1,\ldots,n\}$ cannot be relaxed.

Here, we prove in Theorem 4.1 that (1.1) holds under the more general assumption that the coefficients in (1.2) satisfy for each $i, j \in \{1,\ldots,n\}$,

$$a^{ij}, b^i \in C^{0,\alpha}(\overline{B_1(0)}), \quad c^i \in L^q(B_1(0)), \quad d \in L^{\frac{q}{2}}(B_1(0)) \cap L^{1,\alpha}(B_1(0))$$

for some $q > n$ and $\alpha \in (0,1)$; see Remark 4.2(i). We also assume $a^{ij}(-e_n) = a^{ji}(-e_n)$ for each $i, j \in \{1,\ldots,n\}$ with $\{a^{ij}\}_{i,j=1}^n$ uniformly elliptic over $B_1(0)$ with respect to some $\lambda \in (0,\infty)$. Additionally, we assume $\{b^i\}_{i=1}^n, d$ are weakly non-positive over $B_1(0)$ (see Definition 2.5).

The space $L^{1,\alpha}(B_1(0))$ denotes a Morrey space (see Definition 2.1). Morrey spaces were introduced in [19] to study the existence and regularity of solutions to elliptic systems. Consequently, to prove Theorem 4.1 we must use the $C^{1,\alpha}$ estimate of [19, Theorem 5.5.5′(b)] stated here for convenience as Lemma 3.1.

Since their introduction, Morrey spaces have been studied in and outside the study of partial differential equations. Recent work has been done in the study of elliptic and parabolic partial differential equations involving data in the $L^{1,\alpha}$ Morrey space. We refer to the seminal work in this direction given by [18], which uses Morrey spaces to prove regularity results for solutions to non-linear divergence-form elliptic equations having inhomogeneous term a measure. To see further recent work resulting from and related to [18] using $L^{1,\alpha}$ Morrey spaces to study elliptic and parabolic equations in various settings, we refer the reader to [1–7, 11–17].

Our underlying goal is to illustrate how the Hopf boundary point lemma can be shown in other settings for divergence form equations. To this end, the proof of Theorem 4.1 is given in five steps that demonstrate the necessary theoretical ingredients. The structure of the proof is taken from the proof of [9, Lemma 10.1], which shows one generalization of the Hopf boundary point lemma to divergence form equations.

We only assume working knowledge of real analysis and ready access to the reference [8]. Otherwise, the crucial estimate Theorem 5.5.5′(b) is carefully stated in the current setting as Lemma 3.1. In Section 2, we begin by stating our basic definitions and some preliminary calculations needed in Section 3 to prove the necessary existence result Lemma 3.3. We also state the weak maximum principle needed, Lemma 3.2. In Section 4 we prove the Hopf boundary point lemma, Theorem 4.1.

2 Preliminaries

We will work in $\mathbb{R}^n$ with $n \geq 2$. We denote the volume of the open unit ball $B_1(0) \subset \mathbb{R}^n$ by $\omega_n = \int_{B_1(0)} \mathrm{d}x$. Standard notation for the various spaces of functions shall be used; in particular, $C_c^1(\mathcal{U},[0,\infty))$ will denote the set of non-negative continuously differentiable functions with compact support in an open set $\mathcal{U}\subseteq\mathbb{R}^n$.

We begin by giving the definition of a family of Morrey spaces, to which we will relax the assumptions on the lower-order terms given in [20].

Definition 2.1 Suppose $\alpha\in(0,1)$ and $\mathcal{U}\subseteq\mathbb{R}^n$ is an open set. We say $d\in L^{1,\alpha}(\mathcal{U})$ if $d\in L^1(\mathcal{U})$ with finite $L^{1,\alpha}(\mathcal{U})$ norm, defined by

$$\|d\|_{L^{1,\alpha}(\mathcal{U})}:=\sup_{x\in\mathbb{R}^n,\rho\in(0,\infty)}\frac{1}{\rho^{n-1+\alpha}}\int_{\mathcal{U}\cap B_\rho(x)}|d(y)|\,\mathrm{d}y\,.$$

Remark 2.2 If $q>n$, $\mathcal{U}\subset\mathbb{R}^n$ is a bounded open set, and $c\in L^q(\mathcal{U})$, then $c\in L^{1,\alpha}(\mathcal{U})$ for $\alpha=1-\frac{n}{q}\in(0,1)$, with

$$\|c\|_{L^{1,\alpha}(\mathcal{U})}\le\Big(\int_{\mathcal{U}}\mathrm{d}x\Big)^{1-\frac{1}{q}}\|c\|_{L^q(\mathcal{U})}.$$

Next, we state what it means for u to be a weak supersolution (resp. solution, subsolution) to a linear divergence form equation. The assumptions on the coefficients are to ensure integrability.

Definition 2.3 Let $\mathcal{U}\subset\mathbb{R}^n$ be an open set, and suppose $a^{ij},c^i\in L^2(\mathcal{U})$, $b^i,d,g,f^i\in L^1(\mathcal{U})$ for each $i,j\in\{1,\dots,n\}$. We say $u\in L^\infty(\mathcal{U})\cap W^{1,2}(\mathcal{U})$ *is a weak solution over* $\mathcal{U}$ *of the equation*

$$\sum_{i,j=1}^n D_i(a^{ij}D_ju+b^iu)+\sum_{i=1}^n c^iD_iu+du\le g+\sum_{i=1}^n D_if^i$$

(resp. $\ge,=$) if for all $\zeta\in C_c^1(\mathcal{U};[0,\infty))$

$$\int\sum_{i,j=1}^n a^{ij}D_juD_i\zeta+\sum_{i=1}^n\big(b^iuD_i\zeta-c^i(D_iu)\zeta\big)-du\zeta\,\mathrm{d}x\ge\int-g\zeta+\sum_{i=1}^n f^iD_i\zeta\,\mathrm{d}x$$

(resp. $\le,=$).

The next two definitions hold throughout.

Definition 2.4 Let $\lambda\in(0,\infty)$, $\mathcal{U}\subseteq\mathbb{R}^n$, and suppose we have functions $a^{ij}:\mathcal{U}\to\mathbb{R}$ for each $i,j\in\{1,\dots,n\}$. We say $\{a^{ij}\}_{i,j=1}^n$ are *uniformly elliptic over* $\mathcal{U}$ *with respect to* λ if

$$\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j\ge\lambda|\xi|^2\text{ for each }x\in\mathcal{U}\text{ and }\xi\in\mathbb{R}^n.$$

Definition 2.5 Let $\mathcal{U}\subseteq\mathbb{R}^n$ be an open set, and suppose $b^i,d\in L^1(\mathcal{U})$ for each $i\in\{1,\dots,n\}$. We say $\{b^i\}_{i=1}^n,d$ are *weakly non-positive over* $\mathcal{U}$ if

$$\int d\zeta-\sum_{i=1}^n b^iD_i\zeta\,\mathrm{d}x\le0$$

for each $\zeta\in C_c^1(\mathcal{U},[0,\infty))$.

Proving Theorem 4.1 will require the existence result Lemma 3.1, which in turn we prove using a well-known existence result given by [8, Theorem 8.34].We must thus discuss mollification and Morrey spaces.

Definition 2.6 Let $\Omega = B_1(0) \setminus \overline{B_{\frac{1}{2}}(0)}$, and let $\nu \in C_c^\infty(B_1(0);[0,\infty))$ be a standard mollifier. For $\delta \in (0, \frac{1}{4})$ let $\nu_\delta(x) = \frac{1}{\delta^n}\nu(\frac{x}{\delta})$ and define $\gamma_\delta: \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ by

$$\gamma_\delta(x) = \big((1-4\delta)|x| + 3\delta \big) \frac{x}{|x|} = (1-4\delta)x + 3\delta \frac{x}{|x|}.$$

Using these functions, we make the following definitions.

(i) Given $d \in L^1(\Omega)$, we extend $d(y) = 0$ for $y \in \mathbb{R}^n \setminus \Omega$ and define the usual convolution $d * \nu_\delta: \mathbb{R}^n \to \mathbb{R}$. We also define the weighted convolution $d \circledast \nu_\delta: \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ by $d \circledast \nu_\delta = J\gamma_\delta((d * \nu_\delta) \circ \gamma_\delta)$, where $J\gamma_\delta = |\det(D\gamma_\delta)|$.

(i) Given $\{b^i\}_{i=1}^n \subset C^{0,\alpha}(\overline{\Omega})$, define $b^i \star \nu_\delta: \overline{\Omega} \to \mathbb{R}$ for $i \in \{1,\dots,n\}$ by

$$b^i \star \nu_\delta = \sum_{j=1}^n \Big(\big(D_j(e_i \cdot \gamma_\delta^{-1}) \big) \circ \gamma_\delta \Big) \cdot (b^j \circledast \nu_\delta).$$

We will use these convolutions to prove the existence result, Lemma 3.3. For this, we need the following calculations.

Lemma 2.7 *Denote $\Omega = B_1(0) \setminus \overline{B_{1/2}(0)}$. Suppose $b^i \in C^{0,\alpha}(\overline{\Omega})$ for $i \in \{1,\dots,n\}$ and $d \in L^{1,\alpha}(\Omega)$ with $\alpha \in (0,1)$.*

(i) *For $\delta \in (0, \frac{1}{8})$, the convolutions satisfy*
- *$d * \nu_\delta \in C^\infty(\mathbb{R}^n)$, $\|d * \nu_\delta\|_{L^{1,\alpha}(\Omega)} \le \|d\|_{L^{1,\alpha}(\Omega)}$, and $d * \nu_\delta \to d$ in $L^1(\Omega)$ as $\delta \searrow 0$.*
- *$d \circledast \nu_\delta \in C^\infty(\mathbb{R}^n \setminus \{0\})$, $\|d \circledast \nu_\delta\|_{L^{1,\alpha}(\Omega)} \le 2^n \|d\|_{L^{1,\alpha}(\Omega)}$, and $d \circledast \nu_\delta \to d$ in $L^1(\Omega)$ as $\delta \searrow 0$.*
- *There exists $C_{2.7} = C_{2.7}(n) \in (0,\infty)$ so that*

$$b^i \star \nu_\delta \in C^\infty(\overline{\Omega}), \quad \|b^i \star \nu_\delta\|_{C^{0,\alpha}(\overline{\Omega})} \le C_{2.7} \sum_{j=1}^n \|b^j\|_{C^{0,\alpha}(\overline{\Omega})},$$

and $b^i \star \nu_\delta \to b^i$ as $\delta \searrow 0$ in $L^1(\Omega)$ for each $i \in \{1,\dots,n\}$.

(ii) *If $\{b^i\}_{i=1}^n, d$ are weakly non-positive over Ω, then $\{b^i \star \nu_\delta\}_{i=1}^n, d \circledast \nu_\delta$ are weakly non-positive over Ω.*

Proof We leave some details to the reader, which follow from standard real analysis.

First, consider (i). We discuss each item separately.

- Since $\nu \in C_c^\infty(B_1(0);[0,\infty))$ is a standard mollifier, $d*\nu_\delta \in C^\infty(\mathbb{R}^n)$ and $d*\nu_\delta \to d$ in $L^1(\Omega)$ as $\delta \searrow 0$ are well-known real analysis facts.

  Next, fix $x \in \mathbb{R}^n$ and $\rho \in (0,\infty)$. Using the definition of the convolution, Fubini's theorem, a change of variables and the extension $d(y) = 0$ for $y \in \mathbb{R}^n \setminus \Omega$, the definition of the $L^{1,\alpha}$ norm, and $\int \nu_\delta(z)\,\mathrm{dz} = 1$, since $\nu \in C_c^\infty(B_1(0);[0,\infty))$ is a

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standard mollifier, we compute

$$\begin{aligned}\frac{1}{\rho^{n-1+\alpha}}\int_{\Omega\cap B_\rho(x)}|d*\nu_\delta(y)|\,\mathrm{dy} &\le \frac{1}{\rho^{n-1+\alpha}}\int_{\Omega\cap B_\rho(x)}\int|d(y-z)|\nu_\delta(z)\,\mathrm{dz}\,\mathrm{dy}\\ &\le \int \frac{1}{\rho^{n-1+\alpha}}\int_{\Omega\cap B_\rho(x)}|d(y-z)|\,\mathrm{dy}\,\nu_\delta(z)\,\mathrm{dz}\\ &\le \int \frac{1}{\rho^{n-1+\alpha}}\int_{\Omega\cap B_\rho(x-z)}|d(y)|\,\mathrm{dy}\,\nu_\delta(z)\,\mathrm{dz}\\ &\le \int \|d\|_{L^{1,\alpha}(\Omega)}\nu_\delta(z)\,\mathrm{dz} = \|d\|_{L^{1,\alpha}(\Omega)}.\end{aligned}$$

This verifies $\|d*\nu_\delta\|_{L^{1,\alpha}(\Omega)} \le \|d\|_{L^{1,\alpha}(\Omega)}$.

- Observe that $\gamma_\delta \in C^\infty(\mathbb{R}^n\smallsetminus\{0\})$ and γ_δ converges smoothly over $\overline{\Omega}$ to the identity as $\delta \searrow 0$. Using these facts, Definition 2.6(i), and the previous item, we can show $d\circledast\nu_\delta \in C^\infty(\mathbb{R}^n\smallsetminus\{0\})$ and $d\circledast\nu_\delta \to d$ in $L^1(\Omega)$ as $\delta\searrow 0$.

  Fix any $x\in\mathbb{R}^n$ and $\rho\in(0,\infty)$. By Definition 2.6 we can check that

  $$\gamma_\delta(\Omega) = B_{1-\delta}(0)\smallsetminus\overline{B_{\frac{1}{2}}(0)} \quad\text{and}\quad \gamma_\delta(B_\rho(x))\subseteq B_{(1+8\delta)\rho}(\gamma_\delta(z)).$$

  Using this together with Definition 2.6(i), γ_δ as a change of variables, $\delta\in(0,\frac{1}{8})$, $\alpha\in(0,1)$, and the previous item, we compute

  $$\begin{aligned}\frac{1}{\rho^{n-1+\alpha}}\int_{\Omega\cap B_\rho(x)}|d\circledast\nu_\delta(y)|\,\mathrm{dy} &= \frac{1}{\rho^{n-1+\alpha}}\int_{\gamma_\delta(\Omega\cap B_\rho(x))}|d*\nu_\delta(y)|\,\mathrm{dy}\\ &\le \frac{1}{\rho^{n-1+\alpha}}\int_{\Omega\cap B_{2\rho}(\gamma_\delta(x))}|d*\nu_\delta(y)|\,\mathrm{dy}\\ &\le 2^{n-1+\alpha}\|d\|_{L^{1,\alpha}(\Omega)} \le 2^n\|d\|_{L^{1,\alpha}(\Omega)}.\end{aligned}$$

  We conclude that $\|d\circledast\nu_\delta\|_{L^{1,\alpha}(\Omega)}\le 2^n\|d\|_{L^{1,\alpha}(\Omega)}$.

- Observe that $\gamma_\delta\colon\overline{\Omega}\to(\overline{B_{1-\delta}(0)}\smallsetminus B_{\frac{1}{2}+\delta}(0))$ is invertible, with

  $$\gamma_\delta^{-1}\in C^\infty\big(\overline{B_{1-\delta}(0)}\smallsetminus B_{\frac{1}{2}+\delta}(0);\overline{\Omega}\big);$$

  since γ_δ converges smoothly over $\overline{\Omega}$ to the identity as $\delta\searrow 0$, we conclude that

  $$\big((D_j(e_i\cdot\gamma_\delta^{-1}))\circ\gamma_\delta\big) \longrightarrow \begin{cases}1 & \text{if } j=i,\\ 0 & \text{if } j\ne i,\end{cases}$$

  uniformly over $\overline{\Omega}$ as $\delta\searrow 0$ for each $i,j\in\{1,\dots,n\}$. Using these facts, together with Definition 2.6(ii) and the previous item applied to $b^j\circledast\nu_\delta$ for each $j\in\{1,\dots,n\}$, we can show $b^i\star\nu_\delta\in C^\infty(\overline{\Omega})$ and $b^i\star\nu_\delta\to b^i$ in $L^1(\Omega)$ as $\delta\searrow 0$ for each $i\in\{1,\dots,n\}$.

  Next, using the definition of the $C^{0,\alpha}$ norm and Definition 2.6(ii), we compute for each $i\in\{1,\dots,n\}$,

  $$\|b^i\star\nu_\delta\|_{C^{0,\alpha}(\overline{\Omega})}\le \sum_{j=1}^n\|(D_j(e_i\cdot\gamma_\delta^{-1}))\circ\gamma_\delta\|_{C^{0,\alpha}(\overline{\Omega})}\|J\gamma_\delta\|_{C^{0,\alpha}(\overline{\Omega})}\|(b^j*\nu_\delta)\circ\gamma_\delta\|_{C^{0,\alpha}(\overline{\Omega})}.$$

We also compute, again using the definition of the $C^{0,\alpha}$ norm,

$$\|(b^j * \nu_\delta)\circ\gamma_\delta\|_{C^{0,\alpha}(\overline{\Omega})} \leq \max\Big\{1, \sup_{x,y\in\overline{\Omega},x\neq y}\frac{|\gamma_\delta(x)-\gamma_\delta(y)|^\alpha}{|x-y|^\alpha}\Big\}\|b^j * \nu_\delta\|_{C^{0,\alpha}(\overline{B_{1-\delta}(0)}\setminus B_{\frac{1}{2}+\delta}(0))}$$

for each $j\in\{1,\dots,n\}$. Since $\gamma_\delta(x)=(1-4\delta)x+3\delta\frac{x}{|x|}$ for $x\in\mathbb{R}^n\setminus\{0\}$, we can find $C_{2.7}=C_{2.7}(n)\in(0,\infty)$ so that for each $\delta\in(0,\frac{1}{8})$, we have

$$\|(D_j(e_i\cdot\gamma_\delta^{-1}))\circ\gamma_\delta\|_{C^{0,\alpha}(\overline{\Omega})}\|J\gamma_\delta\|_{C^{0,\alpha}(\overline{\Omega})}\max\Big\{1, \sup_{x,y\in\overline{\Omega},x\neq y}\frac{|\gamma_\delta(x)-\gamma_\delta(y)|^\alpha}{|x-y|^\alpha}\Big\} \leq C_{2.7}$$

for each $i,j\in\{1,\dots,n\}$. These three calculations taken together imply

$$\|b^i \star \nu_\delta\|_{C^{0,\alpha}(\overline{\Omega})}\leq C_{2.7}\sum_{j=1}^n\|b^j * \nu_\delta\|_{C^{0,\alpha}(\overline{B_{1-\delta}(0)}\setminus B_{\frac{1}{2}+\delta}(0))}$$

for each $i\in\{1,\dots,n\}$. To conclude $\|b^i\star\nu_\delta\|_{C^{0,\alpha}(\overline{\Omega})}\leq C_{2.7}\sum_{j=1}^n\|b^j\|_{C^{0,\alpha}(\overline{\Omega})}$ as needed, it therefore suffices to verify

$$\|b^j * \nu_\delta\|_{C^{0,\alpha}(\overline{B_{1-\delta}(0)}\setminus B_{\frac{1}{2}+\delta}(0))}\leq\|b^j\|_{C^{0,\alpha}(\overline{\Omega})}.$$

This follows from the fact that $\nu\in C_c^\infty(B_1(0);[0,\infty))$ is a standard mollifier and the definitions of the convolution and the $C^{0,\alpha}$ norm. For example, given $x,y\in\overline{B_{1-\delta}(0)}\setminus B_{\frac{1}{2}+\delta}(0)$, we can compute

$$\begin{aligned}|b^j * \nu_\delta(x)-b^j * \nu_\delta(y)| &\leq \int|b^j(x+\delta z)-b^j(y+\delta z)|\nu(z)\,\mathrm{d}z\\ &\leq |x-y|^\alpha\|b^j\|_{C^{1,\alpha}(\overline{\Omega})}.\end{aligned}$$

We leave the details to the reader and conclude the required estimate

$$\|b^i\star\nu_\delta\|_{C^{0,\alpha}(\overline{\Omega})}\leq C_{2.7}\sum_{j=1}^n\|b^j\|_{C^{0,\alpha}(\overline{\Omega})}$$

for each $i\in\{1,\dots,n\}$.

Next, consider (ii). Proving that $\{b^i\star\nu_\delta\}_{i=1}^n, d\circledast\nu_\delta$ are weakly non-positive over Ω is done in two steps. First, we check using the definition of the convolution that $\{b^i * \nu_\delta\}_{i=1}^n, d * \nu_\delta$ are weakly non-positive over $B_{1-\delta}(0)\setminus\overline{B_{\frac{1}{2}+\delta}(0)}$; we leave this to the reader. Second, using

$$\gamma_\delta\colon\Omega\longrightarrow B_{1-\delta}(0)\setminus\overline{B_{\frac{1}{2}+\delta}(0)}$$

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as a change of variables, we can check that for $\zeta \in C^1_c(\Omega;[0,\infty))$

$$\begin{aligned}\int (d \circledast \nu_\delta)\zeta - \sum_{i=1}^n (b^i \star \nu_\delta) D_i\zeta \,\mathrm{dx} \\ &= \int (d * \nu_\delta)(\zeta \circ \gamma_\delta^{-1}) - \sum_{i,j=1}^n (b^j * \nu_\delta)(D_j(e_i \cdot \gamma_\delta^{-1}))((D_i\zeta)\circ\gamma_\delta^{-1})\,\mathrm{dx} \\ &= \int (d * \nu_\delta)(\zeta \circ \gamma_\delta^{-1}) - \sum_{j=1}^n (b^j * \nu_\delta) D_j(\zeta \circ \gamma_\delta^{-1})\,\mathrm{dx} \le 0,\end{aligned}$$

since $\{b^j * \nu_\delta\}_{j=1}^n, d * \nu_\delta$ are weakly non-positive over $B_{1-\delta}(0) \setminus \overline{B_{\frac{1}{2}+\delta}(0)}$. ∎

3 Estimate and Existence Lemmas

In this section we prove the necessary a priori gradient estimate, existence, and weak maximum principle results needed to prove Theorem 4.1.

Lemma 3.1 (Morrey estimate) *Suppose $\lambda, J \in (0,\infty)$, $\alpha \in (0,1)$, and let $\Omega = B_1(0) \setminus \overline{B_{\frac{1}{2}}(0)}$. There is $C_{3.1} = C_{3.1}(n,\lambda,J,\alpha) \in (0,\infty)$ such that if*

(i) $a^{ij}, b^i \in C^{0,\alpha}(\overline{\Omega})$ *and* $c^i, d \in L^{1,\alpha}(\Omega)$ *for* $i,j \in \{1,\dots,n\}$,
(ii) $\{a^{ij}\}_{i,j=1}^n$ *are uniformly elliptic over* Ω *with respect to* λ,
(iii) $\sum_{i,j}^n \|a^{ij}\|_{C^{0,\alpha}(\overline{\Omega})} + \sum_{i=1}^n \big(\|b^i\|_{C^{0,\alpha}(\overline{\Omega})} + \|c^i\|_{L^{1,\alpha}(\Omega)}\big) + \|d\|_{L^{1,\alpha}(\Omega)} \le J$,

and if $u \in C^{1,\alpha}(\overline{\Omega})$ is a weak solution over Ω of the equation

$$\sum_{i,j=1}^n D_i(a^{ij}D_ju + b^iu) + \sum_{i=1}^n c^iD_iu + du = g + \sum_{i=1}^n D_if^i$$

with $g \in L^{1,\alpha}(\Omega)$ and $f \in C^{0,\alpha}(\overline{\Omega})$, then

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \le C_{3.1}\Big(\|u\|_{L^1(\Omega)} + \|g\|_{L^{1,\alpha}(\Omega)} + \sum_{i=1}^n \|f^i\|_{C^{0,\alpha}(\overline{\Omega})}\Big).$$

Proof This is [19, Theorem 5.5.5′(b)] (with μ, G, e, f replaced respectively by $\alpha, \Omega, \{f^i\}_{i=1}^n, g$). The $C^{1,\alpha}$-conditions (that is, the "C^1_μ-conditions" as stated in [19, Definition 5.5.2]) are implied by (i). To see the dependence $C_{3.1} = C_{3.1}(n,\lambda,J,\alpha)$ more clearly, *cf.* [19, Theorem 5.5.2(b)]. ∎

Next, we state for convenience the more general version of [8, Theorem 8.16], using the remark on [8, p. 193].

Lemma 3.2 (Weak maximum principle) *Suppose $q > n$ and $\lambda, k \in (0,\infty)$. Denote $\Omega = B_1(0) \setminus \overline{B_{\frac{1}{2}}(0)}$. There is $C_{3.2} = C_{3.2}(n,q,\lambda,k) \in (0,\infty)$ so that if*

(i) $a^{ij} \in L^\infty(\Omega)$, $b^i, c^i \in L^q(\Omega)$ *for* $i,j \in \{1,\dots,n\}$ *and* $d \in L^{\frac{q}{2}}(\Omega)$,
(ii) $\{a^{ij}\}_{i,j=1}^n$ *are uniformly elliptic over* Ω *with respect to* λ,
(iii) $\{b^i\}_{i=1}^n, d$ *are weakly non-positive over* Ω,
(iv) $\sum_{i=1}^n \big(\|b^i\|_{L^q(\Omega)} + \|c^i\|_{L^q(\Omega)}\big) + \|d\|_{L^{\frac{q}{2}}(\Omega)} \le k$,

and if $u \in C(\overline{\Omega}) \cap W^{1,2}(\Omega)$ *is a solution over* Ω *of the equation*

$$\sum_{i,j=1}^{n} D_i(a^{ij}D_j u + b^i u) + \sum_{i=1}^{n} c^i D_i u + du \le g + \sum_{i=1}^{n} D_i f^i$$

with $g \in L^{\frac{q}{2}}(\Omega)$ *and* $f^i \in L^q(\Omega)$ *for each* $i \in \{1,\dots,n\}$, *then*

$$\inf_{\Omega} u \ge \inf_{\partial\Omega} \min\{0,u\} - C_{3.2}\Big(\|g\|_{L^{\frac{q}{2}}(\Omega)} + \sum_{i=1}^{n} \|f^i\|_{L^q(\Omega)} \Big).$$

We use Lemmas 3.1 and 3.2 to show we can solve linear divergence form equations with lower-order terms in a Morrey space. This will allow us to get the barrier functions in step 2 of the proof of Theorem 4.1.

Lemma 3.3 *Suppose* $q > n$ *and* $\lambda \in (0,\infty)$. *Denote* $\alpha = 1 - \frac{n}{q}$ *and* $\Omega = B_1(0) \smallsetminus \overline{B_{\frac{1}{2}}(0)}$. *Also suppose we have functions*

(i) $a^{ij}, b^i \in C^{0,\alpha}(\overline{\Omega})$, $c^i \in L^q(\Omega)$ *for* $i,j \in \{1,\dots,n\}$ *and* $d \in L^{\frac{q}{2}}(\Omega) \cap L^{1,\alpha}(\Omega)$,
(ii) $\{a^{ij}\}_{i,j=1}^{n}$ *are uniformly elliptic over* Ω *with respect to* λ,
(iii) $\{b^i\}_{i=1}^{n}$, d *are weakly non-positive over* Ω.

Then there is a $\varphi \in C^{1,\alpha}(\overline{\Omega})$ *that is a weak solution over* Ω *of the equation*

$$\sum_{i,j=1}^{n} D_i(a^{ij}D_j\varphi + b^i\varphi) + \sum_{i=1}^{n} c^i D_i\varphi + d\varphi = 0$$

with

$$\varphi(x) = \begin{cases} 0 & \text{for } x \in \partial B_1(0), \\ -1 & \text{for } x \in \partial B_{\frac{1}{2}}(0), \end{cases} \quad \text{and} \quad \varphi(x) \in [-1,0] \text{ for each } x \in \overline{\Omega}.$$

Proof We follow the proof of [8, Theorem 8.34, p. 211] and use it directly.

Define for $\delta \in (0,\frac{1}{8})$ and each $i \in \{1,\dots,n\}$

(3.1) $$b^i_\delta = b^i \star \nu_\delta, \quad c^i_\delta = c^i \star \nu_\delta, \quad d_\delta = d \circledast \nu_\delta$$

by Definition 2.6. Now consider the weakly defined operator over Ω,

$$L_\delta u = \sum_{i,j=1}^{n} D_i(a^{ij}D_j u + b^i_\delta u) + \sum_{i=1}^{n} c^i_\delta D_i u + d_\delta u.$$

Then (i),(ii),(iii), (3.1), and Lemma 2.7 imply that L_δ satisfies [8, (8.5),(8.8),(8.85)] over Ω, with

$$K = \sum_{i,j=1}^{n} \|a^{ij}\|_{C^{0,\alpha}(\overline{\Omega})} + \sum_{i=1}^{n} \Big(\|b^i_\delta\|_{C^{0,\alpha}(\overline{\Omega})} + \|c^i_\delta\|_{L^\infty(\Omega)} \Big) + \|d_\delta\|_{L^\infty(\Omega)}.$$

We can thus apply [8, Theorem 8.34] (with b^i, c^i, d, g, f^i replaced respectively by $b^i_\delta, c^i_\delta, d_\delta, 0, 0$) over $\Omega = B_1(0) \smallsetminus \overline{B_{\frac{1}{2}}(0)}$ with operator L_δ to conclude that the generalized Dirichlet problem

$$L_\delta u = 0 \text{ in } \Omega, \quad u = \begin{cases} 0 & \text{over } \partial B_1(0), \\ -1 & \text{over } \partial B_{\frac{1}{2}}(0), \end{cases}$$

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is uniquely solvable in $C^{1,\alpha}(\Omega)$. Letting $\varphi_\delta \in C^{1,\alpha}(\overline{\Omega})$ be this unique solution, and comparing [8, (8.2)] with Definition 2.3, we conclude that φ_δ is a weak solution over Ω of the equation

$$(3.2)\qquad \begin{aligned} &\sum_{i,j=1}^{n} D_i(a^{ij}D_j\varphi_\delta + b^i_\delta\varphi_\delta) + \sum_{i=1}^{n} c^i_\delta D_i\varphi_\delta + d_\delta\varphi_\delta = 0 \\ &\text{with } \varphi_\delta(x) = \begin{cases} 0 & \text{for } x \in \partial B_1(0), \\ -1 & \text{for } x \in \partial B_{\frac{1}{2}}(0). \end{cases} \end{aligned}$$

We also apply Lemma 3.2 (with $f^i, g = 0$ for each $i \in \{1,\dots,n\}$) to get

$$(3.3)\qquad \varphi_\delta(x) \in [-1,0] \text{ for each } x \in \overline{\Omega}.$$

Next, we aim to apply Lemma 3.1 to φ_δ. By (i), (3.1), Remark 2.2, and Lemma 2.7 we can conclude that

$$\sum_{i,j=1}^{n} \|a^{ij}\|_{C^{0,\alpha}(\overline{\Omega})} + \sum_{i=1}^{n}\big(\|b^i_\delta\|_{C^{0,\alpha}(\overline{\Omega})} + \|c^i_\delta\|_{L^{1,\alpha}(\Omega)}\big) + \|d_\delta\|_{L^{1,\alpha}(\Omega)} \le J,$$

where, with $C_{2.7} = C_{2.7}(n)$ by Lemma 2.7, we let

$$J = \sum_{i,j}^{n} \|a^{ij}\|_{C^{0,\alpha}(\overline{\Omega})} + \sum_{i=1}^{n}\big(C_{2.7}\|b^i\|_{C^{0,\alpha}(\overline{\Omega})} + \|c^i\|_{L^{1,\alpha}(\Omega)}\big) + 2^n\|d\|_{L^{1,\alpha}(\Omega)}.$$

Thus, by (3.2), Lemma 3.1, and (3.3), we conclude that

$$\|\varphi_\delta\|_{C^{1,\alpha}(\overline{\Omega})} \le C_{3.1}\|\varphi_\delta\|_{L^1(\Omega)} \le C_{3.1}\omega_n,$$

where $C_{3.1} = C_{3.1}(n,\lambda,J,\alpha) \in (0,\infty)$ does not depend on δ.

We conclude that there is $\varphi \in C^{1,\alpha}(\overline{\Omega})$ so that $\varphi_\delta \to \varphi$ in the $C^1(\overline{\Omega})$-norm as $\delta \searrow 0$. Lemma 2.7(i) and (3.2),(3.3) imply that φ is the desired solution. ∎

4 The Hopf Boundary Point Lemma

We are now ready to state and prove our main result.

Theorem 4.1 (Hopf boundary point lemma) *Suppose $q > n$ and $\lambda \in (0,\infty)$. With $\alpha = 1 - \frac{n}{q}$, suppose*

(i) $a^{ij}, b^i \in C^{0,\alpha}(\overline{B_1(0)})$, $c^i \in L^q(B_1(0))$ *for* $i,j \in \{1,\dots,n\}$ *and* $d \in L^{\frac{q}{2}}(B_1(0)) \cap L^{1,\alpha}(B_1(0))$,

(ii) $\{a^{ij}\}_{i,j=1}^n$ *are uniformly elliptic over* $B_1(0)$ *with respect to* λ,

(iii) $\{b^i\}_{i=1}^n, d$ *are weakly non-positive over* $B_1(0)$,

(iv) $a^{ij}(-e_n) = a^{ji}(-e_n)$ *for each* $i,j \in \{1,\dots,n\}$.

If $u \in C(\overline{B_1(0)}) \cap W^{1,2}(B_1(0))$ *is a weak solution over* $B_1(0)$ *to the equation*

$$\sum_{i,j=1}^{n} D_i(a^{ij}D_ju + b^iu) + \sum_{i=1}^{n} c^iD_iu + du \le 0$$

and $u(x) > u(-e_n) = 0$ *for all* $x \in B_1(0)$, *then* $\liminf_{h\searrow 0} \frac{u((h-1)e_n)}{h} > 0$.

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Proof Set $\Omega = B_1(0) \setminus \overline{B_{\frac{1}{2}}(0)}$, and using (i) define $J, k, K \in (0, \infty)$ by

$$
\begin{aligned}
J &= \sum_{i,j}^{n} \|a^{ij}\|_{C^{0,\alpha}(\overline{\Omega})} + \sum_{i=1}^{n} \big(\|b^i\|_{C^{0,\alpha}(\overline{\Omega})} + \|c^i\|_{L^{1,\alpha}(\Omega)} \big) + \|d\|_{L^{1,\alpha}(\Omega)},\\
k &= \sum_{i=1}^{n} \big(\|b^i\|_{L^q(\Omega)} + \|c^i\|_{L^q(\Omega)} \big) + \|d\|_{L^{\frac{q}{2}}(\Omega)},\\
K &= \sum_{i,j}^{n} \|a^{ij}\|_{C^{0,\alpha}(\overline{\Omega})} + \sum_{i=1}^{n} \big(\|b^i\|_{L^q(\Omega)} + \|c^i\|_{L^q(\Omega)} \big) + \|d\|_{L^{\frac{q}{2}}(\Omega)};
\end{aligned} \tag{4.1}
$$

note that we used Remark 2.2 to conclude $c^i \in L^{1,\alpha}(\Omega)$. The proof now proceeds through five major steps.

Step 1: Freezing at the origin and the barrier φ.

Consider the operator L given by

$$Lu = \sum_{i,j=1}^{n} a^{ij}(-e_n) D_{ij} u \text{ over } \Omega.$$

Then (i),(ii), and (iv) imply that L satisfies [8, (6.1),(6.2)]. Applying [8, Theorem 6.14] (with a^{ij}, b^i, c, f replaced respectively by $a^{ij}(-e_n), 0, 0, 0)$) over $\Omega = B_1(0) \setminus \overline{B_{\frac{1}{2}}(0)}$ with operator L, we conclude that the Dirichlet problem

$$Lu = 0 \text{ in } \Omega, \quad u = \begin{cases} 0 & \text{over } \partial B_1(0), \\ -1 & \text{over } \partial B_{\frac{1}{2}}(0), \end{cases}$$

has a unique solution lying in $C^{2,\alpha}(\overline{\Omega})$. If we let $\varphi \in C^{2,\alpha}(\overline{\Omega})$ be this unique solution, we conclude that φ satisfies

$$\sum_{i,j=1}^{n} a_{ij}(-e_n) D_{ij} \varphi = 0 \text{ over } \Omega \quad \text{with } \varphi(x) = \begin{cases} 0 & \text{for } x \in \partial B_1(0), \\ -1 & \text{for } x \in \partial B_{\frac{1}{2}}(0). \end{cases} \tag{4.2}$$

Using (i),(ii), and (iv) again, we see L satisfies [8, (3.1),(3.2),(3.3)]. Thus, the strong maximum principle (see [8, Theorem 3.5]) (with a^{ij}, b^i, c, f replaced respectively by $a^{ij}(-e_n), 0, 0, 0$), implies $\varphi(x) \in (-1, 0)$ for all $x \in \Omega$. This now means that the classical Hopf boundary point lemma (see [8, Lemma 3.4]) (with x_0, a^{ij}, b^i, c, f replaced respectively by $-e_n, a^{ij}(-e_n), 0, 0, 0)$ implies $D_n \varphi(-e_n) < 0$.

Step 2: Scaling and the barrier φ_ϵ.

For each $\epsilon \in (0, \frac{1}{4})$ and $i, j \in \{1, \dots, n\}$, define (over $\overline{\Omega}$ or Ω)

$$
\begin{aligned}
&u_\epsilon(x) = u(\epsilon(x + e_n) - e_n),\\
&a_\epsilon^{ij}(x) = a^{ij}(\epsilon(x + e_n) - e_n), \quad b_\epsilon^i(x) = \epsilon b^i(\epsilon(x + e_n) - e_n),\\
&c_\epsilon^i(x) = \epsilon c^i(\epsilon(x + e_n) - e_n), \quad d_\epsilon(x) = \epsilon^2 d(\epsilon(x + e_n) - e_n).
\end{aligned} \tag{4.3}
$$

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Observe that (using the change of variables $y = \epsilon x$ and Definition 2.1)

$$\begin{aligned}
&\|a_\epsilon^{ij}\|_{C^{0,\alpha}(\overline{\Omega})} \le \|a^{ij}\|_{C^{0,\alpha}(\overline{\Omega})},\\
&\|b_\epsilon^{i}\|_{C^{0,\alpha}(\overline{\Omega})} \le \epsilon\|b^{i}\|_{C^{0,\alpha}(\overline{\Omega})}, \qquad \|b_\epsilon^{i}\|_{L^q(\Omega)} \le \epsilon^\alpha\|b^{i}\|_{L^q(\Omega)},\\
&\|c_\epsilon^{i}\|_{L^{1,\alpha}(\Omega)} \le \epsilon^\alpha\|c^{i}\|_{L^{1,\alpha}(\Omega)}, \qquad \|c_\epsilon^{i}\|_{L^q(\Omega)} \le \epsilon^\alpha\|c^{i}\|_{L^q(\Omega)},\\
&\|d_\epsilon\|_{L^{1,\alpha}(\Omega)} \le \epsilon^{1+\alpha}\|d\|_{L^{1,\alpha}(\Omega)}, \quad \|d_\epsilon\|_{L^{\frac{q}{2}}(\Omega)} \le \epsilon^{2\alpha}\|d\|_{L^{\frac{q}{2}}(\Omega)},
\end{aligned} \tag{4.4}$$

for each $i, j \in \{1, \dots, n\}$, since $\epsilon \in (0, \frac{1}{4})$ implies $\{\epsilon(x + e_n) - e_n : x \in \Omega\} \subset \Omega$. As well, we have by (ii) and (iii),

$$\begin{aligned}
&\{a_\epsilon^{ij}\} \text{ are uniformly elliptic over } \Omega \text{ with respect to } \lambda,\\
&\{b_\epsilon^{i}\}_{i=1}^n, d_\epsilon \text{ are weakly non-positive over } \Omega.
\end{aligned} \tag{4.5}$$

Using (4.4) and (4.5) we conclude by Lemma 3.3 that for each $\epsilon \in (0, \frac{1}{4})$ there is $\varphi_\epsilon \in C^{1,\alpha}(\overline{\Omega})$ that is a weak solution over Ω of the equation

$$\sum_{i,j=1}^n D_i(a_\epsilon^{ij} D_j\varphi_\epsilon + b_\epsilon^i\varphi_\epsilon) + \sum_{i=1}^n c_\epsilon^i D_i\varphi_\epsilon + d_\epsilon\varphi_\epsilon = 0 \tag{4.6}$$

$$\text{with } \varphi_\epsilon|_{\partial\Omega} = \varphi|_{\partial\Omega} \quad \text{and} \quad \varphi_\epsilon \in [-1, 0] \text{ for each } x \in \overline{\Omega}.$$

Step 3: Comparing φ and φ_ϵ.

Define the functions

$$g_\epsilon = -\sum_{i=1}^n c_\epsilon^i D_i\varphi - d_\epsilon\varphi \quad \text{and} \quad f_\epsilon^i = -\sum_{j=1}^n \big(a_\epsilon^{ij} - a_\epsilon^{ij}(-e_n)\big) D_j\varphi - b_\epsilon^i\varphi \tag{4.7}$$

for $i \in \{1, \dots, n\}$. Then (4.2), (4.6), and (4.7) imply that $\psi_\epsilon = \varphi_\epsilon - \varphi \in C^{1,\alpha}(\overline{\Omega})$ is a weak solution over Ω of the equation

$$\sum_{i,j=1}^n D_i(a_\epsilon^{ij} D_j\psi_\epsilon + b_\epsilon^i\psi_\epsilon) + \sum_{i=1}^n c_\epsilon^i D_i\psi_\epsilon + d_\epsilon\psi_\epsilon = g_\epsilon + \sum_{i=1}^n D_i f_\epsilon^i \tag{4.8}$$

with $\psi_\epsilon|_{\partial\Omega} = 0$.

We wish to apply Lemma 3.1 to ψ_ϵ. Before we do so, we will use Lemma 3.2 to estimate $\|\psi_\epsilon\|_{L^1(\Omega)}$. For this, we make the following three computations.

First, by (4.4), $\epsilon \in (0, \frac{1}{4})$, $\alpha = 1 - \frac{n}{q} > 0$, and with k as in (4.1),

$$\sum_{i=1}^n \big(\|b_\epsilon^i\|_{L^q(\Omega)} + \|c_\epsilon^i\|_{L^q(\Omega)}\big) + \|d_\epsilon\|_{L^{\frac{q}{2}}(\Omega)} \le k.$$

Second, using (4.7), $\varphi \in C^{2,\alpha}(\overline{\Omega})$ by (4.2), Hölder's inequality, and (4.4),

$$\begin{aligned}
\|g_\epsilon\|_{L^{q/2}(\Omega)} &\le \|\varphi\|_{C^1(\Omega)}\Big(\sum_{i=1}^n \|c_\epsilon^i\|_{L^{q/2}(\Omega)} + \|d_\epsilon\|_{L^{q/2}(\Omega)}\Big)\\
&\le \|\varphi\|_{C^1(\Omega)}\Big(\sum_{i=1}^n \epsilon^\alpha\omega_n^{1/q}\|c^i\|_{L^q(\Omega)} + \epsilon^{2\alpha}\|d\|_{L^{q/2}(\Omega)}\Big).
\end{aligned}$$

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Third, we similarly compute for each $i \in \{1,\dots,n\}$ using (4.7), (4.3), and (4.4),

$$\|f_\epsilon^i\|_{L^q(\Omega)} \le \|\varphi\|_{C^1(\Omega)}\Big(\sum_{j=1}^n \omega_n^{1/q}\|a_\epsilon^{ij} - a_\epsilon^{ij}(-e_n)\|_{C(\Omega)} + \|b_\epsilon^i\|_{L^q(\Omega)}\Big)$$
$$\le \|\varphi\|_{C^1(\Omega)}\Big(\sum_{j=1}^n \epsilon^\alpha 2^\alpha \omega_n^{1/q}\|a^{ij}\|_{C^{0,\alpha}(\overline{\Omega})} + \epsilon^\alpha\|b^i\|_{L^q(\Omega)}\Big).$$

These three computations together with (4.8) imply that we can apply Lemma 3.2 (with $u = \psi_\epsilon, -\psi_\epsilon$) to conclude

$$\sup_\Omega |\psi_\epsilon| \le C_{3.2}\Big(\|g_\epsilon\|_{L^{q/2}(\Omega)} + \sum_{i=1}^n \|f_\epsilon^i\|_{L^q(\Omega)}\Big)$$
$$\le C_{3.2}\|\varphi\|_{C^1(\Omega)}\Big(\sum_{i=1}^n \epsilon^\alpha \omega_n^{1/q}\|c^i\|_{L^q(\Omega)} + \epsilon^{2\alpha}\|d\|_{L^{q/2}(\Omega)}\Big)$$
$$+ C_{3.2}\|\varphi\|_{C^1(\Omega)}\Big(\sum_{i,j=1}^n \epsilon^\alpha 2^\alpha \omega_n^{1/q}\|a^{ij}\|_{C^{0,\alpha}(\overline{\Omega})} + \sum_{i=1}^n \epsilon^\alpha\|b^i\|_{L^q(\Omega)}\Big)$$
$$\le \epsilon^\alpha C_{3.2}\max\{2^\alpha\omega_n^{1/q}, 1\}\|\varphi\|_{C^1(\Omega)}K,$$

where $C_{3.2} = C_{3.2}(n, q, \lambda, k)$ and k, K as in (4.1) do not depend on ϵ. Thus

$$\|\psi_\epsilon\|_{L^1(\Omega)} \le \epsilon^\alpha C_{3.2}\max\{2^\alpha\omega_n^{1+\frac{1}{q}}, \omega_n\}\|\varphi\|_{C^1(\Omega)}K. \tag{4.9}$$

Now we shall use Lemma 3.1. For this we make three computations.
First, using (4.4), $\epsilon \in (0, \frac{1}{4})$, $\alpha = 1 - \frac{n}{q} > 0$, and with J as in (4.1),

$$\sum_{i,j=1}^n \|a_\epsilon^{ij}\|_{C^{0,\alpha}(\overline{\Omega})} + \sum_{i=1}^n\Big(\|b_\epsilon^i\|_{C^{0,\alpha}(\overline{\Omega})} + \|c_\epsilon^i\|_{L^{1,\alpha}(\Omega)}\Big) + \|d_\epsilon\|_{L^{1,\alpha}(\Omega)} \le J.$$

Second, using Definition 2.1, (4.7), and (4.4), we compute

$$\|g_\epsilon\|_{L^{1,\alpha}(\Omega)} \le \|\varphi\|_{C^1(\Omega)}\Big(\sum_{i=1}^n \|c_\epsilon^i\|_{L^{1,\alpha}(\Omega)} + \|d_\epsilon\|_{L^{1,\alpha}(\Omega)}\Big)$$
$$\le \|\varphi\|_{C^1(\Omega)}\Big(\sum_{i=1}^n \epsilon^\alpha\|c^i\|_{L^{1,\alpha}(\Omega)} + \epsilon^{2\alpha}\|d\|_{L^{1,\alpha}(\Omega)}\Big).$$

Third, for each $i \in \{1,\dots,n\}$ using (4.3) and (4.4), we compute

$$\|f_\epsilon^i\|_{C^{0,\alpha}(\overline{\Omega})} \le \|\varphi\|_{C^{1,\alpha}(\overline{\Omega})}\Big(\sum_{j=1}^n \|a_\epsilon^{ij} - a_\epsilon^{ij}(-e_n)\|_{C^{0,\alpha}(\overline{\Omega})} + \|b_\epsilon^i\|_{C^{0,\alpha}(\overline{\Omega})}\Big)$$
$$\le \|\varphi\|_{C^{1,\alpha}(\overline{\Omega})}\Big(\sum_{j=1}^n \epsilon^\alpha\|a^{ij}\|_{C^{0,\alpha}(\overline{\Omega})} + \epsilon^\alpha\|b^i\|_{C^{0,\alpha}(\overline{\Omega})}\Big).$$

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These three computations, (4.8), Lemma 3.1, and (4.9) imply that

$$\begin{aligned}\|\psi_\epsilon\|_{C^{1,\alpha}(\overline{\Omega})} &\le C_{3.1}\Big(\|\psi_\epsilon\|_{L^1(\Omega)} + \|g_\epsilon\|_{L^{1,\alpha}(\Omega)} + \sum_{i=1}^n \|f_\epsilon^i\|_{C^{0,\alpha}(\overline{\Omega})}\Big)\\ &\le \epsilon^\alpha C_{3.1}C_{3.2}\max\{2^\alpha\omega_n^{1+\frac{1}{q}}, \omega_n\}\|\varphi\|_{C^1(\Omega)}K\\ &\quad + C_{3.1}\|\varphi\|_{C^1(\Omega)}\Big(\sum_{i=1}^n \epsilon^\alpha\|c^i\|_{L^{1,\alpha}(\Omega)} + \epsilon^{2\alpha}\|d\|_{L^{1,\alpha}(\Omega)}\Big)\\ &\quad + C_{3.1}\|\varphi\|_{C^{1,\alpha}(\overline{\Omega})}\Big(\sum_{i,j=1}^n \epsilon^\alpha\|a^{ij}\|_{C^{0,\alpha}(\overline{\Omega})} + \sum_{i=1}^n \epsilon^\alpha\|b^i\|_{C^{0,\alpha}(\overline{\Omega})}\Big)\\ &\le \epsilon^\alpha C_{3.1}C_{3.2}\max\{2^\alpha\omega_n^{1+\frac{1}{q}}, \omega_n\}\|\varphi\|_{C^1(\Omega)}K\\ &\quad + \epsilon^\alpha C_{3.1}\|\varphi\|_{C^{1,\alpha}(\overline{\Omega})}J,\end{aligned}$$

where $C_{3.1} = C_{3.1}(n,\lambda,J,\alpha)$ and J as in (4.1) do not depend on ϵ. Recalling that $C_{3.2} = C_{3.2}(n,q,\lambda,k)$ and k, K as in (4.1) do not depend on ϵ; then

(4.10) $$\lim_{\epsilon\to 0}|D_n\varphi_\epsilon(-e_n) - D_n\varphi(-e_n)| \le \lim_{\epsilon\to 0}\|\psi_\epsilon\|_{C^{1,\alpha}(\overline{\Omega})} = 0.$$

Step 4: Fixing ϵ and comparing u_ϵ and φ_ϵ.

By **Step 1** and (4.10), we can fix $\epsilon \in (0, \frac{1}{4})$ so that

(4.11) $$D_n\varphi_\epsilon(-e_n) < 0.$$

Recalling $u \in C(\overline{B_1(0)})$ with $u(x) > u(-e_n) = 0$ for $x \in B_1(0)$, we can define $\hat{u}_\epsilon \in C(\overline{\Omega}) \cap W^{1,2}(\Omega)$ by

(4.12) $$\hat{u}_\epsilon = (u_\epsilon + \theta_\epsilon\varphi_\epsilon) \text{ with } \theta_\epsilon = \inf_{\partial B_{\frac{1}{2}}(0)} u_\epsilon > 0.$$

Observe by (4.3) that u_ϵ is a weak solution over Ω of the equation

$$\sum_{i,j=1}^n D_i(a_\epsilon^{ij}D_ju_\epsilon + b_\epsilon^iu_\epsilon) + \sum_{i=1}^n c_\epsilon^iD_iu_\epsilon + d_\epsilon u_\epsilon \le 0.$$

Then (4.2), (4.6), and (4.12) imply $\hat{u}_\epsilon$ is a weak solution over Ω of the equation

$$\sum_{i,j=1}^n D_i(a_\epsilon^{ij}D_j\hat{u}_\epsilon + b_\epsilon^i\hat{u}_\epsilon) + \sum_{i=1}^n c_\epsilon^iD_i\hat{u}_\epsilon + d_\epsilon\hat{u}_\epsilon \le 0 \text{ with } \hat{u}_\epsilon|_{\partial\Omega} \ge 0.$$

We conclude by Lemma 3.2 that $\inf_\Omega \hat{u}_\epsilon \ge 0$.

Step 5: Computing the derivative of u at the origin.

Using (4.3), (4.11), (4.12), and $\inf_\Omega \hat{u}_\epsilon \ge 0$, we conclude that

$$\begin{aligned}\liminf_{h\searrow 0}\frac{u((h-1)e_n)}{h} &= \liminf_{h\searrow 0}\frac{u_\epsilon((\frac{h}{\epsilon}-1)e_n)}{h}\\ &\ge \liminf_{h\searrow 0}\frac{-\theta_\epsilon\varphi_\epsilon((\frac{h}{\epsilon}-1)e_n)}{h} = \frac{-\theta_\epsilon}{\epsilon}D_n\varphi_\epsilon(-e_n) > 0.\end{aligned}$$ ∎

It is typical to make some remarks relaxing some of the assumptions on the coefficients in certain cases; see, for example, [20, Remark 1.2(b)]. We make two more similar remarks.

Remark 4.2 We can relax some of the assumptions of Theorem 4.1.

(i) We need not assume $\alpha = 1 - \frac{n}{q}$; it merely suffices that

$$a^{ij}, b^i \in C^{0,\alpha}(\overline{B_1(0)}), \quad c^i \in L^q(B_1(0)), \quad d \in L^{\frac{q}{2}}(B_1(0)) \cap L^{1,\alpha}(B_1(0))$$

with $q > n$ and general $\alpha \in (0,1)$.

(ii) We can more generally assume $u(-e_n) \leq 0$. We can see this by setting $\hat{u}(x) = u(x) - u(-e_n)$ for $x \in \overline{\Omega}$, and noting that for $\zeta \in C^1_c(\Omega; [0,\infty))$,

$$\begin{aligned} &\int \sum_{i,j=1}^{n} a^{ij} D_j \hat{u} D_i \zeta + \sum_{i=1}^{n} \big(b^i \hat{u} D_i \zeta - c^i (D_i \hat{u}) \zeta \big) - d \hat{u} \zeta \, \mathrm{dx} \\ &\quad = \int \sum_{i,j=1}^{n} a^{ij} D_j u D_i \zeta + \sum_{i=1}^{n} \big(b^i u D_i \zeta - c^i (D_i u) \zeta \big) - d u \zeta \, \mathrm{dx} \\ &\quad\quad + u(-e_n) \int d\zeta - \sum_{i=1}^{n} b^i D_i \zeta \, \mathrm{dx} \geq 0, \end{aligned}$$

since $\{b_i\}_{i=1}^n, d$ are weakly non-positive over Ω.

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