BULL. AUSTRAL. MATH. SOC. 13C13, 13C10, 13G05, 13E15
VOL. 73 (2006) [293–297]

A NOTE ON A THEOREM OF F. WANG AND G. TANG

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In this paper, we refine a theorem of F. Wang and G. Tang concerning freeness of reflexive modules over coherent GCD-domains.

1. INTRODUCTION

Throughout this paper, we assume that all rings are commutative with identity and all modules are unitary.

A ring R is called a *GE-ring* if the special linear group of degree n over R coincides with the elementary linear group of degree n over R for all $n \geqslant 2$. Recently, Wang and Tang [**3**, Theorem 2.13] proved the following theorem, which is the main result of the paper. (For the definitions and properties of a coherent ring and a GCD-domain, see [**1**] and [**2**], respectively.)

THEOREM 1.1. (Wang-Tang) *Let R be a coherent GCD-domain, and $u \in R$ a prime element such that R/uR is both a PID and a GE-ring. Let T be a finitely generated reflexive R-module. If T_u is R_u-free, then T is R-free.*

The purpose of this paper is to remove the assumption that R/uR is a GE-ring; we shall prove the following.

THEOREM 1.2. *Let R be a coherent GCD-domain and T a finitely generated reflexive R-module. Suppose that T_u is R_u-free for some $u \in R$ such that R/uR is a PID. Then T is R-free.*

2. PROOF OF THEOREM 1.2

In this section, we prove Theorem 1.2. For this, we prepare a couple of lemmas.

LEMMA 2.1. *Let R be a ring, and M a finitely presented R-module. Suppose that $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module for all $\mathfrak{p} \in \operatorname{Spec} R$. Then M is a projective R-module.*

PROOF: Let $\phi : X \to Y$ be a surjective homomorphism of R-modules. This map induces an exact sequence

$$\operatorname{Hom}_R(M, X) \xrightarrow{\operatorname{Hom}_R(M,\phi)} \operatorname{Hom}_R(M, Y) \longrightarrow C \longrightarrow 0$$

Received 5th December, 2005

of R-modules. Fix a prime ideal $\mathfrak{p}$ of R. The finite presentation of M yields a natural isomorphism

$$(\operatorname{Hom}_R(M,N))_{\mathfrak{p}} \cong \operatorname{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}},N_{\mathfrak{p}})$$

for any R-module N. Hence we have a commutative diagram

$$\begin{array}{ccccccc} (\operatorname{Hom}_R(M,X))_{\mathfrak{p}} & \xrightarrow{(\operatorname{Hom}_R(M,\phi))_{\mathfrak{p}}} & (\operatorname{Hom}_R(M,Y))_{\mathfrak{p}} & \longrightarrow & C_{\mathfrak{p}} & \longrightarrow & 0 \\ \cong\downarrow & & \cong\downarrow & & & & \\ \operatorname{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}},X_{\mathfrak{p}}) & \xrightarrow{\operatorname{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}},\phi_{\mathfrak{p}})} & \operatorname{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}},Y_{\mathfrak{p}}) & & & & \end{array}$$

where the first row is exact. Since $M_{\mathfrak{p}}$ is $R_{\mathfrak{p}}$-free by assumption, the map $\operatorname{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}},\phi_{\mathfrak{p}})$ in the above diagram is surjective. Thus we have $C_{\mathfrak{p}}=0$ for every $\mathfrak{p}\in\operatorname{Spec}R$, which says that $C=0$. It follows that M is R-projective. ∎

Lemma 2.2. *Let R be a ring, and let $\phi: R^n \to R^m$ be a surjective homomorphism of free R-modules of finite rank. Then one has* $\operatorname{Ker}\phi \cong R^{n-m}$.

Proof: There is a split exact sequence $0\to K\to R^n \xrightarrow{\phi} R^m \to 0$ of R-modules. Let A be a matrix which represents the homomorphism ϕ with respect to the canonical bases; A is an $m\times n$ matrix over R. Since ϕ is a split epimorphism, we have $AB=I_m$ for some $n\times m$ matrix B over R, where I_m denotes the $m\times m$ identity matrix over R. By using column vectors in R^n, we write $B=(b_1,b_2,\dots,b_m)$. If $c_1b_1+c_2b_2+\cdots+c_mb_m=0$ where $c_1,c_2,\dots,c_m\in R$, then

$$B\begin{pmatrix}c_1\\ \vdots\\ c_m\end{pmatrix}=0,\quad\text{hence}\quad \begin{pmatrix}c_1\\ \vdots\\ c_m\end{pmatrix}=AB\begin{pmatrix}c_1\\ \vdots\\ c_m\end{pmatrix}=0.$$

This means that the vectors $b_1,b_2,\dots,b_m$ are linearly independent over R. Hence there exist $b_{m+1},b_{m+2},\dots,b_n\in R^n$ such that $b_1,b_2,\dots,b_m,b_{m+1},b_{m+2},\dots,b_n$ form a free basis of R^n. Letting C be the $n\times(n-m)$ matrix $(b_{m+1},b_{m+2},\dots,b_n)$, we see that the $n\times n$ square matrix $(B\ \ C)$ is invertible. Put

$$D=(B\ \ C)\begin{pmatrix}I_m & -AC\\ 0 & I_{n-m}\end{pmatrix}.$$

Note that D is also an invertible matrix. We have $A=(I_m\ \ 0)D^{-1}$, and get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & R^n & \xrightarrow{A} & R^m & \longrightarrow & 0 \\ & & \downarrow & & D^{-1}\downarrow\cong & & \| & & \\ 0 & \longrightarrow & R^{n-m} & \xrightarrow[\begin{pmatrix}0\\ I_{n-m}\end{pmatrix}]{} & R^n & \xrightarrow[(I_m\ 0)]{} & R^m & \longrightarrow & 0 \end{array}$$

https://doi.org/10.1017/S0004972700038879 Published online by Cambridge University Press

with exact rows. This diagram shows that K is isomorphic to R^{n-m}. □

Now, we can prove Theorem 1.2.

PROOF OF THEOREM 1.2: Let R be a coherent GCD-domain, and let $u \in R$ a element such that R/uR is a PID. (Hence u is automatically a prime element.) Let T be a finitely generated reflexive R-module such that T_u is a free R_u-module. We want to prove that T is a free R-module.

There are elements $x_1, x_2, \ldots, x_n$ of T such that

$$T_u = R_u \cdot x_1 \oplus R_u \cdot x_2 \oplus \cdots \oplus R_u \cdot x_n.$$

Let F be the R-submodule of T which is generated by $x_1, x_2, \ldots, x_n$. Then F is a free R-module with basis $x_1, x_2, \ldots, x_n$. If $F = T$, then there is nothing to prove. Hence we assume that F is a proper submodule of T. Note that $F_u = T_u$. By [**3**, Theorem 2.6], we have

$$(F : T) = \{\, a \in R \mid aT \subseteq F \,\} = u^m R$$

for some $m \geqslant 1$. It follows from [**3**, Theorem 2.11] that there exists a chain of primary submodules of T

$$F = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_t = T \tag{2.1}$$

such that F_i is a maximal uR-prime submodule of F_{i+1} for $0 \leqslant i \leqslant t-1$. According to [**3**, Theorem 2.10], F_i is a finitely generated reflexive R-module for $0 \leqslant i \leqslant t$. Note that F is a finitely generated free R-module which is a maximal uR-prime submodule of F_1. The chain (2.1) says that, to prove the freeness of the R-module T, we can assume without loss of generality that F is a maximal uR-prime submodule of T.

Put $\overline{R} = R/uR$, and set $\overline{M} = M/uM$ for an R-module M. There is an exact sequence

$$0 \to F \to T \to C \to 0 \tag{2.2}$$

of R-modules. Tensoring $\overline{R}$ with this sequence gives an exact sequence

$$0 \to D \to \overline{F} \to \overline{T} \to \overline{C} \to 0.$$

Noting that F is a uR-prime submodule of T, we easily see that

$$D = (uT \cap F)/uF = uT/uF.$$

The R-module T is reflexive, hence is torsionfree. Therefore the homomorphism from $C = T/F$ to $D = uT/uF$ given by $\overline{z} \mapsto \overline{uz}$, where $\overline{z}$ denotes the residue class of $z \in T$ in C, is an isomorphism. We obtain an exact sequence

$$0 \to C \to \overline{F} \to \overline{T} \to C \to 0. \tag{2.3}$$

Since F is a uR-prime submodule of T, it is seen by the definition of a prime submodule that C is a torsionfree $\overline{R}$-module. By [**3**, Corollary 2.7(2)], $\overline{T}$ is also a torsionfree $\overline{R}$-module. The assumption that $\overline{R}$ is a PID implies that both C and $\overline{T}$ are free $\overline{R}$-modules. Hence C is isomorphic to $\overline{R}^m$ for some integer m. Recall that F is a free R-module of rank n. By (2.2) we have an exact sequence

$$0 \to R^n \to T \to \overline{R}^m \to 0. \tag{2.4}$$

On the other hand, we see from (2.3) that $\operatorname{rank}_{\overline{R}} \overline{T} = \operatorname{rank}_{\overline{R}} \overline{F} = \operatorname{rank}_R F = n$. Hence there is an isomorphism $\Phi : \overline{R}^n \to \overline{T}$. Note that we have a natural isomorphism

$$\operatorname{Hom}_{\overline{R}}(\overline{R}^n, \overline{T}) \cong \overline{\operatorname{Hom}_R(R^n, T)}.$$

There exists $\phi \in \operatorname{Hom}_R(R^n, T)$ such that the residue class $\overline{\phi}$ in $\overline{\operatorname{Hom}_R(R^n, T)}$ corresponds to Φ through the isomorphism. Let K and L be the kernel and the cokernel of the map ϕ, respectively. We have an exact sequence

$$0 \to K \to R^n \xrightarrow{\phi} T \to L \to 0. \tag{2.5}$$

Tensoring $\overline{R}$ with this sequence implies that $\overline{L} = 0$, that is, $L = uL$. Since T is finitely generated, it follows from (2.5) that so is L. Applying Nakayama's lemma, we obtain $(1 + au)L = 0$ for some $a \in R$. The element $1 + au \in R$ is nonzero, hence we get $\operatorname{rank}_R L = 0$. On the other hand, since $uC = 0$ and $u \neq 0$, we have $\operatorname{rank}_R C = 0$. Hence it is seen from (2.2) that $\operatorname{rank}_R T = \operatorname{rank}_R F + \operatorname{rank}_R C = n$, and from (2.5) that $\operatorname{rank}_R K = \operatorname{rank}_R R^n - \operatorname{rank}_R T + \operatorname{rank}_R L = 0$. Noting that K is isomorphic to a submodule of a free R-module by (2.5), we must have $K = 0$, and get an exact sequence

$$0 \to R^n \to T \to L \to 0. \tag{2.6}$$

Here, we claim that $T_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for any $\mathfrak{p} \in \operatorname{Spec} R$. In fact, if $\mathfrak{p}$ contains the element u, then $1 + au$ is a unit as an element of $R_{\mathfrak{p}}$, hence $L_{\mathfrak{p}} = 0$ because $(1+au)L_{\mathfrak{p}} = 0$. Localising (2.6) at $\mathfrak{p}$, we obtain an isomorphism $T_{\mathfrak{p}} \cong R_{\mathfrak{p}}^n$. If $\mathfrak{p}$ does not contain u, then $\overline{R}_{\mathfrak{p}} = 0$ and from (2.4) we get an isomorphism $T_{\mathfrak{p}} \cong R_{\mathfrak{p}}^n$. Therefore, in any case, $T_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module.

There is an exact sequence $0 \to R^m \xrightarrow{u} R^m \to \overline{R}^m \to 0$. From this sequence and (2.4), we make the following pullback diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & R^m & = & R^m & & \\
 & & & & \downarrow & & \downarrow{\scriptstyle u} & & \\
0 & \longrightarrow & R^n & \longrightarrow & P & \longrightarrow & R^m & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R^n & \longrightarrow & T & \longrightarrow & \overline{R}^m & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

In this diagram, the middle row splits, and P is isomorphic to R^{n+m}. Hence we obtain an exact sequence

$$0 \to R^n \to R^{n+m} \to T \to 0. \tag{2.7}$$

In particular, T is a finitely presented R-module. Lemma 2.1 implies that T is a projective R-module. Therefore the exact sequence (2.7) splits, and we get an exact sequence $0 \to T \to R^{n+m} \to R^n \to 0$. Lemma 2.2 shows that T is a free R-module. Thus, the proof of the theorem is completed. ∎

References

[1] S. Glaz, *Commutative coherent rings*, Lecture Notes in Mathematics **1371** (Springer-Verlag, Berlin, 1989).

[2] I. Kaplansky, *Commutative rings* (Allyn and Bacon, Inc., Boston, MA, 1970).

[3] F. Wang and G. Tang, 'Reflexive modules over coherent GCD-domains', *Comm. Algebra* **33** (2005), 3283–3292.

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