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Periodic attractors as a result of diffusion

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Abstract. We present a dynamical system in $\mathbb{R}^2$ with a global point attractor but so that two such systems, when coupled by linear diffusion, produce a system in $\mathbb{R}^4$ with no point attractors and yet with all solutions bounded in the positive time direction.

1. *Introduction*

In 1952 an article by Turing appeared on the effects of diffusion between biological cells, [**6**]. He was interested particularly in the question whether diffusion could result in periodic behaviour. Unfortunately the examples of dynamical systems presented in this paper are all linear and it is impossible to have periodic behaviour in any structurally stable linear system.

Smale dealt with that problem in 1976 by giving a structurally stable example, [**5**]. The structural stability implies that such oscillations persist under small perturbations of the dynamic equations and hence that these oscillations are not exceptional.

Before stating the main result of this article we explain what is meant by diffusion between cells. In this we essentially follow the ideas of Turing.

Take two cells whose evolutions are described by

$$\dot{x}^i = f^i(x^i), \qquad x^i \in \mathbb{R}^n \text{ and } i = 1, 2.$$

Assume that these cells are identical, i.e. $f^1 = f^2 = f$. The variable x^i describes the state of the ith cell, so if we think of a cell as a purely chemical system, the components of x^i may be the concentrations of the different chemicals. Coupling of the two cells by a permeable membrane (see figure 1) will result in diffusion, caused by differences in concentrations of the chemicals.

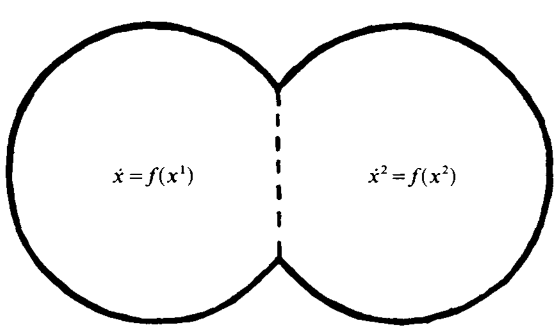

FIGURE 1

We assume that the diffusion is proportional to this difference. So for a particular chemical, the mass transport per unit time from cell 2 to cell 1 equals

$$\mu_k(x_k^2 - x_k^1), \qquad \text{with } \mu_k \geq 0.$$

Coupling thus gives the following system, the so-called Rashevsky-Turing equations:

$$(\ddagger) \qquad \begin{cases} \dot{x}^1 = f(x^1) + D(x^2 - x^1) \\ \dot{x}^2 = f(x^2) + D(x^1 - x^2), \end{cases} \qquad (x^1, x^2) \in \mathbb{R}^{2n}$$

with $D = \operatorname{diag}(\mu_1, \cdots, \mu_n)$, $\mu_i \geq 0$.

In general diffusion, as it occurs in ($\ddagger$), has a damping effect. Yet there are examples of the contrary, cf. [**3**], [**5**].

In this article we shall show some new such examples. We point out that, although ($\ddagger$) gives a realistic model for two cells with diffusion, we do not have, for any $f: \mathbb{R}^n \to \mathbb{R}^n$ and $D = \operatorname{diag}(\mu_1, \cdots, \mu_n)$, a pair of cells and a membrane whose dynamics and diffusion is described by (f, D). We construct examples of (f, D) with which ($\ddagger$) produces periodic dynamics; we do not touch the problem of actually making two interacting cells with periodic dynamics.

In studying the system ($\ddagger$), we take the dynamics of an isolated cell to be simple (i.e. not giving rise to periodic dynamics):
from now on we assume that f has a *global point attractor*, which we may assume to be 0 henceforth. This means that for *every* solution $x(t)$ of $\dot{x} = f(x)$:

$$x(t) \to 0 \qquad \text{as } t \to +\infty.$$

Remark. The cells we thus consider are not already oscillating on their own. This is in contrast with other studies, cf. [**2**], [**4**].

Nevertheless, even under this assumption, diffusion between two identical cells can lead to complex dynamics as we shall see.

Consider the case that f in ($\ddagger$) is linear; so f is represented by a matrix A (one might also think of A as the linearization of f at 0). If the singularity 0 of f is not degenerate, which we assume in the following, then the eigenvalues of A all have negative real part. It will be shown that for a rightly chosen pair (A, D) the zero solution of ($\ddagger$) is unstable. Then at least 0 is not a point attractor of ($\ddagger$). However by destabilizing the zero solution, other point attractors might arise or the system ($\ddagger$) could even have unbounded orbits.

We only want to consider pairs (f, D) for which the corresponding system ($\ddagger$) satisfies:

(P1) the system ($\ddagger$) has no point attractors.

(P2) the orbits of ($\ddagger$) are bounded in the positive time direction.

In the case $n = 4$, there is the example of Smale, cf. [**3**]. For his choice of (f, D) the system ($\ddagger$) has a *global periodic attractor* γ:

for almost every $p \in \mathbb{R}^8$, the orbit $x_p(t)$ through p converges to γ for $t \to +\infty$ and γ is a closed orbit.

One might ask whether similar examples exist with a lower dimension of phase space. In this article it will be shown that this is indeed true, even for $n = 2$.

We conclude this introduction with the simple proof that no such example exists for $n=1$. For let $r(t)=(x(t), y(t)\neq(0,0))$ be a solution of (‡) in the case $n=1$. Then we have:

$$\frac{d}{dt}\frac{1}{2}\|r(t)\|^2=\langle f(x), x\rangle+\langle f(y), y\rangle-\mu\langle x-y, x-y\rangle<0,$$

by assumption on f and μ. As the solution r was arbitrary, 0 is a global point attractor of (‡).

After completing the final version of this paper we became aware of another two dimensional example, recently given by J.C. Alexander, [**1**].

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2. *D-stability*

In this section we take a closer look at the system:

$$(\ddagger)\qquad \begin{cases}\dot{x}^1=f(x^1)+D(x^2-x^1)\\ \dot{x}^2=f(x^2)+D(x^1-x^2),\end{cases}\qquad (x^1, x^2)\in\mathbb{R}^{2n}$$

$D=\operatorname{diag}(\mu_1,\cdots,\mu_n)$, $\mu_i\geq 0$.

By assumption on f, 0 is a solution of (‡). In order to satisfy (P1), 0 should not be a point attractor. Linearizing (‡) at 0 gives the following system:

$$\begin{pmatrix}\dot{x}^1\\ \dot{x}^2\end{pmatrix}=\begin{pmatrix}A-D & D\\ D & A-D\end{pmatrix}\begin{pmatrix}x^1\\ x^2\end{pmatrix}\overset{\text{def}}{=}M\begin{pmatrix}x^1\\ x^2\end{pmatrix},$$

with $A=(Df)|_0$.

Using a linear transformation S,

$$S=\begin{pmatrix}\mathrm{Id} & \mathrm{Id}\\ -\mathrm{Id} & \mathrm{Id}\end{pmatrix}$$

and consequently

$$S^{-1}=\frac{1}{2}\begin{pmatrix}\mathrm{Id} & -\mathrm{Id}\\ \mathrm{Id} & \mathrm{Id}\end{pmatrix}$$

it easily follows that

$$SMS^{-1}=\begin{pmatrix}A & 0\\ 0 & A-2D\end{pmatrix}.$$

So we conclude $\sigma(M)=\sigma(A)\cup\sigma(A-2D)$. Here $\sigma(M)$ denotes the set of eigenvalues of M.

We assumed that 0, the global attractor of f, is not degenerate. This means that $A=(Df)|_0$ satisfies

$$\sigma(A)\subset\mathbb{C}^-\overset{\text{def}}{=}\{z\in\mathbb{C}\,|\,\operatorname{Re} z<0\}.$$

Definition 2.1. An $n\times n$ matrix X, with $\sigma(X)\subset\mathbb{C}^-$, is called *D-stable* if

$$\sigma(A-D)\subset\mathbb{C}^-,\qquad \text{for every } D=\operatorname{diag}(\mu_1,\ldots,\mu_n) \text{ with } \mu_i\geq 0.$$

LEMMA 2.2. *Every symmetric matrix A with $\sigma(A) \subset \mathbb{C}^-$ is D-stable.*

Proof. Let A be a symmetric $n \times n$ matrix. For an arbitrary diffusion matrix $D = \operatorname{diag}(\mu_1, \cdots, \mu_n)$, with $\mu_i \geq 0$, also the matrix $A - D$ is symmetric and consequently $\sigma(A - D) \subset \mathbb{R}$. Therefore it suffices to show that $\det(A - D) \neq 0$, for every $n \times n$ diffusion matrix D. (If $(A - D)$ has a positive eigenvalue then $\det(A - \mu D) = 0$ for some $\mu \in (0, 1)$).

Suppose $\det(A - \tilde{D}) = 0$ for a certain diffusion matrix $\tilde{D}$. This would imply the existence of a vector $v \neq 0$, so that $Av = \tilde{D}v$. Then $\langle Av, v\rangle = \langle \tilde{D}v, v\rangle \geq 0$, which gives a contradiction, since $v \mapsto \langle Av, v\rangle$ is a negative definite quadratic function. □

Remark. A matrix which is not D-stable is called D-unstable.

So for a pair (f, D) for which the system (‡) satisfies (P1), it is necessary for $(Df)|_0$ to be D-unstable.

Therefore we are going to characterize the D-stable 2×2- and 3×3 matrices. First we need some notation. Let Σ_i be the set whose elements are all subsets of $\{1, \ldots, n\}$ with cardinality i. Here n is fixed. For $\sigma = \{a_1, \ldots, a_i\} \in \Sigma_i$, we shall denote by A_σ an $(n - i) \times (n - i)$ matrix obtained from an $n \times n$ matrix A by deleting the a_kth row and column, $k = 1, \cdots, i$.

The next identity can be easily checked:

$$\frac{d^i}{d\lambda^i} \det(A - \lambda \operatorname{Id})|_{\lambda=0} = i!(-1)^i \sum_{\sigma \in \Sigma_{n-i}} \det A_\sigma. \tag{2.3}$$

So by writing the characteristic polynomial of A in the form:

$$(-1)^n \lambda^n + s_{n-1}\lambda^{n-1} + \cdots + s_1\lambda + s_0,$$

we have

$$s_i = (-1)^i \sum_{\sigma \in \Sigma_{n-i}} \det A_\sigma, \qquad i = 0, \ldots, n-1. \tag{2.4}$$

Before we formulate a proposition on D-stable matrices there are a few lemmas we need.

LEMMA 2.5. *Let A be an $n \times n$ matrix and $\sigma \in \Sigma_1$. If* $\det(A - D) \leq 0$ $[\geq 0]$ *for every $n \times n$ diffusion matrix D, then* $\det(A_\sigma - \tilde{D}) \geq 0$ $[\leq 0]$ *for every $(n-1) \times (n-1)$ diffusion matrix $\tilde{D}$.*

Proof. We prove the lemma by contradiction. Let A be an $n \times n$ matrix with $\det(A - D) \leq 0$, for every $n \times n$ diffusion matrix D. Suppose that $\det(A_\sigma - \tilde{D}) < 0$, for certain $\sigma \in \Sigma_1$ and $\tilde{D} = \operatorname{diag}(\mu_1, \cdots, \mu_{n-1})$. Without loss of generality we may assume that $\sigma = \{n\}$. Now let $D(\mu) = \operatorname{diag}(\mu_1, \cdots, \mu_{n-1}, \mu)$, $\mu \geq 0$. In that case we have

$$\det(A - D(\mu)) = \text{const} + k(a_{nn} - \mu), \qquad \text{with } k = \det(A_\sigma - \tilde{D}) < 0.$$

So for μ big enough $\det(A - D(\mu)) > 0$. This is the required contradiction which proves the lemma for $\det(A - D) \leq 0$. The proof in the other case is similar. □

LEMMA 2.6. *Let A be an $n \times n$ matrix with $\sigma(A) \subset \mathbb{C}^-$, then for $i = 0, \cdots, n-1$ the following holds:*

$$s_i \begin{cases} < 0, & \text{if } n \text{ is odd} \\ > 0, & \text{if } n \text{ is even.} \end{cases}$$

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Proof. Let the set of n eigenvalues of A be given by $\{\lambda_1, \cdots, \lambda_n\}$. The λ_i are roots of the characteristic equation of A, so

$$\det(A-\lambda\,\mathrm{Id}) = \prod_{i=1}^{n} (\lambda_i - \lambda).$$

We obtain, using the notation of above,

$$s_i = (-1)^i \sum_{\sigma\in\Sigma_{n-i}} \lambda_{\sigma_1}\cdot\,\cdots\,\cdot\lambda_{\sigma_{n-i}}, \quad \text{with } \sigma=\{\sigma_1,\cdots,\sigma_{n-i}\}.$$

If all eigenvalues of A are real, then the conclusion of the lemma is obvious. Now suppose that A has one pair $(\lambda, \bar\lambda)$ of complex eigenvalues. Then we determine the sign of the sum above by combining the right terms $\lambda_{\sigma_1}\cdot\,\cdots\,\cdot\lambda_{\sigma_{n-i}}$.

Terms containing both λ and $\bar\lambda$ are comparable with terms only consisting of negative reals because $\lambda\cdot\bar\lambda>0$. A term which contains λ but not $\bar\lambda$, we combine with the term in the sum which equals it, except that λ is replaced by $\bar\lambda$. Adding these two terms gives a term which is the product of negative reals. This finishes the proof in the case of one pair of complex eigenvalues. If A has more complex eigenvalues the proof is analogous. □

So for matrices B, satisfying the premisses of Lemma 2.5 and Lemma 2.6, either $\det B_\sigma$ is non-negative for every $\sigma\in\Sigma_i$, i fixed, or $\det B_\sigma$ is non-positive for all σ. Further there is at least one $\sigma\in\Sigma_i$ for which $\det B_\sigma$ differs from zero.

For an $n\times n$ matrix X we introduce the following notation, as far as defined: let

$$t(X) = \operatorname{sgn}(n-1)\operatorname{sign}(n-2)\cdots\operatorname{sgn}(0),$$

where we write for sgn(i) + or − according to the following convention:

$$\operatorname{sgn}(i) \text{ is } \begin{cases} +, & \text{if } \exists\bar\sigma\in\Sigma_i \text{ with } \det X_{\bar\sigma}>0 \text{ and further } \det X_\sigma\ge 0, \forall\sigma\in\Sigma_i \\ -, & \text{if } \exists\bar\sigma\in\Sigma_i \text{ with } \det X_{\bar\sigma}<0 \text{ and further } \det X_\sigma\le 0, \forall\sigma\in\Sigma_i. \end{cases}$$

Combining the previous results we obtain:

Proposition 2.7. *Let A be a D-stable $n\times n$ matrix, then A satisfies the following property*:

$$t(A) = \begin{cases} -+-\cdots+, & \textit{if n is even} \\ -+-\cdots-, & \textit{if n is odd.} \end{cases}$$

Proof. In the case n is even, lemma 2.6 implies:

$$s_i>0 \qquad \text{for } i=0,\cdots,n-1.$$

By Lemma 2.5 we know that $\det A_\sigma\ge 0$ (≤ 0) if and only if i is even(odd), where $\sigma\in\Sigma_i$. This together with (2.4) proves the proposition for n is even. The proof for n odd is analogous. □

For $n\le 3$ the property mentioned in Proposition 2.7 is even necessary and sufficient for A with $\sigma(A)\subset\mathbb{C}^-$ to be D-stable. In the case $n=1$ this is trivial; for $n=3$ we prove it below. The simpler proof for $n=2$ is left to the reader.

Proposition 2.8. *If A is a 3×3 matrix with $t(A)$ defined and*

(i) $\sigma(A)\subset\mathbb{C}^-$;

(ii) $t(A)=-+-$;

then A is D-stable.

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Proof. Let A satisfy the two properties mentioned in the proposition Write $A^\mu = A - D(\mu)$, with $\mu = (\mu_1, \mu_2, \mu_3)$ and $\mu_i \geq 0$. We need to prove that

$$(*) \qquad \sigma(A^\mu) \cap \text{imaginary axis} = \varnothing, \qquad \forall \mu.$$

Using $t(A) = -+-$, we get by straightforward calculation

$$\det(A^\mu) \leq \det A < 0, \qquad \forall \mu.$$

Thus $0 \notin \sigma(A^\mu)$, for all $\mu = (\mu_1, \mu_2, \mu_3)$.

Now suppose that for certain $\mu = s$, the matrix A^s has a couple of purely imaginary eigenvalues. Then the remaining real eigenvalue of A^s is $\operatorname{tr} A^s$. For an arbitrarily chosen μ the eigenvalues of A^μ are the roots of the characteristic equation of A^μ:

$$0 = -\lambda^3 + \lambda^2 \operatorname{tr} A^\mu - \lambda \sum_{\sigma \in \Sigma_1} \det A^\mu_\sigma + \det A^\mu \stackrel{\text{def}}{=} \Phi^\mu(\lambda).$$

From $\sigma(A) \subset \mathbb{C}^-$ we conclude that $\Phi^0(\operatorname{tr} A^0) = \Phi^0(\operatorname{tr} A) > 0$. For if $\Phi^0(\operatorname{tr} A) \leq 0$ then the polynomial Φ^0 would have a zero on the interval $(-\infty, \operatorname{tr} A]$. In that case Φ^0 must also have a non-negative zero, because $\operatorname{tr} A$ equals the sum of the zeros of Φ^0. This contradicts the assumption $\sigma(A) \subset \mathbb{C}^-$.

Further we have

$$\frac{\partial}{\partial \mu_i} \Phi^\mu(\operatorname{tr} A^\mu) > 0, \qquad i = 1, 2, 3.$$

For instance, if we denote $\det A^\mu_\sigma$ by Δ^μ_i, where $\sigma = \{i\}$, then

$$\frac{\partial}{\partial \mu_1} \Phi^\mu(\operatorname{tr} A^\mu) = \frac{\partial}{\partial \mu_1} \left(-\operatorname{tr} A^\mu \sum_{i=1}^{3} \Delta^\mu_i + \det A^\mu \right)$$
$$= \Delta^\mu_2 + \Delta^\mu_3 + (a_{22} - \mu_2 + a_{33} - \mu_3) \cdot \operatorname{tr} A^\mu > 0.$$

This is because of the property $t(A) = -+-$, by which

$$\Delta^\mu_k, -a_{kk} > 0, \qquad k = 1, 2, 3.$$

Hence $\Phi^\mu(\operatorname{tr} A^\mu) > 0$, for every $\mu = (\mu_1, \mu_2, \mu_3)$ with $\mu_i \geq 0$. Here we reach a contradiction because $\Phi^s(\operatorname{tr} A^s) = 0$. Therefore assertion $(*)$ is true and the proposition is proved. □

Conjecture. In the case $n \geq 4$, the property from Proposition 2.7 is also necessary and sufficient for A, with $\sigma(A) \subset \mathbb{C}^-$, to be D-stable.

Before we give an example of (‡), satisfying (P1) and (P2), we state the next useful lemma.

LEMMA 2.9. *Let A be a D-stable $n \times n$ matrix and suppose there exists a constant $c > 0$ such that f, occurring in (‡), satisfies:*

$$\|Ax - f(x)\| \leq c, \qquad \textit{for all } x \in \mathbb{R}^n.$$

Then for every $n \times n$ diffusion matrix D, the system (‡) has bounded orbits in the positive time direction.

Proof. We study the system (‡) for the pair (f, D) where D is an arbitrary diffusion matrix. The vector field X associated with (‡) can be written as:

$$X(x) = B(x) + C(x) = \sum_{1 \leq i,j \leq 2n} b_{ij} x_j \frac{\partial}{\partial x_i} + \sum_{1 \leq i \leq 2n} C_i(x) \frac{\partial}{\partial x_i}.$$

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Here

$$(b_{ij})_{1\le i,j\le 2n} = \begin{pmatrix} A-D & D \\ D & A-D \end{pmatrix}$$

and the functions C_i are uniformly bounded. Because A is D-stable, there exists a quadratic Lyapunov function $\mathscr{L}$, so that $B(\mathscr{L})<0$ and $B(\mathscr{L})$ is quadratic. For this function $\mathscr{L}$,

$$C(\mathscr{L}) = \sum_{1\le i\le 2n} C_i(x)\frac{\partial \mathscr{L}}{\partial x_i}$$

and $\partial\mathscr{L}/\partial x_i$ is only linear. So for x with $\|x\|$ sufficiently big, $X(\mathscr{L})(x)<0$. This gives the conclusion of the lemma. □

Remark. For every matrix B with $\sigma(B)\subset\mathbb{C}^-$ there exists a set W of diffusion matrices so that $\sigma(B-2D)\subset\mathbb{C}^-$ for $D\in W$. Lemma 2.9 can obviously be reformulated for such a matrix B, restricting the choice of D to the set W.

The following conditions on f are imposed to make the study of the dynamics of (‡) easier:
We require that f is odd. Then Δ and $\Delta^\perp$ are invariant under the flow of (‡), where Δ and $\Delta^\perp$ are given by:

$$\Delta = \{(x^1, x^2)\in\mathbb{R}^{2n} \mid x^1 = x^2\}$$
$$\Delta^\perp = \{(x^1, x^2)\in\mathbb{R}^{2n} \mid x^1 = -x^2\}.$$

We assume f to be injective as well, then the singularities of (‡) are all contained in $\Delta^\perp$.

3. *An example in dimension* 2

We now show the existence of a dynamical system (‡), see the introduction, on $\mathbb{R}^4$ which satisfies the properties (P1) and (P2):

(P1) the system (‡) has no point attractors;

(P2) the orbits of (‡) are bounded in the positive time direction.

The dynamical system f on $\mathbb{R}^2$ we shall take, describing the dynamics for each cell apart, is quite simple. It is linear with a disturbance in a vertical strip around the y-axis. More precisely, except for the vertical strip $S=\{(x,y)\in\mathbb{R}^2 \mid |x|\le 3\}$, f is linear:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix} -c & 8 \\ -4.1 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = A\begin{pmatrix}x\\ y\end{pmatrix},$$

with $c = 4.1+\frac{4}{3}\sin 3 \approx 4.29$.

On the strip S the dynamics is given by

$$\begin{cases} \dot{x} = -4\sin x - 4.1x + 8y \\ \dot{y} = -4.1x + 4y \end{cases}$$

With this choice f is odd and injective. Because the dynamics for the system (‡) appears to be persistent for this f, the example can be made C^1 by perturbing f a little. Now f is fixed, it is only the appropriate diffusion matrix that further matters. For the moment we take the diffusion matrix of the form

$$\operatorname{diag}(\lambda, 0) \qquad \text{with } \lambda \ge 0.$$

We use Lemma 2.9 to conclude that for certain matrices of this form, the system (‡) satisfies (P2).

We have

$$\|Ax - f(x)\| \le \max_{|t|\le 3} |4 \sin t - t\tfrac{4}{3} \sin 3|, \qquad \text{for all } x \in \mathbb{R}^2.$$

So $\|Ax - f(x)\|$ is uniformly bounded on $\mathbb{R}^2$. As $\operatorname{tr}(A-2D)<0$, it is not possible for the 2×2 matrix $A-2D$ to have two eigenvalues with non-negative real part. Therefore $\sigma(A-2D) \subset \mathbb{C}^-$ if and only if $\det(A-2D)>0$, which is equivalent to

$$\lambda < 2.05 - \tfrac{2}{3}\sin 3 = \lambda_4 \approx 1.96.$$

So by Lemma 2.9:

the system (‡) satisfies (P2) for $0 \le \lambda < \lambda_4$.

There remains property (P1). For that we take a look at the singularities of (‡) with $\lambda \in [0, \lambda_4)$. In particular 0 is a singularity of (‡) and

$$(Df)|_0 = \begin{pmatrix} -8.1 & 8 \\ -4.1 & 4 \end{pmatrix}.$$

By Proposition 2.7 this matrix is D-unstable, so the zero solution of (‡) can be destabilized. For this it is necessary and sufficient that

$$\det((Df)|_0 - 2D) < 0 \quad \text{which is equivalent to} \quad \lambda > 0.05 = \lambda_0.$$

So for the system (‡) to satisfy (P1) and (P2), it is necessary that $\lambda \in (\lambda_0, \lambda_4)$ We now check whether this condition is also sufficient.

As mentioned before f is odd, so $\Delta^\perp$ is invariant under the flow of (‡). In $\Delta^\perp$ we know of singularities of (‡) different from 0, for $\lambda \in (\lambda_0, \lambda_4)$. This follows directly from an index argument. To see this, assign to each singularity a number -1 or $+1$, the so-called index, whose value depends on the character of the singularity. The index of a saddle for instance is -1 and of an attractor $+1$. After we have chosen a suitable neighbourhood of 0 in $\Delta^\perp$, the sum of the indices of the singularities of (‡) in this neighbourhood must remain $+1$, for $\lambda \in [0, \lambda_4]$. As the zero solution destabilizes for $\lambda > \lambda_0$, the existence of singularities different from 0 follows for $\lambda \in (\lambda_0, \lambda_4)$. In fact there are two extra singularities in $\Delta^\perp$, as we shall see below. Considering these as singularities of the flow on $\Delta^\perp$, none of them is a saddle point. We remark that in the following, we often drop the variable λ for convenience.

Let $\lambda \in (\lambda_0, \lambda_4)$ and suppose $z \ne 0$ is a singularity of (‡). Because f is odd and injective, z is of the form $(p, -p)$, with p a singularity of $f-2D$. So it suffices to look for singularities $p=(x, y)$ of $f-2D$ and because f is odd we may assume $x>0$. It is clear that for $\lambda \in (\lambda_0, \lambda_4)$ the singularities of $f-2D$ are in the strip S.

We have that $p=(x, y)$ is a singularity of $f-2D$ if and only if

$$\begin{cases} (4.1-2\lambda)x = 4 \sin x \\ -4.1x + 4y = 0. \end{cases} \tag{3.1}$$

This system of equations has a unique pair of solutions, (x, y) and $(-x, -y) \ne (0, 0)$ depending on λ, provided $\lambda \in (\lambda_0, \lambda_4)$. We conclude:
for $\lambda \in (\lambda_0, \lambda_4)$ there is a unique singularity $(x(\lambda), y(\lambda))$ of $f-2D$ with $x(\lambda)>0$. The function $\lambda \mapsto x(\lambda)$ is monotonically increasing.

We next investigate whether for every $\lambda \in (\lambda_0, \lambda_4)$, the pair of singularities different from 0 are point attractors of (‡). By symmetry we only have to care about one singularity; we take the one with $x > 0$.

The eigenvalues of the linearized system at $(p, -p)$ are given by $\sigma(B) \cup \sigma(B - 2D)$ with

$$B = (Df)|_p = \begin{pmatrix} -4\cos x - 4.1 & 8 \\ -4.1 & 4 \end{pmatrix}.$$

For B we have $\det B > 0$ and $\operatorname{tr} B = -4\cos x - 0.1$. So

$$\operatorname{tr} B = 0 \text{ if and only if } x = \pi - \arccos \tfrac{1}{40} = x_1. \tag{3.2}$$

To x_1 corresponds a unique $\lambda_1 \in (\lambda_0, \lambda_4)$ such that $x_1 = x(\lambda_1)$, see above.

The eigenvalues of B, which depend on λ, cross the imaginary axis with positive speed at $\lambda = \lambda_1 \approx 0.8$:

$$\frac{d}{d\lambda} \operatorname{tr} B|_{\lambda_1} = 4 \sin x_1 \cdot \frac{d}{d\lambda} x|_{\lambda_1} > 0.$$

So we can picture $\sigma(B)$ as follows:

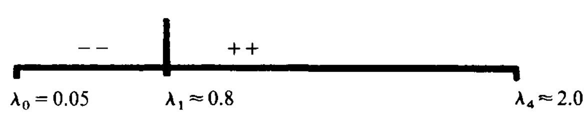


FIGURE 2. Dependence of $\sigma(B)$ on the variable λ. Here the sign − or + corresponds to an eigenvalue with negative or positive real part respectively.

By choosing a diffusion matrix with $\lambda \in (\lambda_1, \lambda_4)$, we already achieve that none of the stationary solutions of (‡) is a point attractor.

MAIN RESULT. *There exists a dynamical system f on $\mathbb{R}^2$ with a global point attractor, so that for a properly chosen D, the system* (‡) *has*

(i) *no point attractors*;

(ii) *bounded orbits in the positive time direction.*

Moreover the system (‡) is persistent, i.e. small perturbations of it, such as for instance a system with a slightly different diffusion matrix, also have the two properties mentioned above. Numerical observations give the existence of periodic attractors for the system (‡). For this see figure 7.

Of course we can exploit the dynamics of (‡) further. Assume D is still of the form $\operatorname{diag}(\lambda, 0)$, $\lambda \geq 0$. For knowing the local dynamics of the singularities completely, we also have to look at $\sigma(B - 2D)$. The details of this analysis are given in the appendix (§ I).

It results in the following:

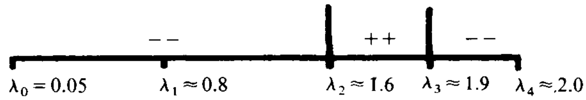


FIGURE 3. Dependence of $\sigma(B-2D)$ on the variable λ.

Thus it is even possible to force the singularities different from 0 to be repellers: just take $\lambda \in (\lambda_2, \lambda_3)$.

Finally if D is of the form $D = \operatorname{diag}(\lambda, \mu)$ with $\lambda, \mu \geq 0$, we get figure 4. For details see the appendix (§ II).

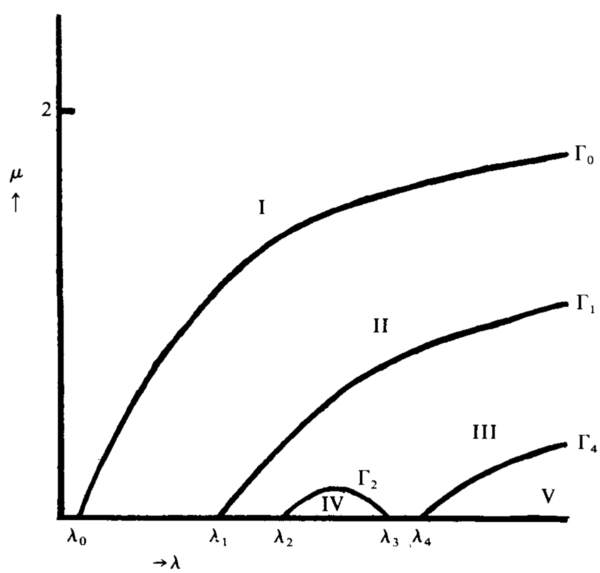


FIGURE 4. Various areas in the λ, μ-plane, all with different dynamics for (‡). For each area the sign of the real part of the eigenvalues of the linearized system at 0 and at one of the other singularities, if these at least exist, are given beneath. The first two eigenvalues concern internal behaviour in $\Delta^{\perp}$, the remaining pair normal behaviour. With the sign $-$ or $+$ corresponds an eigenvalue with negative or positive real part respectively.

I: $----$, 0 is the sole singularity of (‡).
II: $-+--$, $----$.
III: $-+--$, $++--$.
IV: $-+--$, $++++$.
V: $-+--$, 0 is the sole singularity of (‡) and there exist unbounded orbits in $\Delta^{\perp}$.

In the last part of this section we give some details about the dynamics of (‡) for a pair $(\lambda, \mu) \in$ IV (see figure 4). For such a pair the dynamics on Δ and $\Delta^{\perp}$ is simple. On Δ we have f dynamics; all orbits spiral towards 0, see figure 5.

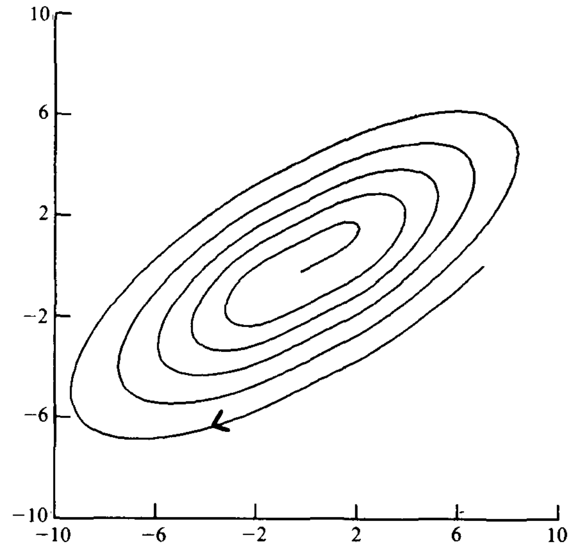


FIGURE 5. Dynamics of (‡) on Δ.

And a sketch of the phase portrait on $\Delta^{\perp}$ is the following:

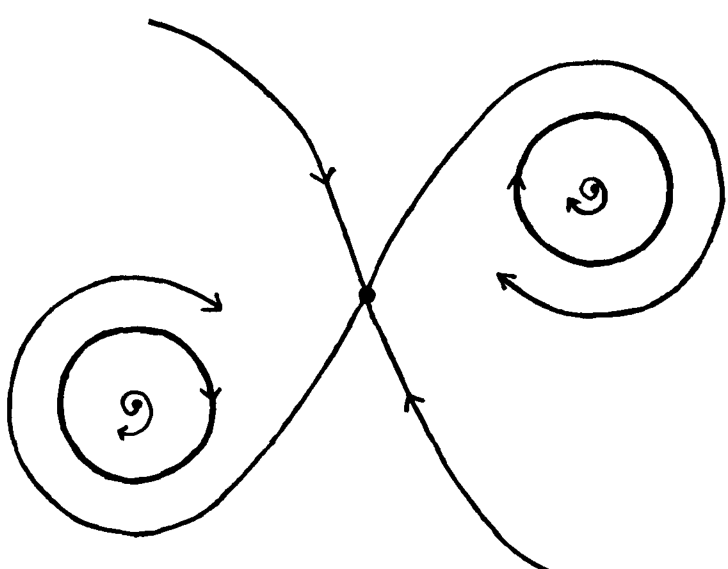
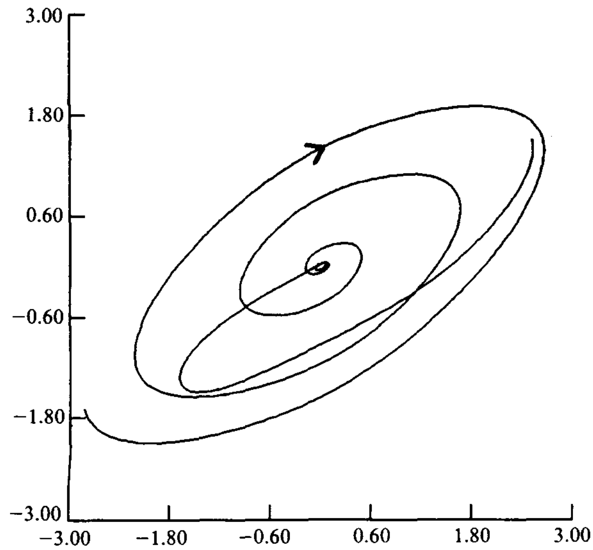


FIGURE 6. Dynamics of (‡) on $\Delta^{\perp}$ for a pair $(\lambda, \mu) \in$ IV.

Even with this we do not yet know the complete dynamics of (‡). However, we at least know that the orbits are bounded in the positive time direction and point attractors are absent. The system could have periodic behaviour: an attracting periodic orbit, created at Γ_1 by Hopf bifurcation and which survived the stronger diffusion. Also the system might exhibit a strange attractor. In order to get more information about the dynamics we did some numerical work. Therefore we took a set of initial values in $\mathbb{R}^4$ not lying on Δ or $\Delta^\perp$, and iterated these in forward time direction. Then the orbits were projected orthogonally onto Δ and $\Delta^\perp$. In figure 7 some parts of a projected orbit are shown.

(A)



(B)

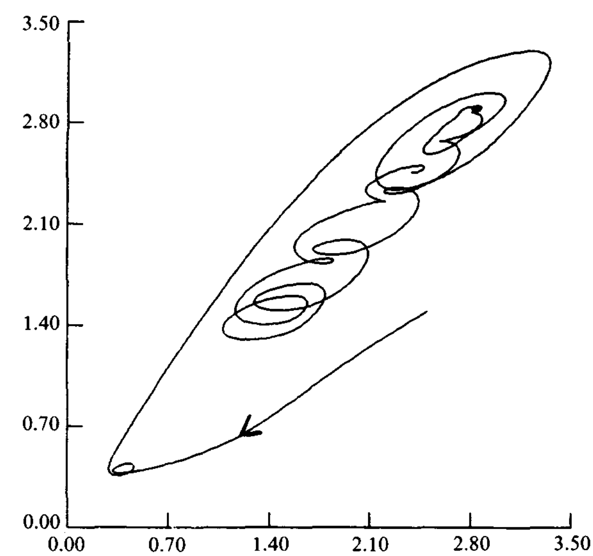


FIGURE 7. Some projected parts of an orbit with initial value (5, 3, 0, 0) in the case $D = \text{diag}(1.7, 0.005)$. The plots (A) (C) and (E) are projections onto Δ and (B), (D) and (F) are projections onto $\Delta^\perp$.

(A) $T = 0\text{–}9$ (B) $T = 0\text{–}15$
(C) $T = 30\text{–}35$ (D) $T = 30\text{–}35$
(E) $T = 110\text{–}115$ (F) $T = 110\text{–}112$

Notice the long stabilization time compared with the period of the closed orbit. This is probably due to the weak character of the attractor. The system might be close to a T^2-bifurcation.

Other initial values gave at last the same two closed orbits, or by symmetry of (‡) the two orbits rotated 180° around 0 in Δ and $\Delta^{\perp}$. Moreover different projections showed also two closed orbits.

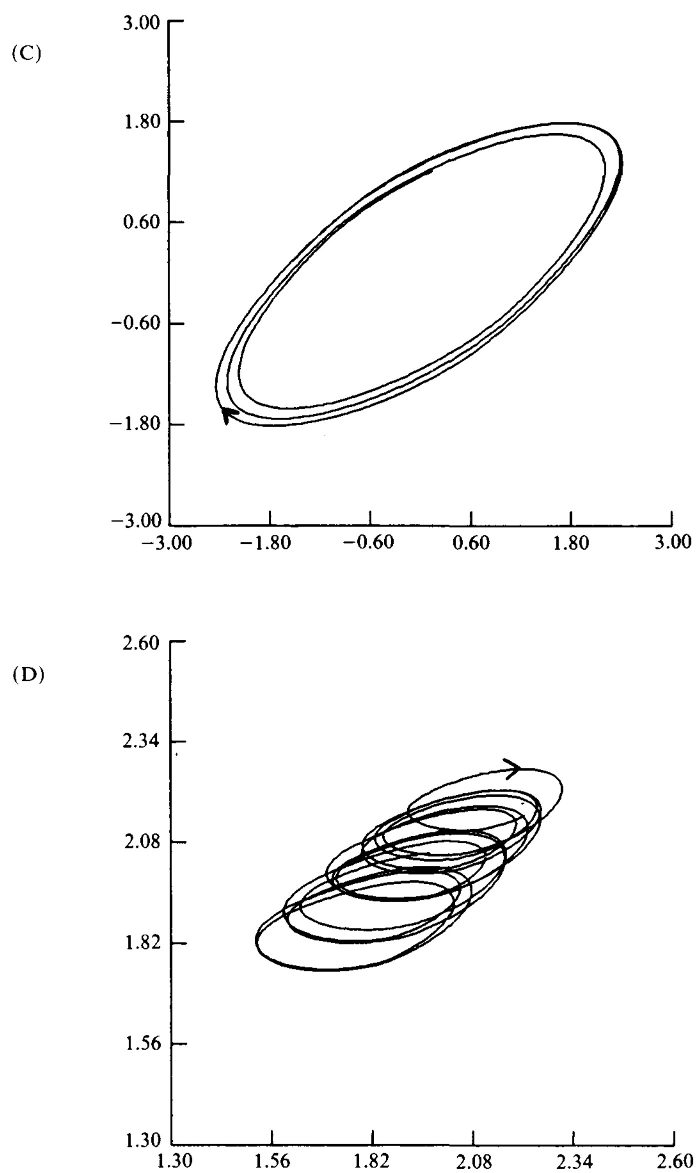


FIGURE 7 (continued).

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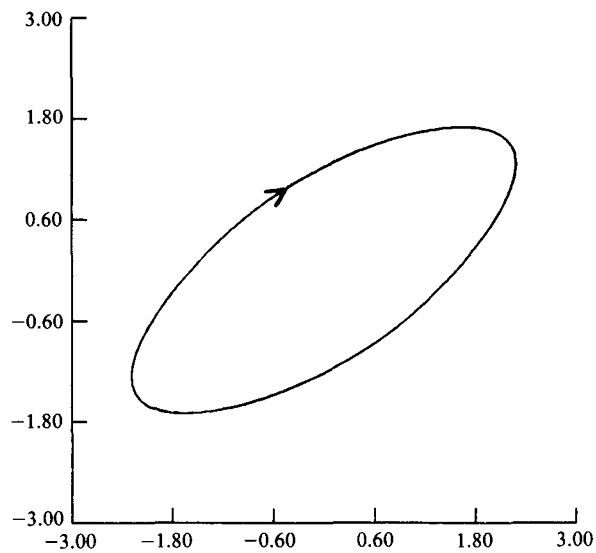
(E)



(F)

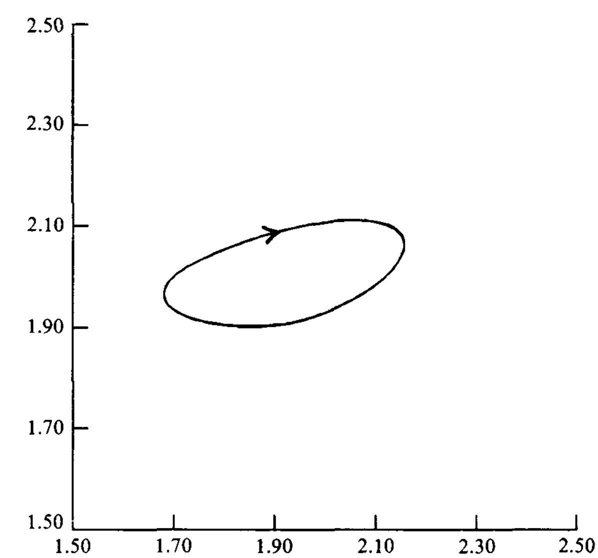


FIGURE 7 (continued).

So there is numerical evidence of the existence of a periodic attractor for (‡). This result seems to hold for every (λ, μ) between the arcs Γ_1 and Γ_4.

4. *Appendix*

I. In this part we study $\sigma(B-2D)$. We assume D is of the form $\mathrm{diag}(\lambda, 0)$, $\lambda \geq 0$. Recall that $B = (Df)|_p$, with p a singularity of $f-2D$. Let $p=(x, y)$ with $0<x<3$,

p depending on λ. We have

$$\operatorname{tr}(B) \geq \operatorname{tr}(B-2D) = -4\cos x - 0.1 - 2\lambda = -4\cos x - 4.2 + \frac{4\sin x}{x},$$

see (3.1). Define a function g on $(0, 3)$ by

$$g(x) = -4\cos x - 4.2 + \frac{4\sin x}{x}.$$

Regarding both singularities of (‡) different from 0 as singularities of the flow on $\Delta^{\perp}$, none of them is a saddle point. So the zeros of g correspond with purely imaginary eigenvalues of $B-2D$. We already know that $g<0$ on $(0, x_1)$, see (3.2). The function φ with $\varphi(x) = (\sin x)/x$ is decreasing on $(0, \pi)$, so for $3 > x > x_1 > 0$ we get

$$g(x) < 4\varphi(x_1) - 4\cos x - 4.2 < -4\cos x - 1.68$$

and consequently $g<0$ on $(0, 2)$. Repeating this procedure, we eventually obtain $g<0$ on $(0, 2.3]$.

The function g certainly has zeros on $(0, 3)$. For instance $g\,(2.7)>0$ and $g\,(3)<0$. In fact we have:

LEMMA 4.1. *The function g has exactly two zeros on* $(0, 3)$.

Proof. It suffices to show that g' has at most one zero on $(2.3, 3)$. Differentiating g we get

$$g'(x) = -\frac{4\cos x}{x^2}(tgx - x^2 tgx - x), \qquad x \in (2.3, 3).$$

For zeros of g' on $(2.3, 3)$ we only need to look at the expression between parentheses.

Define h by $h(x) = tgx - x^2 tgx - x$. Then for $x \in (2.3, 3)$:

$$h'(x) = \frac{1-x^2}{\cos^2 x} - 2xtgx - 1 < 1 - x^2 - 2xtgx - 1 = -x(x + 2tgx) < 0,$$

because $tg\,(2.3) > -1.12$.

So h is monotonically decreasing on $(2.3, 3)$ and thus g' has at most one zero on $(2.3, 3)$. □

With the two zeros of g, which we denote by x_2 and x_3, correspond two unique and different values for λ, $\lambda_2 \approx 1.62$ and $\lambda_3 \approx 1.91$ respectively. This concludes the analysis leading to figure 3.

II. We now take diffusion matrices of the form

$$\operatorname{diag}(\lambda, \mu) \qquad \text{with } \lambda, \mu \geq 0.$$

A diffusion matrix D destabilizes the zero solution of (‡) if and only if

$$\det((Df)|_0 - 2D) < 0, \quad \text{which is equivalent to} \quad 4\lambda(\mu - 2) + 0.4 < 0.$$

As the system (‡) has to satisfy (P1) it follows that $0 \leq \mu < 2$.

Let Γ_0 be the graph of γ_0, where

$$\gamma_0(\mu) = \frac{0.1}{2-\mu}, \qquad 0 \leq \mu < 2,$$

so $\gamma_0(0) = \lambda_0$.

For (λ, μ) on the left of Γ_0, 0 is a point attractor of (‡), and for (λ, μ) on the right the zero solution of (‡) is unstable.

For the system (‡) to have bounded orbits it is necessary and sufficient that

$$\det(A-2D)>0$$

which is equivalent to

$$(4\lambda+2c)(\mu-2)+32.8>0, \qquad c=4.1+\tfrac{4}{3}\sin 3.$$

Let Γ_4 be the graph of γ_4, where

$$\gamma_4(\mu)=\frac{8.2}{2-\mu}-\frac{c}{2}, \qquad 0\le\mu<2.$$

As the system (‡) must satisfy (P1) and (P2) we thus necessarily have to choose (λ, μ) between the arcs Γ_0 and Γ_4. Choosing (λ, μ) in that way gives rise to a pair of singularities besides 0. For if we take $D=\operatorname{diag}(\lambda, \mu)$, then $p=(x, y)\neq 0$ is a singularity of $f-2D$ if and only if

$$\begin{cases} -4\sin x+\left(\dfrac{16.4}{2-\mu}-4.1-2\lambda\right)x=0 \\ -4.1x+(4-2\mu)y=0 \end{cases} \tag{5.2}$$

with $\lambda\ge 0$, $0\le\mu<2$.

From this the existence of the two singularities easily follows. To investigate their stability we linearize in $(p, -p)$. The eigenvalues of the linearized system are given by

$$\sigma(B)\cup\sigma(B-2D)$$

with

$$B=(Df)|_p=\begin{pmatrix} -4\cos x-4.1 & 8 \\ -4.1 & 4\end{pmatrix}.$$

We already know that $\operatorname{tr} B=0$ if and only if $x=x_1=\pi-\arccos\frac{1}{40}$. Substituting this value for x in (5.2), we define Γ_1 to be the graph of γ_1 where

$$\gamma_1(\mu)=\frac{8.2}{2-\mu}-2.05-\frac{2\sin x_1}{x_1}, \qquad 0\le\mu<2.$$

Then for (λ, μ) on Γ_1 we have $\operatorname{tr} B=0$. It is easily checked that $\gamma_0'<\gamma_1'=\gamma_4'$. This fixes the mutual position of Γ_0, Γ_1 and Γ_4.

There remains the arc Γ_2. Let Γ_2 be the arc on which $\operatorname{tr}(B-2D)=0$. So for (λ, μ) on Γ_2, we get by (5.2) that $p=(x, y)$ must satisfy

$$-4\cos x+4+\frac{4\sin x}{x}-\frac{16.4}{2-\mu}-2\mu=0,$$

or equivalently

$$g(x)=2\left(\mu+\frac{8.2}{2-\mu}\right)-8.2, \tag{5.3}$$

with g as defined in § I. This implies that

$$\Gamma_2\subset\{(\lambda, \mu)\in\mathbb{R}^2 \mid \lambda\ge 0 \text{ and } 0\le\mu\le\hat{\mu}\},$$

where $\hat{\mu}$ is uniquely determined by

$$2\left(\hat{\mu}+\frac{8.2}{2-\hat{\mu}}\right)-8.2=\max_{t\in(0,3)} g(t).$$

And for every $\mu\in[0,\hat{\mu}]$ there exist x_l^μ and x_r^μ satisfying (5.3) and $x_2\le x_l^\mu\le x_r^\mu\le x_3$. In particular $x_l^{\hat{\mu}}=x_r^{\hat{\mu}}$, $x_l^0=x_2$ and $x_r^0=x_3$.

For a solution (x, y) of (5.2) we obtain by differentiating (5.2)

$$\frac{\partial x}{\partial\lambda}>0 \quad\text{and}\quad \frac{\partial x}{\partial\mu}<0. \tag{5.4}$$

Write $x_l^\mu=x(\lambda_l(\mu),\mu)$ and $x_r^\mu=x(\lambda_r(\mu),\mu)$ $\mu\in[0,\hat{\mu}]$, we have by (5.4)

$$\lambda_l(\mu)\le\lambda_r(\mu),\qquad \mu\in[0,\hat{\mu}].$$

Now let $0\le\mu<\bar{\mu}<\hat{\mu}$, then

$$x(\lambda_l(\mu),\mu)<x(\lambda_l(\bar{\mu}),\bar{\mu}),$$

and by (5.4) we conclude $\lambda_l(\mu)<\lambda_l(\bar{\mu})$. In a similar way from $0\le\mu<\bar{\mu}\le\hat{\mu}$ it follows that

$$\lambda_r(\mu)>\lambda_r(\bar{\mu}).$$

It is obvious that $\Gamma_1\cap\Gamma_2=\emptyset$. Thus we have dealt with the arc Γ_2 as well. This explains figure 4.

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