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L-Functoriality for Local Theta Correspondence of Supercuspidal Representations with Unipotent Reduction

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Abstract. The preservation principle of local theta correspondences of reductive dual pairs over a p-adic field predicts the existence of a sequence of irreducible supercuspidal representations of classical groups. Adams and Harris–Kudla–Sweet have a conjecture about the Langlands parameters for the sequence of supercuspidal representations. In this paper we prove modified versions of their conjectures for the case of supercuspidal representations with unipotent reduction.

1 Introduction

1.1 Let F be a non-archimedean local field of odd residual characteristic, (G, G') a reductive dual pair over F consisting of either (1) two unitary groups (with respect to a quadratic extension D/F); or (2) an even orthogonal group and a symplectic group. There exists a one-to-one correspondence (called the *local theta correspondence*) between some irreducible admissible representations of G and some irreducible admissible representations of G' with respect to a fixed splitting of the metaplectic cover of $G \times G'$ (*cf.* [MVW87, Wal90]).

It is very interesting to know how the Langlands parameters of the paired representations are related. In [Ral82], Rallis matches the Satake parameters of the unramified representations for an unramified reductive dual pair. In [Aub91], Aubert matches the parameters of the representations with non-trivial vectors fixed by an Iwahori subgroup for the case of a split reductive dual pair of the similar size. The representations considered by Rallis and Aubert are all in the principal series, but in this paper we focus on the correspondence of certain supercuspidal representations. However, all the representations considered in [Ral82], [Aub91] and here all belong to the class called *representations with unipotent reduction* and characterized by Lusztig [Lus95, Lus02]. (Lusztig called these representations *unipotent*. Here we follow the terminology of Mœglin–Waldspurger in [MW03].)

1.2 We now describe our result in more detail. One of the most interesting phenomena of the local theta correspondence is the so-called *preservation principle* or *conservation relation*. More precisely, denote G by $G(V)$ if G is the group of isometries of

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the (quadratic, symplectic or Hermitian) space V. For a given quadratic (symplectic, or Hermitian, respectively) space V, we have two Witt towers $\{V'^{+}\}$ and $\{V'^{-}\}$ of symplectic (quadratic, or skew-Hermitian, respectively) spaces, called the *related Witt towers* (*cf.* Subsections 2.3). Suppose that π is an irreducible supercuspidal representation of $G(V)$. Let $n'^{\pm}(\pi)$ denote the minimal dimensions of $V'^{\pm}$ varying in their respective Witt towers such that π occurs in the theta correspondence for the reductive dual pairs $(G(V), G(V'^{\pm}))$. Then it is known in [KR05, HKS96] that

$$n'^{+}(\pi) + n'^{-}(\pi \otimes \mathrm{sgn}) = 2\dim(V) + \begin{cases} 0, & \text{if } G(V) \text{ is even orthogonal,} \\ 2, & \text{if } G(V) \text{ is unitary,} \\ 4, & \text{if } G(V) \text{ is symplectic,} \end{cases} \tag{1.1}$$

where "sgn" denotes the sign character (*cf.* Subsection 2.1) of the group $G(V)$. Note that the formulation here is slightly different from the original one in [HKS96] for unitary group cases due to the choice of different splittings of the metaplectic covers.

If we start with any irreducible supercuspidal representation π of a group $G(V)$, then we can obtain two irreducible supercuspidal represnetations π'^{+} and π'^{-} of $G(V'^{+})$ and $G(V'^{-})$, respectively, such that $\pi \otimes (\pi'^{\pm} \otimes \mathrm{sgn})$ first occurs in the theta correspondence for the dual pair $(G(V), G(V'^{\pm}))$. Then we switch the roles of V and $V'^{\pm}$ and repeat the same process. We will obtain a sequence of spaces $\{V_i\}$ (indexed by $\mathbb{Z}$) and irreducible supercuspidal representations π_i of $G(V_i)$ such that $\pi_i \otimes (\pi_{i+1} \otimes \mathrm{sgn})$ is a first occurrence of irreducible supercuspidal representations for the reductive dual pair $(G(V_i), G(V_{i+1}))$ for each $i \in \mathbb{Z}$. Note that the spaces V_i and V_j are in the same Witt tower if $i \equiv j \pmod 4$. From (1.1) we have the relation $\dim(V_{i-1}) + \dim(V_{i+1}) = 2\dim(V_i) + \delta_i$, where δ_i is 0, 2, or 4 depending on the space V_i. Hence, we can and will normalize the index i such that

$$\cdots > \dim(V_{-2}) > \dim(V_{-1}) \geq \dim(V_0) \leq \dim(V_1) < \dim(V_2) < \cdots$$

(*cf.* Subsections 2.3).

1.3 Let W_F denote the Weil group of F and let ${}^L G = {}^\vee G \rtimes W_F$ be the L-group of G where ${}^\vee G$ denotes the complex dual group of G. Suppose that π is an irreducible admissible representation of G. According to the local Langlands conjecture ([Bor79]), there should be a unique (up to conjugation) homomorphism (called the *Langlands parameter*)

$$\varphi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L G$$

satisfying certain conditions associated with π. Let $\theta(\pi)$ denote the irreducible admissible representation of G' paired with π in the theta correspondence for the reductive dual pair (G, G'). Adams [Ada89] makes a conjecture on the Langlands parameter of $\theta(\pi)$ provided the Langlands parameter φ of π is known (see Conjecture 3.1). Suppose that the Langlands parameter φ_0 of π_0 in the sequence $\{\pi_i\}$ is known. Then Harris, Kudla, and Sweet [HKS96] make a conjecture on the Langlands parameter φ_i of π_i modified from Adams's conjecture (see Subsection 3.3).

A special class of irreducible admissible representations called *having unipotent reduction* are defined by Lusztig [Lus83]. If G is adjoint simple, Lusztig [Lus95, Lus02] constructs a bijection between (the isomorphism classes of) the irreducible admissible representations π with unipotent reduction and (the equivalence classes of) the refined *unramified* Langlands parameters φ (*i.e.,* the parameters φ whose restriction to the initial group of F is trivial). A unramified Langlands parameter is characterized by a pair (y, N) (up to conjugation) such that

- y is an semisimple element in the complex dual group ${}^\vee G$;
- N is a nilpotent element in the Lie algebra of ${}^\vee G$;
- $\mathrm{Ad}_y(N) = qN$,

where q is the cardinality of the residue field of F. Such a pair (y, N) will also be called the *Langlands parameter* of π. Following Lusztig's construction, we will write down the Langlands parameter (y, N) of an irreducible supercuspidal representation π with unipotent reduction of a classical group $G(V)$, which is not necessarily adjoint or even connected (*cf.* Section 6).

1.4 It is known that the representations with unipotent reduction are preserved by the local theta correspondence in some cases. In particular, for dual pairs of orthogonal-symplectic cases, if one of the representation in sequence $\{\pi_i\}$ has unipotent reduction, then every representation in the sequence has unipotent reduction. The case for unitary groups is a little more involved (*cf.* Subsection 4.4.1).

The aim of this article is to analyze the Langlands parameters of the sequence $\{\pi_i\}$ of supercuspidal representations with unipotent reductions. More precisely, for $k \in \mathbb{N}$, let ρ_k be the symmetric tensor representation of $\mathrm{SL}_2(\mathbb{C})$ on a k-dimensional complex vector space. Let $\mathrm{d}\rho_k\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}_k(\mathbb{C})$ be the associated representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Then we define

$$y_k = \rho_k\left(\begin{bmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2}\end{bmatrix}\right) \quad \text{and} \quad \delta_k = \mathrm{d}\rho_k\left(\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right]\right).$$

The main results (Theorems 5.3, 5.5, 5.7, and 5.9) of this paper are the following:

(I) Suppose that $G(V_0)$ is a unitary group and the dimensions of V_0, V_1 are of the same parity. Let (y_i, N_i) be the Langlands parameter for the supercuspidal representation $\pi_i \otimes \mathrm{sgn}^{((i+1)(i+2)+t_0)/2}$ with unipotent reduction if D/F is unramified; for π_i if D/F is ramified (where t_0 is given in (4.5)). Then we have

$$y_i = \mathrm{diag}(y_0, y_2, y_4, \dots, y_{2i^\sharp}) \quad \text{and} \quad N_i = \mathrm{diag}(N_0, \delta_2, \delta_4, \dots, \delta_{2i^\sharp}),$$

where $i^\sharp = \min(|i|, |i+1|)$.

(II) Suppose that $G(V_0)$ is a unitary group and the dimensions of V_0, V_1 are of the opposite parity. Then (y_i, N_i) for the representation in (5.4) where i odd is given by

$$y_i = \mathrm{diag}(y_0, y_1, y_3, \dots, y_{2|i|-1}) \quad \text{and} \quad N_i = \mathrm{diag}(N_0, \delta_1, \delta_3, \dots, \delta_{2|i|-1}).$$

(III) Suppose that $G(V_0)$ is an even orthogonal group. Then (y_i, N_i) for π_i is given by

$$y_i = \operatorname{diag}(y_0, \gamma_1, \gamma_3, \ldots, \gamma_{2|i|-1}) \quad \text{and} \quad N_i = \operatorname{diag}(N_0, \delta_1, \delta_3, \ldots, \delta_{2|i|-1}).$$

(IV) Suppose that $G(V_0)$ is a symplectic group. Then (y_i, N_i) for π_i is given by

$$y_i = \begin{cases} \operatorname{diag}(y_0, -\gamma_1, -\gamma_3, \ldots, -\gamma_{2|i|-1}), & \text{if } i \text{ is even,} \\ \operatorname{diag}(-y_0, \gamma_1, \gamma_3, \ldots, \gamma_{2|i|-1}), & \text{if } i \text{ is odd,} \end{cases}$$
$$N_i = \operatorname{diag}(N_0, \delta_1, \delta_3, \ldots, \delta_{2|i|-1}).$$

For the case of orthogonal-symplectic groups, the result is a consequence of a modification of Adams' conjecture (see Conjecture 3.2). Thus, the theorems imply that the modified Adams' conjecture holds when the representations are supercuspidal with unipotent reduction. However, for the case of unitary groups, Adams' conjecture (see Conjecture 3.1) and hence Harris–Kudla–Sweet's conjecture [HKS96, speculations 7.7 and 7.8] need to be modified slightly due to the fact that representations with unipotent reductions are not really preserved by theta correspondence.

1.5 The basic idea of the proof is very simple. The irreducible supercuspidal representation with unipotent reduction of G is so special and must be induced from an irreducible representation of an open compact subgroup, which is inflated from an irreducible unipotent cuspidal representation of a product of finite classical groups. In this situation, the Langlands parameter of the representation can be constructed explicitly. Also the analog of the preservation principle holds for finite reductive dual pairs (*cf.* (4.2)) and the correspondence of unipotent cuspidal representations for finite dual pairs is known (*cf.* [AM93]). Then Proposition 4.6 provides a compatibility result between the correspondence of supercuspidal representations with unipotent reduction for reductive dual pairs over a nonarchimedean local field and the correspondence of unipotent cuspidal representations for reductive dual pairs over its residue field. Finally, the relation of the Langlands parameter of the representations paired by theta correspondence is established.

1.6 The content of the paper is as follows. In Section 2, we give the notation and basic results of local theta correspondence, in particular, the preservation principle by Kudla and Rallis in [KR05]. The homomorphisms between L-groups are given in Section 3. The definition is adapted from [Ada89, KR94, HKS96]. In Section 4, we give the description of the sequence of irreducible cuspidal representations of finite classical groups in which unipotent representations occur. Proposition 4.6 is the key relation between the theta correspondence of unipotent cuspidal representations of finite classical groups and the theta correspondence of supercuspidal representations of p-adic classical groups. Then in Section 5 we state the main results of the paper, Theorems 5.3, 5.5, 5.7, and 5.9. In Section 6 we give an explicit description of the Langlands parameter (y, N) associated with an irreducible supercuspidal representation π with unipotent reduction of a classical group $G(V)$. The description follows closely the construction in [Lus95, Lus02, Mor96] for adjoint simple groups. In the final section, we give the proofs of the main theorems and provide a few examples.

2 Local Theta Correspondence

2.1 Basic Notation

Let F be a non-archimedean local field with odd residual characteristic, let $\mathbf{f}$ be the residue field of F, and let q be the cardinality of $\mathbf{f}$. We fix a nontrivial additive character ψ of F.

Let D be F itself or a quadratic extension E of F, let V a (non-degenerate) ϵ-Hermitian space (with $\epsilon = 1$ or -1) over D, and let $G(V)$ be the group of isometries. We consider the following cases:

(a) If $D = E$ and

  (a.1) if $\dim_E(V) = n$ is odd, then $G(V)$ is the unitary group $\mathrm{U}_n(F)$;

  (a.2) if $\dim_E(V) = n$ is even, then $G(V)$ is denoted by $\mathrm{U}^+_n(F)$ (resp. $\mathrm{U}^-_n(F)$) when the anisotropic kernel of V is trivial (resp. 2-dimensional).

  If E/F is a ramified, we always assume that V is even-dimensional.

(b) If $D = F$, $\epsilon = -1$ and $\dim(V) = 2n$, then $G(V)$ is the symplectic group $\mathrm{Sp}_{2n}(F)$.

(c) If $D = F$, $\epsilon = 1$, and $\dim(V) = 2n$, then the even orthogonal group $G(V)$ is denoted by $\mathrm{O}^+_{2n}(F)$ (resp. $\mathrm{O}^-_{2n}(F)$, $\mathrm{O}'_{2n}(F)$) when the anisotropic kernel of V is trivial (resp. four-dimensional, two-dimensional). For the case $G(V) = \mathrm{O}'_{2n}(F)$ we will assume that the center of the even Clifford algebra of V is an unramified quadratic extension of F.

If $G(V)$ is a nontrivial unitary group (resp. nontrivial orthogonal group), let sgn denote the linear character of $G(V)$ of order two whose restriction to the special unitary group (resp. orthogonal group) is trivial. If $G(V)$ is a symplectic group or the trivial group, let sgn be the trivial character.

2.2 Reductive Dual Pairs

Let $(G(V), G(V'))$ be a *reductive dual pair* over F where $G(V)$ and $G(V')$ are the classical groups considered in the previous subsection. The dual pair $(G(V), G(V'))$ splits; *i.e.,* there exists a splitting homomorphism

$$\beta\colon G(V) \times G(V') \longrightarrow \mathrm{Mp}(V \otimes_D V')$$

where $\mathrm{Mp}(V \otimes_D V')$ denotes the metaplectic cover of the symplectic group $\mathrm{Sp}(V \otimes_D V')$. Let ω_ψ be the Weil representation of $\mathrm{Mp}(V \otimes_D V')$ with respect to the nontrivial character ψ. We regard ω_ψ as a representation of $G(V) \times G(V')$ via the splitting β.

Suppose π is an irreducible admissible representation of $G(V)$. We write

$$\omega_\psi \Big/ \Big(\bigcap_{f \in \mathrm{Hom}_{G(V)}(\omega_\psi|_{G(V)}, \pi)} \ker(f) \Big) \simeq \pi \otimes \Theta(\pi, V')$$

for some $\Theta(\pi, V')$ equal to 0 or a smooth representation of $G(V')$. If $\Theta(\pi, V') \neq 0$, then it is a finitely generated representation of $G(V')$ with a unique irreducible quotient $\theta(\pi, V')$. The correspondence that is denoted by $\pi \leftrightarrow \theta(\pi, V')$ is a bijection between subsets of (isomorphism classes of) irreducible admissible representations of $G(V)$ and irreducible admissible representations of $G(V')$, which occurs in the correspondence [MVW87, Wal90].

The following basic facts for the theta correspondence are well known ([MVW87, Kud86]):

(a) If $\Theta(\pi, V') \neq 0$, then $\Theta(\pi, V' + V'_{k,k}) \neq 0$ for any $k \geq 1$.
(b) $\Theta(\pi, V'_{\mathrm{an}} + V'_{k,k}) \neq 0$ if $k \geq \dim_D(V)$.
(c) If π is supercuspidal, $\Theta(\pi, V'_{\mathrm{an}} + V'_{k-1,k-1}) = 0$ and $\Theta(\pi, V'_{\mathrm{an}} + V'_{k,k}) \neq 0$, then $\pi' := \theta(\pi, V'_{\mathrm{an}} + V'_{k,k})$ is also supercuspidal. Moreover, $\theta(\pi, V'_{\mathrm{an}} + V'_{l,l})$ is never supercuspidal if $l > k$.

Here V'_{an} is an anisotropic ϵ'-Hermitian space and $V'_{k,k}$ denotes the direct sum of k copies of ϵ'-Hermitian hyperbolic planes. The correspondence $\pi \leftrightarrow \pi'$ of supercuspidal representations in (c) is called a *first occurrence* (of supercuspidal representations).

2.3 Preservation Principle

From now on we consider the theta correspondence under the splitting with respect to a generalized lattice model of the Weil representation ω_ψ given in [Pan01] (see also [Pan02, section 11]).

For a fixed ϵ-Hermitian space V over D we consider the following *related Witt towers* $\{V'^{\pm}_{\mathrm{an}} + V'_{k,k} \mid k = 0, 1, 2, \ldots\}$:

(a) $D = E$ and V'^{+}_{an} (resp. V'^{-}_{an}) is the trivial (resp. a two-dimensional anisotropic) ϵ'-Hermitian space;
(b) $D = E$ and both V'^{+}_{an} and V'^{-}_{an} are one-dimensional anisotropic ϵ'-Hermitian spaces with non-isomorphic forms;
(c) $D = F$, $\epsilon = 1$, and both V'^{+}_{an} and V'^{-}_{an} are the trivial;
(d) $D = F$, $\epsilon = -1$, and the anisotropic space V'^{+}_{an} (resp. V'^{-}_{an}) is the trivial (resp. four-dimensional);
(e) $D = F$, $\epsilon = -1$, and both V'^{+}_{an} and V'^{-}_{an} are two-dimensional with non-isomorphic quadratic forms.

Let π be an irreducible supercuspidal representation of $G(V)$. Then $\pi \otimes \mathrm{sgn}$ is also irreducible supercuspidal. Let $n^{+}(\pi)$ (resp. $n^{-}(\pi \otimes \mathrm{sgn})$) denote the smallest dimension of $V'^{+}_{\mathrm{an}} + V'_{k,k}$ (resp. $V'^{-}_{\mathrm{an}} + V'_{k,k}$) in its Witt tower such that π (resp. $\pi \otimes \mathrm{sgn}$) occurs in the theta correspondence for the pair $(G(V), G(V'^{+}_{\mathrm{an}} + V'_{k,k}))$ (resp. $(G(V), G(V'^{-}_{\mathrm{an}} + V'_{k,k}))$). The following relation

$$n^{+}(\pi) + n^{-}(\pi \otimes \mathrm{sgn}) = 2\dim(V) + \begin{cases} 0 & \text{if } G(V) \text{ is even orthogonal,} \\ 2 & \text{if } G(V) \text{ is unitary,} \\ 4 & \text{if } G(V) \text{ is symplectic,} \end{cases} \tag{2.1}$$

is called the *preservation principle* (or *conservation relation*) (see [HKS96, KR05, Pan02]). The preservation principle suggests that there exist irreducible supercuspidal representations π_i of $G(V_i)$ for $i \in \mathbb{Z}$ such that π is isomorphic to π_{i_0} for some integer i_0 (the integer i_0 might not be unique), and $\pi_i \leftrightarrow \pi_{i+1} \otimes \mathrm{sgn}$ is a first occurrence in the local theta correspondence for the dual pair $(G(V_i), G(V_{i+1}))$ for $i \in \mathbb{Z}$.

Let n_i denote the dimension of V_i. We assume that π_0 does not come from a smaller group via the theta correspondence; *i.e.,* we normalize the index i such that

$\cdots > n_{-2} > n_{-1} \geq n_0 \leq n_1 < \cdots$. Then from (2.1), it is not difficult to see that we have the following cases:

(I) $n_i = n + i(i+1)$ with $n_{-1} = n_0 = n$ for some n, and $G(V_i)$ is unitary for each i;

(II) $n_i = n + i^2$ with $n_0 = n$ for some n, and $G(V_i)$ is unitary for each i;

(III) $n_i = n + i^2 - \frac{1+(-1)^{i+1}}{2}$ with $n_{-1} = n_0 = n_1 = n$ for some n, and $G(V_i)$ is orthogonal (resp. symplectic) if i is even (resp. odd); or

(IV) $n_i = n + i^2 + \frac{1+(-1)^{i+1}}{2}$ with $n_0 = n$ for some n, and $G(V_i)$ is symplectic (resp. orthogonal) if i is even (resp. odd).

Note that $n_i = n_{-i-1}$ for case (I), and $n_i = n_{-i}$ otherwise.

Remark 2.1 (i) For case (I) with $n = 0$, we have $n_{-1} = n_0 = 0$. The anisotropic kernel of V_i is zero-dimensional if $i \equiv 0, 3 \pmod 4$, and two-dimensional if $i \equiv 1, 2 \pmod 4$.

(ii) For case (II) with $n = 0$, we have $n_0 = 0$, $n_{-1} = n_1 = 1$. The anisotropic kernel of V_i is zero-dimensional if $i \equiv 0 \pmod 4$, two-dimensional if $i \equiv 2 \pmod 4$, and one-dimensional otherwise.

(iii) For case (III) with $n = 0$, we have $n_{-1} = n_0 = n_1 = 0$. The quadratic space V_i has trivial (resp. four-dimensional) anisotropic kernel if $i \equiv 0 \pmod 4$ (resp. $i \equiv 2 \pmod 4$).

(iv) For case (IV) with $n = 0$, we have $n_0 = 0$ and $n_{-1} = n_1 = 2$. The quadratic space V_i has two-dimensional anisotropic kernel for every odd i.

Let n_i^* denote the dimension of the complex vector space $\mathbf{V}_i$ such that $G(\mathbf{V}_i)$ is the complex dual group of $G(V_i)$. Then we have

$$n_i^* = \begin{cases} n_i, & \text{if } G(V_i) \text{ is unitary or even orthogonal;} \\ n_i + 1, & \text{if } G(V_i) \text{ is symplectic.} \end{cases} \tag{2.2}$$

It is easy to check that $n_i^* = n_0^* + i^2$ for cases (III) and (IV).

3 L-functoriality of Local Theta Correspondence

Let F, V, and $G(V)$ be defined as in Subsection 2.1.

3.1 L-groups

Let W_F be the Weil group of F, ${}^\vee G(V)$ the complex dual group of $G(V)$. The *L-group* ${}^L G(V) = {}^\vee G(V) \rtimes W_F$ of (the inner class of) $G(V)$ is defined with respect to the action of W_F on ${}^\vee G(V)$ described as follows (*cf.* [Ada89, section 3]).

(a) If $G(V) = \mathrm{U}_n(F)$ or $\mathrm{U}_n^{\pm}(F)$, the subgroup W_E of W_F acts trivially on ${}^\vee G(V) = \mathrm{GL}_n(\mathbb{C})$ and an element $w_\sigma \in W_F \smallsetminus W_E$ acts on $\mathrm{GL}_n(\mathbb{C})$ by $g \mapsto \Phi_n {}^{\mathrm{t}}g^{-1} \Phi_n^{-1}$, where

$$\Phi_n = \begin{bmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ (-1)^{n-1} & & & \end{bmatrix} \tag{3.1}$$

and ${}^{\mathrm{t}}g$ denotes the transpose of the matrix g.

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(b) If $G(V) = \mathrm{Sp}_{2n}(F)$, then the action of W_F on ${}^\vee G(V) = \mathrm{SO}_{2n+1}(\mathbb{C})$ is trivial and hence ${}^L G(V) = \mathrm{SO}_{2n+1}(\mathbb{C}) \times W_F$, the direct product.
(c) If $G(V) = \mathrm{O}^+_{2n}(F)$ or $\mathrm{O}^-_{2n}(F)$, then the action of W_F on ${}^\vee G(V) = \mathrm{O}_{2n}(\mathbb{C})$ is trivial and hence ${}^L G(V) = \mathrm{O}_{2n}(\mathbb{C}) \times W_F$.
(d) Suppose that $G(V) = \mathrm{O}'_{2n}(F)$. Recall that we assume that the center E of the even Clifford algebra of V is an unramified extension of F. The subgroup W_E acts trivially on ${}^\vee G(V) = \mathrm{O}_{2n}(\mathbb{C})$ and an element $w_\sigma \in W_F \smallsetminus W_E$ acts on $\mathrm{O}_{2n}(\mathbb{C})$ via $g \mapsto \Phi'_{2n} g \Phi'^{-1}_{2n}$, where

$$\Phi'_{2n} = \mathrm{diag}(-1, 1, \dots, 1) \in \mathrm{O}_{2n}(\mathbb{C}) \smallsetminus \mathrm{SO}_{2n}(\mathbb{C}). \tag{3.2}$$

3.2 Homomorphisms of L-groups

Let $(G(V), G(V'))$ be a reductive dual pair. Slightly modified from [HKS96, section 7] and [Ada89, section 4], the homomorphism α between L-groups ${}^L G(V)$ and ${}^L G(V')$ is defined as follows.

3.2.1 Unitary Cases

For positive integers $a_1, \dots, a_k$ we define

$$\Phi_{a_1, a_2, \cdots, a_k} = \mathrm{diag}(\Phi_{a_1}, \Phi_{a_2}, \dots, \Phi_{a_k}) = \begin{bmatrix} \Phi_{a_1} & & & \\ & \Phi_{a_2} & & \\ & & \ddots & \\ & & & \Phi_{a_k} \end{bmatrix}, \tag{3.3}$$

where Φ_{a_i} is given in (3.1).

Let n (resp. n') denote the dimension of V (resp. V') and suppose that $n \le n'$. Choose a pair of characters χ_1, χ_2 of $E^\times$ such that $\chi_1|_{F^\times} = \epsilon_{E/F}^{\dim(V')}$ and $\chi_2|_{F^\times} = \epsilon_{E/F}^{\dim(V)}$ where $\epsilon_{E/F}$ is the quadratic character with respect to the extension E/F. We regard χ_1, χ_2 as characters of W_E via the local class field theory. The map $\alpha\colon \mathrm{GL}_n(\mathbb{C}) \rtimes W_F \to \mathrm{GL}_{n'}(\mathbb{C}) \rtimes W_F$ is defined by

$$\begin{aligned} g \times w &\longmapsto \chi_2(w)\, \mathrm{diag}\big(\chi_1(w)^{-1} g, \mathrm{id}_{n'-n}\big) \times w \\ 1 \times w_\sigma &\longmapsto \Phi_{n, n'-n} \Phi_{n'}^{-1} \times w_\sigma, \end{aligned} \tag{3.4}$$

where $g \in \mathrm{GL}_n(\mathbb{C})$, $w \in W_E$, $w_\sigma \in W_F \smallsetminus W_E$ and $\mathrm{id}_{n'-n}$ denotes the identity matrix of size $n' - n$.

3.2.2 Split Orthogonal-symplectic Case

In this subsection we consider that case where V (resp. V') is a $2n$-dimensional (resp. $2n'$-dimensional) quadratic (resp. symplectic) space such that the anisotropic kernels of V is either trivial or four-dimensional. This situation is called the *split case*.

Suppose that $n \le n'$. We define $\alpha\colon \mathrm{O}_{2n}(\mathbb{C}) \times W_F \to \mathrm{SO}_{2n'+1}(\mathbb{C}) \times W_F$ by

$$g \times w \longmapsto \mathrm{diag}\big(g, \det(g)\, \mathrm{id}_{2(n'-n)+1}\big) \times w \tag{3.5}$$

for $g \in \mathrm{O}_{2n}(\mathbb{C})$ and $w \in W_F$.

Suppose that $n' < n$. We define $\alpha\colon \mathrm{SO}_{2n'+1}(\mathbb{C}) \times W_F \to \mathrm{O}_{2n}(\mathbb{C}) \times W_F$ by

$$g \times w \longmapsto \mathrm{diag}(g, \mathrm{id}_{2(n-n')-1}) \times w \tag{3.6}$$

for $g \in \mathrm{SO}_{2n'+1}(\mathbb{C})$ and $w \in W_F$.

3.2.3 Non-split Orthogonal-symplectic Case

Now we consider the case where V (resp. V') is a $2n$-dimensional (resp. $2n'$-dimensional) quadratic (resp. symplectic) space such that the anisotropic kernels of V is two-dimensional. This situation is called the *non-split case*.

Suppose that $n \le n'$. We define $\alpha\colon \mathrm{O}_{2n}(\mathbb{C}) \rtimes W_F \to \mathrm{SO}_{2n'+1}(\mathbb{C}) \times W_F$ by

$$\begin{aligned} g \times w &\longmapsto \mathrm{diag}(g, \det(g)\,\mathrm{id}_{2(n'-n)+1}) \times w \\ 1 \times w_\sigma &\longmapsto \mathrm{diag}(-\Phi'_{2n}, -\mathrm{id}_{2(n'-n)+1}) \times w_\sigma \end{aligned} \tag{3.7}$$

for $g \in \mathrm{O}_{2n}(\mathbb{C})$, $w \in W_E$ and $w_\sigma \in W_F \smallsetminus W_E$. Here Φ'_{2n} is defined in (3.2).

Suppose that $n' < n$. We define $\alpha\colon \mathrm{SO}_{2n'+1}(\mathbb{C}) \times W_F \to \mathrm{O}_{2n}(\mathbb{C}) \rtimes W_F$ by

$$\begin{aligned} g \times w &\longmapsto \mathrm{diag}(g, \mathrm{id}_{2(n-n')-1}) \times w \\ 1 \times w_\sigma &\longmapsto \mathrm{diag}(-\mathrm{id}_{2n'+1}, \mathrm{id}_{2(n-n')-1})\Phi'_{2n} \times w_\sigma \end{aligned} \tag{3.8}$$

for $g \in \mathrm{SO}_{2n'+1}(\mathbb{C})$, $w \in W_E$ and $w_\sigma \in W_F \smallsetminus W_E$.

3.3 L-functoriality for Local Theta Correspondence

According to the local Langlands conjecture (*cf.* [Bor79]) with each irreducible admissible representation π of $G(V)$ one can associate a unique admissible homomorphism

$$\varphi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L G(V)$$

up to conjugation under ${}^\vee G(V)$. The (conjugacy class of the) homomorphism φ is called the *Langlands parameter* of the representation π.

For $l \in \mathbb{N}$, let ρ_l denote the symmetric tensor representation of $\mathrm{SL}_2(\mathbb{C})$ on the l-dimensional complex vector space. We also write id_k for the k-copies of trivial representation of $\mathrm{SL}_2(\mathbb{C})$.

3.3.1 Unitary Case

Regard $\mathrm{id}_k \oplus \rho_l$ as a homomorphism from $\mathrm{SL}_2(\mathbb{C})$ to $\mathrm{GL}_{k+l}(\mathbb{C}) \rtimes W_F$ via

$$\mathrm{id}_k \oplus \rho_l\colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_k(\mathbb{C}) \times \mathrm{GL}_l(\mathbb{C}) \hookrightarrow \mathrm{GL}_{k+l}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{k+l}(\mathbb{C}) \rtimes W_F.$$

The following conjecture from [HKS96, section 7] is modified from Adams' conjecture in [Ada89].

Conjecture 3.1 *Let $(G(V), G(V'))$ be a reductive dual pair of unitary groups. Let n and n' denote the dimensions of V and V', respectively. Suppose that $n \le n'$, π is an irreducible supercuspidal representation of $G(V)$ with Langlands parameter $\varphi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G(V)$, and the correspondence $\pi \leftrightarrow \theta(\pi, V')$ is a first occurrence of supercuspidal representations. Then the Langlands parameter $\theta(\varphi)\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G(V')$*

of $\theta(\pi, V')$ is given by

$$\theta(\varphi)|_{W_F} = \alpha \circ \varphi|_{W_F},$$
$$\theta(\varphi)|_{\mathrm{SL}_2(\mathbb{C})} = (\alpha \circ \varphi|_{\mathrm{SL}_2(\mathbb{C})}) \cdot (\mathrm{id}_n \oplus \rho_{n'-n}),$$

where $\alpha\colon {}^L G(V) \to {}^L G(V')$ is given in (3.4).

3.3.2 Orthogonal-symplectic Case

It is known that $\rho_l(\mathrm{SL}_2(\mathbb{C})) \subset \mathrm{SO}_l(\mathbb{C})$ for a positive odd integer l, and $\mathrm{id}_k \oplus \rho_l$ is regarded as a homomorphism from $\mathrm{SL}_2(\mathbb{C})$ to $\mathrm{SO}_{k+l}(\mathbb{C}) \rtimes W_F$ via

$$\mathrm{id}_k \oplus \rho_l\colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_k(\mathbb{C}) \times \mathrm{SO}_l(\mathbb{C}) \hookrightarrow \mathrm{SO}_{k+l}(\mathbb{C}) \hookrightarrow \mathrm{SO}_{k+l}(\mathbb{C}) \rtimes W_F.$$

The following conjecture is modified from a conjecture in [Ada89, section 4]. The modification is the analog in the case of symplectic-orthogonal groups of Harris-Kudla-Sweet's conjecture in [HKS96, section 7].

Conjecture 3.2 *Let $(G(V), G(V'))$ be a reductive dual pair of an even orthogonal group and a symplectic group. Let $2n$ and $2n'$ denote the dimension of V and V', respectively. Suppose that $n' \geq n$ (resp. $n' > n$) if $G(V')$ is symplectic (resp. even orthogonal). Suppose also that π is an irreducible supercuspidal representation of $G(V)$ with Langlands parameter $\varphi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G(V)$, and the correspondence $\pi \leftrightarrow \theta(\pi, V')$ is a first occurrence of supercuspidal representations. Then the Langlands parameter $\theta(\varphi)\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G(V')$ of $\theta(\pi, V')$ is given by*

$$\theta(\varphi)|_{W_F} = \alpha \circ \varphi|_{W_F},$$
$$\theta(\varphi)|_{\mathrm{SL}_2(\mathbb{C})} = (\alpha \circ \varphi|_{\mathrm{SL}_2(\mathbb{C})}) \cdot (\mathrm{id}_k \oplus \rho_l)$$

where $k = 2n$, $l = 2n' - 2n + 1$ (resp. $k = 2n + 1$, $l = 2n' - 2n - 1$) if $G(V')$ is symplectic (resp. even orthogonal), and $\alpha\colon {}^L G(V) \to {}^L G(V')$ is given in (3.5)–(3.8).

3.4 Consequences of the Conjectures

Keep the setting in Subsection 2.3. For unitary cases, note that $\dim(V_i)$ and $\dim(V_j)$ are of the same parity if i and j are of the same parity. Now the pair of characters χ_1, χ_2 of $E^\times$ are chosen such that $\chi_1|_{F^\times} = \epsilon_{E/F}^{\dim(V_1)}$, and $\chi_2|_{F^\times} = \epsilon_{E/F}^{\dim(V_0)}$. Harris–Kudla–Sweet [HKS96] describe the Langlands parameter φ_i of π_i in terms of the Langlands parameter $\varphi_0\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_{n_0}(\mathbb{C}) \rtimes W_F$ of π_0 as follows. Let $\overline{\varphi}_0(w)$, $\overline{\varphi}_0(w_\sigma)$, and $\overline{\varphi}_0(x)$ be the elements in $\mathrm{GL}_{n_0}(\mathbb{C})$ defined by

$$\varphi_0(w) = \overline{\varphi}_0(w) \times w,$$
$$\varphi_0(w_\sigma) = \overline{\varphi}_0(w_\sigma) \times w_\sigma,$$
$$\varphi_0(x) = \overline{\varphi}_0(x) \times 1$$

for $w \in W_E$, $w_\sigma \in W_F \smallsetminus W_E$, and $x \in \mathrm{SL}_2(\mathbb{C})$.

3.4.1 Unitary Case (I)

Suppose that the dimensions of V_0 and V_1 are all of the same parity. Then $n_i = n_0 + i(i+1)$. According to Conjecture 3.1 and (3.4),

$$\varphi_i\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_{n_0+i(i+1)}(\mathbb{C}) \rtimes W_F$$

should be given by

$$\begin{aligned} w &\longmapsto \begin{cases} \mathrm{diag}(\overline{\varphi}_0(w), \chi_1(w)\,\mathrm{id}_{i(i+1)}) \times w & \text{if } i \text{ is even,}\\ \chi_2(w)\chi_1(w)^{-1}\,\mathrm{diag}(\overline{\varphi}_0(w), \chi_1(w)\,\mathrm{id}_{i(i+1)}) \times w & \text{if } i \text{ is odd,}\end{cases}\\ w_\sigma &\longmapsto \mathrm{diag}(\overline{\varphi}_0(w_\sigma), \mathrm{id}_{i(i+1)})\Phi_{n_0,2,4,\dots,2i^\sharp}\Phi^{-1}_{n_0+i(i+1)} \times w_\sigma,\\ x &\longmapsto \mathrm{diag}(\overline{\varphi}_0(x), \rho_2(x), \rho_4(x), \dots, \rho_{2i^\sharp}(x)) \times 1 \end{aligned} \tag{3.9}$$

for $w \in W_E$, $w_\sigma \in W_F \smallsetminus W_E$, $x \in \mathrm{SL}_2(\mathbb{C})$, and $i^\sharp = \min(|i|, |i+1|)$.

Remark 3.3 Since in this case $\dim(V_i)$ are either all even or all odd, one may choose $\chi_1 = \chi_2$ and simplify the expression in (3.9). This is the original formulation in [HKS96]. However, to make the expression in (3.9) and (3.10) more symmetric, we do not make such a choice here.

3.4.2 Unitary Case (II)

Suppose that the dimensions of V_0 and V_1 are of the opposite parity. Then $n_i = n_0 + i^2$. According to Conjecture 3.1 and (3.4),

$$\varphi_i\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_{n_0+i^2}(\mathbb{C}) \rtimes W_F$$

should be given by

$$\begin{aligned} w &\longmapsto \begin{cases} \mathrm{diag}(\overline{\varphi}_0(w), \chi_1(w)\,\mathrm{id}_{i^2}) \times w & \text{if } i \text{ is even,}\\ \chi_2(w)\chi_1(w)^{-1}\,\mathrm{diag}(\overline{\varphi}_0(w), \chi_1(w)\,\mathrm{id}_{i^2}) \times w & \text{if } i \text{ is odd,}\end{cases}\\ w_\sigma &\longmapsto \mathrm{diag}(\overline{\varphi}_0(w_\sigma), \mathrm{id}_{i^2})\Phi_{n_0,1,3,\dots,2|i|-1}\Phi^{-1}_{n_0+i^2} \times w_\sigma,\\ x &\longmapsto \mathrm{diag}(\overline{\varphi}_0(x), \rho_1(x), \rho_3(x), \dots, \rho_{2|i|-1}(x)) \times 1 \end{aligned} \tag{3.10}$$

for $w \in W_E$, $w_\sigma \in W_F \smallsetminus W_E$, and $x \in \mathrm{SL}_2(\mathbb{C})$.

3.4.3 Split Orthogonal-symplectic Case

Now $n_i^* = n_0^* + i^2$ where n_i^* is defined in (2.2). According to Conjecture 3.2, (3.5), and (3.6), for $i \in \mathbb{Z} \smallsetminus \{0\}$,

$$\varphi_i\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^LG(V_i) = \begin{cases} \mathrm{O}_{n_0^*+i^2}(\mathbb{C}) \times W_F, & \text{or}\\ \mathrm{SO}_{n_0^*+i^2}(\mathbb{C}) \times W_F \end{cases}$$

should be given by

$$\begin{aligned} w &\longmapsto \mathrm{diag}(\overline{\varphi}_0(w), \det(\overline{\varphi}_0(w)), \mathrm{id}_{i^2-1}) \times w\\ x &\longmapsto \mathrm{diag}(\overline{\varphi}_0(x), \det(\overline{\varphi}_0(x))\rho_1(x), \rho_3(x), \dots, \rho_{2|i|-1}(x)) \times 1 \end{aligned} \tag{3.11}$$

for $w \in W_F$ and $x \in \mathrm{SL}_2(\mathbb{C})$.

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3.4.4 Non-split Orthogonal-symplectic Case

Note that n_0^* is odd, since V_0 is symplectic. According to Conjecture 3.2, (3.7), and (3.8), we have

$$\begin{aligned}\varphi_1(w_\sigma) &= \operatorname{diag}\big(-\overline{\varphi}_0(w_\sigma), \mathrm{id}_1\big)\,\Phi'_{n_0^*+1} \times w_\sigma,\\ \varphi_2(w_\sigma) &= \operatorname{diag}\big(-\overline{\varphi}_1(w_\sigma)\Phi'_{n_0^*+1}, -\det(\overline{\varphi}_1(w_\sigma))\,\mathrm{id}_3\big) \times w_\sigma\\ &= \operatorname{diag}\big(\overline{\varphi}_0(w_\sigma), -\mathrm{id}_1, -\mathrm{id}_3\big) \times w_\sigma.\end{aligned}$$

Then it is easy to see that, for $i \in \mathbb{Z} \smallsetminus \{0\}$,

$$\varphi_i \colon W_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^LG(V_i) = \begin{cases} \mathrm{O}_{n_0^*+i^2}(\mathbb{C}) \rtimes W_F, & \text{or}\\ \mathrm{SO}_{n_0^*+i^2}(\mathbb{C}) \times W_F \end{cases}$$

should be given by

$$(3.12)\qquad \begin{aligned} w &\longmapsto \operatorname{diag}(\overline{\varphi}_0(w), \mathrm{id}_{i^2}) \times w,\\ w_\sigma &\longmapsto \begin{cases} \operatorname{diag}(\ \ \overline{\varphi}_0(w_\sigma), -\mathrm{id}_{i^2}) \times w_\sigma & \text{if } i \text{ even},\\ \operatorname{diag}(-\overline{\varphi}_0(w_\sigma), \ \ \mathrm{id}_{i^2})\Phi'_{n_0^*+i^2} \times w_\sigma & \text{if } i \text{ odd}, \end{cases}\\ x &\longmapsto \operatorname{diag}(\overline{\varphi}_0(x), \rho_1(x), \rho_3(x), \dots, \rho_{2|i|-1}(x)) \times 1 \end{aligned}$$

for $w \in W_E$, $w_\sigma \in W_F \smallsetminus W_E$, and $x \in \mathrm{SL}_2(\mathbb{C})$.

4 Supercuspidal Representations with Unipotent Reduction

4.1 Unipotent Representations of Finite Classical Groups

Let $\mathbf{v}$ be a (non-degenerate) symplectic space or an even quadratic space over $\mathbf{f}$, or a Hermitian space over a quadratic extension of $\mathbf{f}$. Let $G(\mathbf{v})$ denote the the group of isometries, and let $G^0(\mathbf{v})$ denote the identity component of $G(\mathbf{v})$. Deligne–Lusztig [DL76] define a very special class of irreducible representations of $G^0(\mathbf{v})$ called *unipotent cuspidal*. Lusztig [Lus77] shows that the group $G^0(\mathbf{v})$ has a unique unipotent cuspidal representation if and only if the dimension of $\mathbf{v}$ is one of the forms:

$$(4.1)\qquad \dim(\mathbf{v}) = \begin{cases} \frac{1}{2}s(s+1) & \text{for some integer } s \text{ if } G(\mathbf{v}) = \mathrm{U}(\mathbf{v}),\\ 2s(s+1) & \text{for some integer } s \text{ if } G(\mathbf{v}) = \mathrm{Sp}(\mathbf{v}),\\ 2s^2 & \text{for some even integer } s \text{ if } G^0(\mathbf{v}) = \mathrm{SO}^+(\mathbf{v}),\\ 2s^2 & \text{for some odd integer } s \text{ if } G^0(\mathbf{v}) = \mathrm{SO}^-(\mathbf{v}). \end{cases}$$

This unipotent cuspidal representation will be denoted by ζ_s. Note that the dimension of $\mathbf{v}$ is allowed to be zero. In this situation, by our convention, the trivial representation of the trivial group is also called *unipotent* (cuspidal).

(a) If $G(\mathbf{v})$ is unitary or symplectic, we extend the index of ζ_s to whole integers by letting $\zeta_s = \zeta_{-s-1}$.

(b) For the finite even orthogonal group $\mathrm{O}(\mathbf{v})$, the induced representation $\mathrm{Ind}_{\mathrm{SO}(\mathbf{v})}^{\mathrm{O}(\mathbf{v})}\zeta_s$ is decomposed into $\zeta \oplus (\zeta \otimes \mathrm{sgn})$ for a cuspidal representation ζ of $\mathrm{O}(\mathbf{v})$ when $s \neq 0$. Following [AM93], we call both ζ and $\zeta \otimes \mathrm{sgn}$ the *unipotent cuspidal representations* of $\mathrm{O}(\mathbf{v})$. They will be denoted by ζ_s and ζ_{-s}, *i.e.*, $\zeta_{-s} = \zeta_s \otimes \mathrm{sgn}$.

Suppose that ξ_i is an irreducible representation of the group $G(\mathbf{v}_i)$ for $i = 1, \dots, k$. The irreducible representation $\xi_1 \otimes \cdots \otimes \xi_k$ of $G(\mathbf{v}_1) \times \cdots \times G(\mathbf{v}_k)$ is called *unipotent* if each ξ_i is unipotent.

4.2 Theta Correspondences for Finite Reductive Dual Pairs

Recall that we fix a nontrivial additive character ψ of F. Without loss of generality, we can assume that $\psi|_{\mathfrak{p}}$ is trivial and $\psi|_{\mathfrak{o}}$ is nontrivial, where $\mathfrak{o}$ is the ring of integers of F and $\mathfrak{p}$ is the maximal ideal of $\mathfrak{o}$. Then ψ induces a nontrivial character $\bar{\psi}$ of $\mathbf{f}$. We consider the correspondence for the reductive dual pairs defined over $\mathbf{f}$ with respect to the Weil representation $\bar{\omega}$ associated with $\bar{\psi}$.

Suppose η is an irreducible cuspidal representation of a finite classical group $G(\mathbf{v})$. Then the "preservation principle" (*cf.* [Pan02, theorem 12.3]) for finite reductive dual pairs suggests that there exist irreducible cuspidal representations η_i of $G(\mathbf{v}_i)$ for $i \in \mathbb{Z}$ such that $\eta \simeq \eta_{i_0}$ for some integer i_0, and $\eta_i \leftrightarrow (\eta_{i+1} \otimes \mathrm{sgn})$ is a first occurrence of irreducible cuspidal representations for the dual pair $(G(\mathbf{v}_i), G(\mathbf{v}_{i+1}))$. Here "sgn" is the sign character if $G(\mathbf{v})$ is a nontrivial unitary or orthogonal group, and it is the trivial character if $G(\mathbf{v})$ is trivial or symplectic. We know that

$$\dim(\mathbf{v}_{i-1}) + \dim(\mathbf{v}_{i+1}) = \begin{cases} 2\dim(\mathbf{v}_i) + 1 & \text{if } G(\mathbf{v}_i) \text{ is unitary,} \\ 2\dim(\mathbf{v}_i) & \text{if } G(\mathbf{v}_i) \text{ is orthogonal,} \\ 2\dim(\mathbf{v}_i) + 2 & \text{if } G(\mathbf{v}_i) \text{ is symplectic,} \end{cases} \tag{4.2}$$

for each $i \in \mathbb{Z}$. From (4.2) the index i can be normalized such that we have the following three cases:

(i) $\mathbf{v}_i$ is unitary for each i, and $\dim(\mathbf{v}_i) = \dim(\mathbf{v}_0) + \frac{i(i+1)}{2}$.

(ii) $\mathbf{v}_i$ is symplectic if i is odd, split orthogonal if $i \equiv 0 \pmod 4$, orthogonal with 2-dimensional anisotropic kernel if $i \equiv 2 \pmod 4$, and

$$\dim(\mathbf{v}_i) = \dim(\mathbf{v}_0) + \begin{cases} 2(\frac{i}{2})^2 & \text{if } i \text{ is even,} \\ 2(\frac{i-1}{2})(\frac{i+1}{2}) & \text{if } i \text{ is odd.} \end{cases}$$

(iii) $\mathbf{v}_i$ is symplectic when i is even, orthogonal when i is odd, and

$$\dim(\mathbf{v}_i) = \dim(\mathbf{v}_0) + \begin{cases} 2(\frac{i}{2})^2 & \text{if } i \text{ is even,} \\ 2(\frac{i-1}{2})(\frac{i+1}{2}) + 1 & \text{if } i \text{ is odd.} \end{cases}$$

In particular, $\dim(\mathbf{v}_0)$ is the minimum among all dimensions.

Next we want to know how a unipotent cuspidal representation η of $G(\mathbf{v})$ occurs in its sequence $\{\eta_i \mid i \in \mathbb{Z}\}$. We consider the following cases.

4.2.1 Finite Unitary Groups

In this subsection, suppose $G(\mathbf{v})$ is unitary and $\eta \leftrightarrow \eta'$ is a first occurrence in $\bar{\omega}$ for the dual pair $(G(\mathbf{v}), G(\mathbf{v}'))$. Then from [AMR96] §1.C we know that $(\eta \otimes \mathrm{sgn}^{\dim(\mathbf{v}')}) \leftrightarrow (\eta' \otimes \mathrm{sgn}^{\dim(\mathbf{v})})$ occurs in $\bar{\omega}^\flat$, which denotes the Weil representation defined in [Gér77]. Assume that $\eta = \zeta_j$ for some integer j.

(a) If $\dim(\mathbf{v}')$ is even, then $\eta \otimes \mathrm{sgn}^{\dim(\mathbf{v}')} = \zeta_j$. By [AM93, theorem 4.1] we know that $\eta' \otimes \mathrm{sgn}^{\dim(\mathbf{v})} = \zeta_{j+1}$ or ζ_{j-1}, *i.e.*,

$$\eta' = \begin{cases} \zeta_{j-1} \otimes \mathrm{sgn}^{\dim(\mathbf{v})}, & \text{if } j \equiv 0,1 \pmod 4; \\ \zeta_{j+1} \otimes \mathrm{sgn}^{\dim(\mathbf{v})}, & \text{if } j \equiv 2,3 \pmod 4). \end{cases}$$

(b) If both $\dim(\mathbf{v}')$ and $\dim(\mathbf{v})$ are odd, then $\eta \otimes \mathrm{sgn}^{\dim(\mathbf{v}')} = \zeta_j \otimes \mathrm{sgn}$. Because ζ_j is unipotent, $\zeta_j \otimes \mathrm{sgn}$ is in the Lusztig series $\mathcal{E}(G(\mathbf{v}), s)$, where s is a semisimple element in the dual group of $G(\mathbf{v})$ such that no eigenvalue of s is equal to 1. Then by [AMR96, théorème 2.6] we see that $\dim(\mathbf{v}') = \dim(\mathbf{v})$ and $\eta' \otimes \mathrm{sgn}^{\dim(\mathbf{v})} = \eta \otimes \mathrm{sgn}^{\dim(\mathbf{v}')}$, *i.e.*, $\eta' = \eta$, if we identify $G(\mathbf{v})$ and $G(\mathbf{v}')$.

(c) If $\dim(\mathbf{v}')$ is odd and $\dim(\mathbf{v})$ is even, then $\dim(\mathbf{v}') = \dim(\mathbf{v}) + 1$ by (b) and preservation principle (4.2). In this case, η' is not unipotent unless $\dim(\mathbf{v}) = 0$.

Lemma 4.1 *Let $\{\eta_i \mid i \in \mathbb{Z}\}$ be a sequence of cuspidal representations given by preservation principle such that each $G(\mathbf{v}_i)$ is unitary and $\dim(\mathbf{v}_0) = 0$. Then $\eta_i \otimes \mathrm{sgn}^{\dim(\mathbf{v}_{i+1})}$ is unipotent for each $i \in \mathbb{Z}$.*

Proof By assumption, $\eta_i \leftrightarrow (\eta_{i+1} \otimes \mathrm{sgn})$ occurs in the Weil representation $\bar{\omega}$. Then from [AMR96, §1.C] we know that $(\eta_i \otimes \mathrm{sgn}^{\dim(\mathbf{v}_{i+1})}) \leftrightarrow (\eta_{i+1} \otimes \mathrm{sgn}^{\dim(\mathbf{v}_i)+1})$ occurs in $\bar{\omega}^\flat$. Note that $\dim(\mathbf{v}_i) + 1 \equiv \dim(\mathbf{v}_{i+2}) \pmod 2$ from (4.2). Hence, $\eta_{i+1} \otimes \mathrm{sgn}^{\dim(\mathbf{v}_i)+1} = \eta_{i+1} \otimes \mathrm{sgn}^{\dim(\mathbf{v}_{i+2})}$. From [AM93, theorem 3.5] we know that $\eta_i \otimes \mathrm{sgn}^{\dim(\mathbf{v}_{i+1})}$ is unipotent if and only if $\eta_{i+1} \otimes \mathrm{sgn}^{\dim(\mathbf{v}_{i+2})}$ is unipotent. Because $\dim(\mathbf{v}_0) = 0$, $\eta_0 = \eta_0 \otimes \mathrm{sgn}^{\dim(\mathbf{v}_1)}$ is the trivial representation of the trivial unitary group. Thus η_0 is unipotent, and hence each $\eta_i \otimes \mathrm{sgn}^{\dim(\mathbf{v}_{i+1})}$ is unipotent. ■

Remark 4.2 From this lemma and the above argument, we conclude the following about the position of the unipotent cuspidal representation ζ_j of $\mathrm{U}_{\frac{j(j+1)}{2}}(q)$ in its sequence $\{\eta_i \mid i \in \mathbb{Z}\}$:

(i) If $j \equiv 2,3 \pmod 4$, then $\dim(\mathbf{v}_{j+1}) = \frac{(j+1)(j+2)}{2}$ is even, and hence $\zeta_j = \eta_j$ with $\dim(\mathbf{v}_0) = 0$ by Lemma 4.1.

(ii) If $j \equiv 1 \pmod 4$, then $\dim(\mathbf{v}) = \frac{j(j+1)}{2}$ is odd and $\zeta_j \otimes \mathrm{sgn}$ first occurs in $\bar{\omega}^\flat$ with $\dim(\mathbf{v}') = \dim(\mathbf{v})$, *i.e.*, ζ_j first occurs in $\bar{\omega}$ with $\dim(\mathbf{v}') = \dim(\mathbf{v})$. Therefore $\zeta_j = \eta_0 \otimes \mathrm{sgn} = \eta_{-1}$ with $\dim(\mathbf{v}_0) = \frac{j(j+1)}{2}$.

(iii) If $j \equiv 0 \pmod 4$, then $\dim(\mathbf{v}) = \frac{j(j+1)}{2}$ is even and $\zeta_j \otimes \mathrm{sgn}$ first occurs in $\bar{\omega}^\flat$ with $\dim(\mathbf{v}') = \dim(\mathbf{v}) + 1$ which is odd. This means that ζ_j first occurs in $\bar{\omega}$ with $\dim(\mathbf{v}') = \dim(\mathbf{v}) + 1$, and therefore $\zeta_j = \eta_0$ with $\dim(\mathbf{v}_0) = \frac{j(j+1)}{2}$.

We can check that $\eta_{-i-1} = \eta_i \otimes \mathrm{sgn}$ for all cases. From the above, we also see that if $\dim(\mathbf{v}_0) \neq 0$, then there is at most one unipotent representation, namely η_0, in the sequence $\{\eta_i \mid i \in \mathbb{Z}\}$.

4.2.2 Finite Symplectic and Even Orthogonal Groups

Lemma 4.3 *Let* $\{\eta_i \mid i \in \mathbb{Z}\}$ *be a sequence of cuspidal representations given by preservation principle such that* $\dim(\mathbf{v}_0) = 0$ *and each* $G(\mathbf{v}_i)$ *is either symplectic or even orthogonal. Then all* η_i *and* $\eta_i \otimes \mathrm{sgn}$ *are unipotent.*

Proof Now the occurrence of $\eta_i \leftrightarrow (\eta_{i+1} \otimes \mathrm{sgn})$ in $\bar{\omega}$ implies the occurrence of $\eta_i \leftrightarrow (\eta_{i+1} \otimes \mathrm{sgn})$ in $\bar{\omega}^\flat$ since $\bar{\omega} = \bar{\omega}^\flat$ for this case. From [AM93, theorem 3.5] we know that η_i is unipotent if and only if η_{i+1} and $\eta_{i+1} \otimes \mathrm{sgn}$ are unipotent. (Of course, if $G(\mathbf{v}_{i+1})$ is a symplectic group, then sgn is trivial and $\eta_{i+1}, \eta_{i+1} \otimes \mathrm{sgn}$ are the same representation.) Now $\dim(\mathbf{v}_0) = 0$, so η_0 is the trivial representation of the trivial orthogonal group $G(\mathbf{v}_0)$. Thus, η_0 is unipotent, and hence each η_i and $\eta_i \otimes \mathrm{sgn}$ are unipotent. ∎

Remark 4.4 In [AM93, theorem 3.5(2)], there is an assumption that the residual characteristic q has to be large enough so that the decompostion of the Weil representation for $(\mathrm{Sp}(\mathbf{v}), \mathrm{SO}(\mathbf{v}'))$ in [Sri79] holds. However, if we look at the proof in [Sri79, p. 151] carefully, we see that the assumption on q is only needed when both $\mathrm{Sp}(\mathbf{v}), \mathrm{SO}(\mathbf{v}')$ have the same rank. In our situation, that will happen only when both have zero rank by the preservation principle. This means that the assumption on q in [AM93, theorem 3.5(2)] is not necessary if we only consider the correspondence of unipotent cuspidal representations.

Remark 4.5 This is case (ii) of Subsection 4.2. Suppose that $\dim(\mathbf{v}_0) = 0$. It is known from case (b) of Subsection 4.1 that $\eta_{-i} \simeq \eta_i \otimes \mathrm{sgn}$ for each $i \in \mathbb{Z}$. If i is odd, then $\mathbf{v}_i$ is symplectic and η_i is the unique unipotent cuspidal representation of $\mathrm{Sp}(\mathbf{v}_i)$ with $\dim(\mathbf{v}_i) = 2(\frac{i-1}{2})(\frac{i+1}{2})$, *i.e.*, $\eta_i = \zeta_{\frac{i-1}{2}}$. If i is even and not zero, then $\eta_i, \eta_i \otimes \mathrm{sgn}$ are the two unipotent cuspidal representations of $\mathrm{O}^\pm(\mathbf{v}_i)$ with $\dim(\mathbf{v}_i) = 2(\frac{i}{2})^2$.

Since all unipotent cuspidal representations of finite symplectic groups and finite even orthogonal groups occur in the sequence $\{\eta_i \mid i \in \mathbb{Z}\}$ if $\dim(\mathbf{v}_0) = 0$, we see that no representation in $\{\eta_i \mid i \in \mathbb{Z}\}$ is unipotent if $\dim(\mathbf{v}_0) \neq 0$.

4.3 Supercuspidal Representations of Depth Zero

Let F, D, and V be as in Subsection 2.1. A lattice L in V is called a *good lattice* if $L^*\mathfrak{p}_D \subseteq L \subseteq L^*$ where

$$L^* = \{v \in V \mid \langle v, l \rangle \in \mathfrak{o}_D \text{ for all } l \in L\},$$

and $\mathfrak{o}_D$ denotes the ring of integers of D and $\mathfrak{p}_D$ denotes the maximal ideal in $\mathfrak{o}_D$.

Let L be a good lattice in V. Then the quotients $\mathbf{v}^* := L^*/L$ and $\mathbf{v} := L/L^*\mathfrak{p}_D$ are vector spaces over $\mathbf{f}_D := \mathfrak{o}_D/\mathfrak{p}_D$ with non-degenerate ϵ-Hermitian (or ϵ-symmetric) forms. The stabilizer $G(V)_L$ of L in $G(V)$ is a maximal compact subgroup (*cf.* [Tit79, 3.2]), and there is a surjective homomorphism

$$G(V)_L \longrightarrow G(\mathbf{v}) \times G(\mathbf{v}^*) \tag{4.3}$$

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with kernel denoted by $G(V)_{L,0^+}$. If η and η^* are representations of $G(\mathbf{v})$ and $G(\mathbf{v}^*)$, respectively, then $\eta\otimes\eta^*$ is a representation of $G(\mathbf{v})\times G(\mathbf{v}^*)$ and can be pulled back as a representation of $G(V)_L$ via the homomorphism in (4.3).

When $G(V)$ is connected (*i.e.,* $G(V)$ is not orthogonal), from [MP96, Mor99] we know that the compactly induced representation

$$c\text{-Ind}_{G(V)_L}^{G(V)}(\eta\otimes\eta^*)$$

is an irreducible supercuspidal representation of $G(V)$ if and only if η and η^* are irreducible cuspidal representations of $G(\mathbf{v})$ and $G(\mathbf{v}^*)$, respectively. Moreover, all irreducible supercuspidal representations of $G(V)$ of depth zero are obtained in this way.

Now we want to show that the same result still holds for an orthogonal group. Let $G=\mathrm{O}(V)$, $H=\mathrm{SO}(V)$, and let π be an irreducible supercuspidal representation of H of depth zero. Then $\mathrm{Ind}_H^G\pi$ is either an irreducible representation or the direct sum of two irreducible representations $\pi^1\oplus\pi^2$. Now every irreducible supercuspidal representation π_G of G of depth zero is either $\mathrm{Ind}_H^G\pi$ or π^i for some irreducible supercuspidal representation π of H of depth zero. Because H is connected, we have $\pi=c\text{-Ind}_{H_L}^H\eta$ for some good lattice L in V and some irreducible representation η of H_L inflated from an irreducible cuspidal representation of $H_L/H_{L,0^+}$. Now H_L is a subgroup of G_L of index 2. If $\pi_G=\mathrm{Ind}_H^G\pi$, then $\eta_G:=\mathrm{Ind}_{H_L}^{G_L}\eta$ is irreducible and $\pi_G=c\text{-Ind}_{G_L}^G\eta_G$. If $\pi_G=\pi^i$ for $i=1,2$, then $\mathrm{Ind}_{H_L}^{G_L}\eta$ is the direct sum of two irreducible representations $\eta^1\oplus\eta^2$, and $\pi_G=c\text{-Ind}_{G_L}^G\eta^i$.

Proposition 4.6 *Suppose the supercuspidal representation π_r in a sequence given by preservation principle is of the form $c\text{-Ind}_{G(V_r)_{L_r}}^{G(V_r)}(\eta\otimes\eta^*)$ for some good lattice L_r in some space V_r and some cuspidal representations η,η^*. Then there exist integers s,t such that $\eta=\eta_s$, $\eta^*=\eta_t^*$ and*

$$\pi_{r+i}=c\text{-Ind}_{G(V_{r+i})_{L_{r+i}}}^{G(V_{r+i})}(\eta_{s+i}\otimes\eta_{t+i}^*)\tag{4.4}$$

for $i\in\mathbb{Z}$ where L_{r+i} is a good lattice in V_{r+i} and η_{s+i} (resp. η_{t+i}^*) is the i-th term after η_s (resp. η_t^*) in its sequence given by preservation principle.

Proof Suppose that:
- $(G(V),G(V'))$ is a reductive dual pair;
- π is the induced representation $c\text{-Ind}_{G(V)_L}^{G(V)}(\eta\otimes\eta^*)$ where η (resp. η^*) is an irreducible cuspidal representation of $G(L/L^*\mathfrak{p}_D)$ (resp. $G(L^*/L)$) where L is a good lattice in V;
- π' is the induced representation $c\text{-Ind}_{G(V')_{L'}}^{G(V')}(\eta'\otimes\eta'^*)$ where η'^* (resp. η') is an irreducible cuspidal representation of $G(L'^*/L')$ (resp. $G(L'/L'^*\mathfrak{p}_D)$) where L' is a good lattice in V'.

Then by [Pan02, theorems 9.3 and 9.5] we know that $\pi\leftrightarrow\pi'$ is a first occurrence in the local theta correspondence if and only if both $\eta\leftrightarrow\eta'^*$ and $\eta^*\leftrightarrow\eta'$ are first occurrences in the theta correspondence for finite reductive dual pairs.

Now back to our situation: $\pi_r \leftrightarrow (\pi_{r+1} \otimes \mathrm{sgn})$ first occurs in the theta correspondence. Write

$$\pi_r = \pi = c\text{-}\mathrm{Ind}_{G(V)_L}^{G(V)}(\eta \otimes \eta^*) \quad \text{and} \quad \pi_{r+1} \otimes \mathrm{sgn} = \pi' = c\text{-}\mathrm{Ind}_{G(V')_{L'}}^{G(V')}(\eta' \otimes \eta'^*).$$

Then $\eta \leftrightarrow \eta'^*$ first occurs in the theta correspondence. Hence, we can choose an integer s (could be positive or negative) such that $\eta = \eta_s$ and $\eta'^* = \eta_{s+1} \otimes \mathrm{sgn}$. Similarly, there is an integer t such that $\eta^* = \eta_t^*$ and $\eta' = \eta_{t+1}^* \otimes \mathrm{sgn}$. Then we have

$$\pi_{r+1} \otimes \mathrm{sgn} = c\text{-}\mathrm{Ind}_{G(V_{r+1})_{L_{r+1}}}^{G(V_{r+1})}\big((\eta_{s+1} \otimes \mathrm{sgn}) \otimes (\eta_{t+1}^* \otimes \mathrm{sgn})\big),$$

and hence

$$\pi_{r+1} \simeq c\text{-}\mathrm{Ind}_{G(V_{r+1})_{L_{r+1}}}^{G(V_{r+1})}(\eta_{s+1} \otimes \eta_{t+1}^*).$$

Therefore, (4.4) holds for $i = 1$, and consequently, it also holds for any positive integer i by induction.

Next consider the case $i = -1$: $\pi_{r-1} \leftrightarrow (\pi_r \otimes \mathrm{sgn})$. If we write

$$\pi_{r-1} = c\text{-}\mathrm{Ind}_{G(V')_{L'}}^{G(V')}(\eta' \otimes \eta'^*) \quad \text{and} \quad \pi_r = c\text{-}\mathrm{Ind}_{G(V)_L}^{G(V)}(\eta \otimes \eta^*),$$

then $\eta' \leftrightarrow \eta^*$, $\eta'^* \leftrightarrow \eta$ are first occurrences, and hence $\eta' = \eta_{s-1}$, $\eta^* \otimes \mathrm{sgn} = \eta_s$, and $\eta'^* = \eta_{t-1}^*$, $\eta \otimes \mathrm{sgn} = \eta_t^*$ for some integers s, t. Therefore, (4.4) holds for $i = -1$, and consequently, it also holds for any negative integer i again by an induction argument. Thus, the proposition is proved. ∎

4.4 Correspondence of Supercuspidal Representations with Unipotent Reduction

An irreducible admissible representation is said to be *unipotent* or *having unipotent reductions* if it admits a nonzero invariant vector by the pro-p-unipotent radical of a parahoric subgroup (*cf.* [Lus83, MW03]). From the results of Moy–Prasad and Morris mentioned in Subsection 4.3, we know that the supercuspidal representation

$$c\text{-}\mathrm{Ind}_{G(V)_L}^{G(V)}(\eta \otimes \eta^*)$$

has unipotent reduction if both η, η^* are unipotent cuspidal representations. All supercuspidal representations with unipotent reductions are constructed in the way.

Suppose that $\{\pi_i \mid i \in \mathbb{Z}\}$ is a sequence of irreducible supercuspidal representations by preservation principle in which at least one representation have unipotent reduction. So we can write

$$\pi_0 = c\text{-}\mathrm{Ind}_{G(V_0)_{L_0}}^{G(V_0)}(\eta_{s_0} \otimes \eta_{t_0}^*) \tag{4.5}$$

for some integers s_0, t_0 where sequences $\{\eta_i \mid i \in \mathbb{Z}\}$ and $\{\eta_i^* \mid i \in \mathbb{Z}\}$ are as in Remarks 4.2 or 4.5. From these two remarks, we know that if $\dim(\mathbf{v}_0) \neq 0$, then at most one representation in $\{\eta_i \mid i \in \mathbb{Z}\}$ is unipotent. Therefore, for our purpose, we will consider the cases such that $\dim(\mathbf{v}_0) = \dim(\mathbf{v}_0^*) = 0$ for the sequences $\{\eta_i \mid i \in \mathbb{Z}\}$ and $\{\eta_i^* \mid i \in \mathbb{Z}\}$. We have the following cases.

4.4.1 Unitary Case I: Unramified Extension

We assume that E is an unramified extension of F. Because $\eta_i = \zeta_i \otimes \operatorname{sgn}^{\frac{(i+1)(i+2)}{2}}$ from Lemma 4.1, we have

$$\begin{aligned}\pi_i &= c\text{-Ind}_{G(V_i)_{L_i}}^{G(V_i)}(\eta_{s_0+i}\otimes\eta^*_{t_0+i})\\ &= c\text{-Ind}_{G(V_i)_{L_i}}^{G(V_i)}\Big(\big(\zeta_{s_0+i}\otimes \operatorname{sgn}^{\frac{(i+1)(i+2)}{2}+\frac{s_0^2+s_0(2i+3)}{2}}\big)\otimes\big(\zeta^*_{t_0+i}\otimes\operatorname{sgn}^{\frac{(i+1)(i+2)}{2}+\frac{t_0^2+t_0(2i+3)}{2}}\big)\Big)\end{aligned}$$

by Proposition 4.6. Consider the following situations (the details of which can be found in Section 7):

(a) If $t_0 - s_0 \equiv 0 \pmod 4$, then $s_0 + t_0 = 0$. Hence both s_0, t_0 are even and
$$\frac{s_0^2+s_0(2i+3)}{2} \equiv \frac{t_0^2+t_0(2i+3)}{2} \equiv \frac{t_0}{2} \pmod 2.$$
Thus, $\pi_i \otimes \operatorname{sgn}^{\frac{(i+1)(i+2)+t_0}{2}}$ has unipotent reduction for each $i \in \mathbb{Z}$.

(b) If $t_0 - s_0 \equiv 1 \pmod 4$, then $s_0 + t_0 = -1$, and hence t_0 is even. We can see that
$$\begin{aligned}\frac{(i+1)(i+2)}{2} + \frac{s_0^2+s_0(2i+3)}{2} &\equiv \frac{i(i+1)}{2} + \frac{t_0}{2} \pmod 2,\\ \frac{(i+1)(i+2)}{2} + \frac{t_0^2+t_0(2i+3)}{2} &\equiv \frac{(i+1)(i+2)}{2} + \frac{t_0}{2} \pmod 2.\end{aligned}$$
Hence, $\pi_i \otimes \operatorname{sgn}^{\frac{t_0+2}{2}}$ (resp. $\pi_i \otimes \operatorname{sgn}^{\frac{t_0}{2}}$) has unipotent reduction for $i \equiv 1 \pmod 4$ (resp. $i \equiv 3 \pmod 4$).

(c) If $t_0 - s_0 \equiv 2 \pmod 4$, then $s_0 + t_0 = 0$. We can see that
$$\frac{s_0^2+s_0(2i+3)}{2} \not\equiv \frac{t_0^2+t_0(2i+3)}{2} \pmod 2$$
for any i. Hence, none of π_i or $\pi_i \otimes \operatorname{sgn}$ has unipotent reduction unless $i \in \{t_0, t_0-1, -t_0, -t_0-1\}$.

(d) If $t_0 - s_0 \equiv 3 \pmod 4$, then $s_0 + t_0 = -1$, and hence t_0 is odd. We can see that
$$\begin{aligned}\frac{(i+1)(i+2)}{2} + \frac{s_0^2+s_0(2i+3)}{2} &\equiv \frac{(i+1)(i+2)}{2} + \frac{t_0+1}{2} \pmod 2,\\ \frac{(i+1)(i+2)}{2} + \frac{t_0^2+t_0(2i+3)}{2} &\equiv \frac{i(i+1)}{2} + \frac{t_0-1}{2} \pmod 2.\end{aligned}$$
Hence, $\pi_i \otimes \operatorname{sgn}^{\frac{t_0-1}{2}}$ (resp. $\pi_i \otimes \operatorname{sgn}^{\frac{t_0+1}{2}}$) has unipotent reduction for $i \equiv 1 \pmod 4$ (resp. $i \equiv 3 \pmod 4$).

It is easy to see that $\dim(V_0)$, $\dim(V_1)$ are of the same parity for cases (a) and (c), and of the opposite parity otherwise.

4.4.2 Unitary Case II: Ramified Extension

Now we assume that E is a ramified extension of F. Then one of η_{s_0} and $\eta^*_{t_0}$ is a unipotent cuspidal representation of a finite even orthogonal group, and the other is a unipotent cuspidal representation of a symplectic group. From Remark 4.5 we see that every representation in $\{\eta_i \mid i \in \mathbb{Z}\}$ or $\{\eta_i^* \mid i \in \mathbb{Z}\}$ is unipotent, and hence by Proposition 4.6 all representations π_i and $\pi_i \otimes \mathrm{sgn}$ have unipotent reduction.

4.4.3 Orthogonal-symplectic Case

The situation for reductive dual pairs of even orthogonal groups and symplectic groups is similar to the case considered in Subsection 4.4.2, *i.e.,* if one representation in the sequence $\{\pi_i \mid i \in \mathbb{Z}\}$ has unipotent reduction, then all π_i and $\pi_i \otimes \mathrm{sgn}$ have unipotent reduction.

5 The Main Results

5.1 Unramified Langlands Parameters

A Langlands parameter $\varphi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^\vee G(V) \rtimes W_F$ is called *unramified* if it is trivial on the inertia subgroup I_F of W_F. It is known that W_F/I_F is an infinite cyclic group generated by a Frobenius element Fr (*cf.* [Tat79] (1.4.1)), and hence it is not difficult to see that an unramified Langlands parameter φ is characterized by the conjugacy class of a pair (y, N) where

- y is a semisimple element of the dual group ${}^\vee G(V)$,
- N is a nilpotent element of the Lie algebra ${}^\vee\mathfrak{g}(V)$, and
- $\mathrm{Ad}_y(N) = qN$

where Ad denotes the adjoint action of ${}^\vee G(V)$ on its Lie algebra ${}^\vee\mathfrak{g}(V)$, and q is the cardinality of $\mathbf{f}$ (*cf.* [Lus83] 1.2). Details can be found in Subsections 5.1.1 and 5.1.2. We will also call such a pair (y, N) an (unramified) *Langlands parameter* for $G(V)$.

A correspondence between the set (of isomorphism classes) of irreducible admissible representations of an adjoint simple group with unipotent reduction and the set (of equivalence classes) of unramified Langlands parameters is given in [Lus95] and [Lus02] via the isomorphisms of certain Hecke algebras. Because we only concern supercuspidal representations, those Hecke algebras degenerate and do not play any role here. The explicit description of an extension of Lusztig's result in our situation (*i.e.,* for a group not necessarily adjoint) is given in Section 6.

For a positive integer k, let ρ_k be given in Subsection 3.3. Then we define

$$(5.1)\qquad \begin{aligned} y_k &= \rho_k\left(\begin{bmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2}\end{bmatrix}\right) = \mathrm{diag}(q^{(k-1)/2}, q^{(k-3)/2}, \dots, q^{(1-k)/2}) \in \mathrm{GL}_k(\mathbb{C}),\\ \delta_k &= d\rho_k\left(\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right]\right) \in \mathfrak{gl}_k(\mathbb{C}).\end{aligned}$$

Note that $\rho_k(y_2) = y_k$ and $\mathrm{Ad}_{y_k}(\delta_k) = q\delta_k$.

5.1.1 Unitary Case

Suppose that $G(V)$ is a unitary group. Let $\varphi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G(V)$ be an unramified Langlands parameter, and let $\mathrm{Fr} \in W_F$ denote a fixed geometric Frobenius element. Let $\overline{\varphi}$ be defined as in Subsection 3.4.1. Because $\overline{\varphi}|_{\mathrm{SL}_2(\mathbb{C})}$ is a (finite-dimensional) algebraic representation, there are positive integers $a_1, \dots, a_k$ such that

$$\overline{\varphi}|_{\mathrm{SL}_2(\mathbb{C})} \simeq \rho_{a_1} \oplus \cdots \oplus \rho_{a_k},$$

where ρ_{a_i} is given in Subsection 3.3. Define

$$(5.2)\qquad \begin{aligned} y &= \begin{cases} \overline{\varphi}\Big(\mathrm{Fr}, \Big[\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\Big]\Big) & \text{if } \mathrm{Fr} \in W_E, \\ \overline{\varphi}\Big(\mathrm{Fr}, \Big[\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\Big]\Big) \Phi_{a_1+\cdots+a_k}\Phi^{-1}_{a_1,\dots,a_k} & \text{if } \mathrm{Fr} \in W_F \smallsetminus W_E, \end{cases} \\ N &= \mathrm{d}(\overline{\varphi}|_{\mathrm{SL}_2(\mathbb{C})})\left(\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]\right), \end{aligned}$$

where $\Phi_{a_1,\dots,a_k}$ is given in (3.3), and the $\mathfrak{sl}_2(\mathbb{C})$-representation $\mathrm{d}(\overline{\varphi}|_{\mathrm{SL}_2(\mathbb{C})})$ denotes the differential of $\overline{\varphi}|_{\mathrm{SL}_2(\mathbb{C})}$. Then N is a nilpotent element in ${}^\vee\mathfrak{g}(V)$ and y is a semisimple element in ${}^\vee G(V)$, which does not depend on the choice of Fr, because φ is unramified. It is easy to check that $\Phi^{-1}_{a_1,\dots,a_k} N \Phi_{a_1,\dots,a_k} = -{}^{\mathrm{t}}N$.

Lemma 5.1 *Let y and N be defined as in (5.2). Then we have* $\mathrm{Ad}_y(N) = qN$.

Proof First suppose that Fr is in W_E. Then

$$\begin{aligned} \mathrm{Ad}_y(N) &= \overline{\varphi}\Big(\mathrm{Fr}, \Big[\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\Big]\Big) \cdot \mathrm{d}(\overline{\varphi}|_{\mathrm{SL}_2(\mathbb{C})})\left(\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]\right) \cdot \overline{\varphi}\Big(\mathrm{Fr}, \Big[\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\Big]\Big)^{-1} \\ &= \mathrm{d}(\overline{\varphi}|_{\mathrm{SL}_2(\mathbb{C})})\left(\Big[\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\Big]\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]\Big[\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\Big]^{-1}\right) \\ &= \mathrm{d}(\overline{\varphi}|_{\mathrm{SL}_2(\mathbb{C})})\left(q\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]\right) = qN. \end{aligned}$$

Next suppose that $\mathrm{Fr} \in W_F \smallsetminus W_E$. By the action given in Subsection 3.1, we have

$$\begin{aligned} \mathrm{Ad}_y(N) &= \overline{\varphi}\Big(\mathrm{Fr}, \Big[\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\Big]\Big) \Phi_n \Phi^{-1}_{a_1,\dots,a_k} N \Phi_{a_1,\dots,a_k} \Phi_n^{-1} \overline{\varphi}\Big(\mathrm{Fr}, \Big[\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\Big]\Big)^{-1} \\ &= \overline{\varphi}\Big(\mathrm{Fr}, \Big[\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\Big]\Big) \Phi_n(-{}^{\mathrm{t}}N)\Phi_n^{-1} \overline{\varphi}\Big(\mathrm{Fr}, \Big[\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\Big]\Big)^{-1} \\ &= \overline{\varphi}\Big(\mathrm{Fr}, \Big[\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\Big]\Big) \mathrm{Fr}(N) \overline{\varphi}\Big(\mathrm{Fr}, \Big[\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\Big]\Big)^{-1} \\ &= \mathrm{d}(\overline{\varphi}|_{\mathrm{SL}_2(\mathbb{C})})\left(\Big[\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\Big]\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]\Big[\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\Big]^{-1}\right) = qN, \end{aligned}$$

where $n = \dim(V)$. ∎

5.1.2 Orthogonal-symplectic Case

Suppose that $G(V)$ is an even orthogonal group or a symplectic group. For a given unramified parameter φ we define

$$(5.3)\qquad \begin{aligned} y &= \begin{cases} \overline{\varphi}\Big(\mathrm{Fr}, \Big[\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\Big]\Big) & \text{if } G(V) \text{ is } \mathrm{Sp}_{2n}(F) \text{ or } \mathrm{O}^{\pm}_{2n}(F), \\ \overline{\varphi}\Big(\mathrm{Fr}, \Big[\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\Big]\Big) \Phi'_{2n} & \text{if } G(V) \text{ is } \mathrm{O}'_{2n}(F), \end{cases} \\ N &= \mathrm{d}(\overline{\varphi}|_{\mathrm{SL}_2(\mathbb{C})})\left(\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]\right) \end{aligned}$$

where Φ'_{2n} is defined in (3.2).

Lemma 5.2 *Let y and N be defined as in (5.3). Then we have* $\mathrm{Ad}_y(N) = qN$.

Proof First, if $G(V)$ is $\mathrm{Sp}_{2n}(F)$ or $\mathrm{O}^{\pm}_{2n}(F)$, then W_F acts trivial on ${}^{\vee}G(V)$ and ${}^{\vee}\mathfrak{g}(V)$. The proof is exactly the same as the proof of the first part of Lemma 5.1. Next suppose $G(V) = \mathrm{O}'_{2n}(F)$. Then Fr is in $W_F \smallsetminus W_E$ where E is defined in Subsection 3.1(d). Hence from Subsection 3.1, we have

$$\begin{aligned}
\mathrm{Ad}_y(N) &= \overline{\varphi}\left(\mathrm{Fr}, \begin{bmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{bmatrix}\right) \Phi'_{2n} N \Phi'^{-1}_{2n} \overline{\varphi}\left(\mathrm{Fr}, \begin{bmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{bmatrix}\right)^{-1} \\
&= \overline{\varphi}\left(\mathrm{Fr}, \begin{bmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{bmatrix}\right) \mathrm{Fr}(N) \overline{\varphi}\left(\mathrm{Fr}, \begin{bmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{bmatrix}\right)^{-1} \\
&= \mathrm{d}(\overline{\varphi}|_{\mathrm{SL}_2(\mathbb{C})})\left(\begin{bmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{bmatrix} \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \begin{bmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{bmatrix}^{-1}\right) = qN. \qquad \blacksquare
\end{aligned}$$

5.2 The Main Results

Keep the setting of Subsection 2.3 and assume that there are infinitely many representations in the sequence $\{\pi_i \mid i \in \mathbb{Z}\}$ that have unipotent reductions. As in Subsection 4.4, we write $\pi_0 = c\text{-}\mathrm{Ind}^{G(V_0)}_{G(V_0)_{L_0}}(\eta_{s_0} \otimes \eta^*_{t_0})$ for some integers s_0, t_0.

5.2.1 Unitary Case (I)

In this subsection we suppose that $\dim(V_0), \dim(V_1)$ are of the same parity. If $D = E$ is an unramified extension of F, we see that we are in case (a) of Subsection 4.4.1, and hence t_0 is even, each $\pi_i \otimes \mathrm{sgn}^{\frac{(i+1)(i+2)+t_0}{2}}$ has unipotent reduction. If E is a ramified extension of F, then every representation π_i has unipotent reduction from Subsection 4.4.2.

Theorem 5.3 *Suppose that $G(V_0)$ is a unitary group and $\dim(V_0), \dim(V_1)$ are of the same parity. Let (y_i, N_i) be the Langlands parameter for the supercuspidal representation with unipotent reduction $\pi_i \otimes \mathrm{sgn}^{\frac{(i+1)(i+2)+t_0}{2}}$ (resp. π_i) for D/F unramified (resp. ramified). Then we have*

$$y_i = \mathrm{diag}(y_0, \gamma_2, \gamma_4, \dots, \gamma_{2i^\sharp}) \quad \text{and} \quad N_i = \mathrm{diag}(N_0, \delta_2, \delta_4, \dots, \delta_{2i^\sharp}),$$

where $i^\sharp = \min(|i|, |i+1|)$, and γ_k, δ_k are defined as in (5.1).

Note that $2 + 4 + \cdots + 2i^\sharp = i(i+1)$ for $i \in \mathbb{Z}$. The proof of the theorem will be given in Subsection 7.1.

Remark 5.4 First suppose that E is an unramified quadratic extension of F. We have $\mathrm{Fr} \in W_F \smallsetminus W_E$. By definition, we have $y_0 = \overline{\varphi}(\mathrm{Fr}, \gamma_2)\Phi_{n_0}\Phi^{-1}_{a_1,\dots,a_k}$, where $n_0 = \dim(V_0)$ and $[a_1, \dots, a_k]$ is the partition of n_0 associated with N_0. Then from (3.9) and (5.2) we have

$$\begin{aligned}
&\operatorname{diag}\big(\overline{\varphi}_0(\mathrm{Fr},\gamma_2),\rho_2(\gamma_2),\rho_4(\gamma_2),\dots,\rho_{2i^\sharp}(\gamma_2)\big)\\
&\quad\times\Phi_{n_0,2,4,\dots,2i^\sharp}\Phi^{-1}_{n_0+i(i+1)}\Phi_{n_0+i(i+1)}\Phi^{-1}_{a_1,\dots,a_k,2,4,\dots,2i^\sharp}\\
&=\operatorname{diag}\big(y_0\Phi_{a_1,\dots,a_k}\Phi^{-1}_{n_0},\rho_2(\gamma_2),\rho_4(\gamma_2),\dots,\rho_{2i^\sharp}(\gamma_2)\big)\\
&\quad\times\Phi_{n_0,2,4,\dots,2i^\sharp}\Phi^{-1}_{a_1,\dots,a_k,2,4,\dots,2i^\sharp}\\
&=\operatorname{diag}\big(y_0,\rho_2(\gamma_2),\rho_4(\gamma_2),\dots,\rho_{2i^\sharp}(\gamma_2)\big)\\
&=\operatorname{diag}(y_0,y_2,y_4,\dots,y_{2i^\sharp}).
\end{aligned}$$

Next, if E is a ramified quadratic extension of F, then $\mathrm{Fr}\in W_E$, $y_0=\overline{\varphi}(\mathrm{Fr},\gamma_2)$ and

$$\begin{aligned}
&\operatorname{diag}\big(\overline{\varphi}_0(\mathrm{Fr},\gamma_2),\rho_2(\gamma_2),\rho_4(\gamma_2),\dots,\rho_{2i^\sharp}(\gamma_2)\big)\\
&\quad=\operatorname{diag}\big(y_0,\rho_2(\gamma_2),\rho_4(\gamma_2),\dots,\rho_{2i^\sharp}(\gamma_2)\big)\\
&\quad=\operatorname{diag}(y_0,y_2,y_4,\dots,y_{2i^\sharp}).
\end{aligned}$$

Therefore, Theorem 5.3 is consistent with (3.9) (*cf.* [HKS96, speculation 7.7]) up to a twisting of the sgn character.

5.2.2 Unitary Case (II)

Now we suppose that $\dim(V_0),\dim(V_1)$ are of the opposite parity. For this case by our assumption in Subsection 2.1, E is an unramified extension of F. From Subsection 4.4.1 we see that t_0-s_0 is odd, π_i or $\pi_i\otimes\mathrm{sgn}$ do not have unipotent reduction for even i, and

$$\begin{cases}\pi_i\otimes\mathrm{sgn}^{\frac{t_0+2}{2}}\\ \pi_i\otimes\mathrm{sgn}^{\frac{t_0}{2}}\\ \pi_i\otimes\mathrm{sgn}^{\frac{t_0-1}{2}}\\ \pi_i\otimes\mathrm{sgn}^{\frac{t_0+1}{2}}\end{cases}\quad\text{has unipotent reduction if}\quad\begin{cases}t_0-s_0\equiv1,\ i\equiv1\pmod 4,\\ t_0-s_0\equiv1,\ i\equiv3\pmod 4,\\ t_0-s_0\equiv3,\ i\equiv1\pmod 4,\\ t_0-s_0\equiv3,\ i\equiv3\pmod 4.\end{cases}\tag{5.4}$$

Let (y_0,N_0) be the Langlands parameter of the representation $c\text{-}\mathrm{Ind}^{G(V_0)}_{G(V_0)_{L_0}}(\zeta_{s_0}\otimes\zeta_{t_0})$.

Theorem 5.5 *Suppose that $G(V_0)$ is a unitary group and $\dim(V_0),\dim(V_1)$ are of the opposite parity. Then the Langlands parameter (y_i,N_i) for the supercuspidal representation in (5.4) for i odd is given by*

$$y_i=\operatorname{diag}(y_0,y_1,y_3,\dots,y_{2|i|-1})\quad\textit{and}\quad N_i=\operatorname{diag}(N_0,\delta_1,\delta_3,\dots,\delta_{2|i|-1}).$$

Note that $1+3+\cdots+(2|i|-1)=i^2$ for $i\in\mathbb{Z}$. The proof will be given in Subsection 7.2.

Remark 5.6 Again, Theorem 5.5 is consistent with (3.10) (*cf.* [HKS96, speculation 7.8]) up to a twisting of the sgn character.

5.2.3 Split Orthogonal-symplectic Case

Theorem 5.7 *Suppose that $G(V_0)$ is an even orthogonal group. Then the Langlands parameter (y_i, N_i) for π_i, $i \in \mathbb{Z}$, satisfies*

$$y_i = \operatorname{diag}(y_0, \gamma_1, \gamma_3, \ldots, \gamma_{2|i|-1}) \quad \textit{and} \quad N_i = \operatorname{diag}(N_0, \delta_1, \delta_3, \ldots, \delta_{2|i|-1}).$$

The proof will be given in Subsection 7.3.

Remark 5.8 In this case we have

$$y_0 = \overline{\varphi}_0\left(\mathrm{Fr}, \begin{bmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{bmatrix}\right) \quad \text{and} \quad N_0 = d\overline{\varphi}_0\left(\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]\right)$$

from the definition in (5.2). Then by (3.11), we should have

$$y_i = \operatorname{diag}(y_0, \det(y_0)\gamma_1, \gamma_3, \ldots, \gamma_{2|i|-1}) \quad \text{and} \quad N_i = \operatorname{diag}(N_0, \delta_1, \delta_3, \ldots, \delta_{2|i|-1}).$$

We will see in Remark 7.2 that $\det(y_0) = 1$, and hence Theorem 5.7 is consistent with Conjecture 3.2.

5.2.4 Non-split Orthogonal-symplectic Case

Theorem 5.9 *Suppose that $G(V_0)$ is a symplectic group. Then the Langlands parameter (y_i, N_i) for π_i, $i \in \mathbb{Z}$, satisfies*

$$y_i = \begin{cases} \operatorname{diag}(y_0, -\gamma_1, -\gamma_3, \ldots, -\gamma_{2|i|-1}) & \textit{if } i \textit{ is even}, \\ \operatorname{diag}(-y_0, \gamma_1, \gamma_3, \ldots, \gamma_{2|i|-1}) & \textit{if } i \textit{ is odd}, \end{cases}$$
$$N_i = \operatorname{diag}(N_0, \delta_1, \delta_3, \ldots, \delta_{2|i|-1}).$$

The proof will be given in Subsection 7.4.

Remark 5.10 In this case, by our assumption in Subsection 2.1, E is an unramified quadratic extension of F, and hence Fr is in $W_F \smallsetminus W_E$. Then according to (3.12), we should have

$$\begin{aligned} y_1 &= \overline{\varphi}_1\left(\mathrm{Fr}, \begin{bmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{bmatrix}\right)\Phi'_{n_0^*+1} \\ &= \operatorname{diag}\left(-\overline{\varphi}_0\left(\mathrm{Fr}, \begin{bmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{bmatrix}\right), \rho_1\left(\begin{bmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{bmatrix}\right)\right)\Phi'_{n_0^*+1}\Phi'_{n_0^*+1} \\ &= \operatorname{diag}(-y_0, \gamma_1), \\ y_2 &= \overline{\varphi}_2\left(\mathrm{Fr}, \begin{bmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{bmatrix}\right) = \operatorname{diag}(-\gamma_1, -\gamma_3) = \operatorname{diag}(y_0, -\gamma_1, -\gamma_3) \end{aligned}$$

by (5.3). Then by induction we can see that the pair (y_i, N_i) associated with φ_i should be given by

$$y_i = \begin{cases} \operatorname{diag}(y_0, -\gamma_1, -\gamma_3, \ldots, -\gamma_{2|i|-1}), & \text{if } i \text{ is even}, \\ \operatorname{diag}(-y_0, \gamma_1, \gamma_3, \ldots, \gamma_{2|i|-1}), & \text{if } i \text{ is odd}, \end{cases}$$
$$N_i = \operatorname{diag}(N_0, \delta_1, \delta_3, \ldots, \delta_{2|i|-1}).$$

Again, Theorem 5.9 is consistent with Conjecture 3.2.

6 Langlands Parameters for Supercuspidal Representations with Unipotent Reduction

A construction of the unramified Langlands parameter associated with an irreducible representation with unipotent reduction of an adjoint group G is in [Lus95, Lus02], see also [Mor96]. If G is a unitary group with respect to an unramified quadratic extension, a similar construction can also be found in [Mœg05]. Now we follow their construction closely and give a description of the parameter (y, N) associated with a supercuspidal representation with unipotent reduction

$$\pi = c\text{-Ind}_{G(V)_L}^{G(V)}(\eta \otimes \eta^*)$$

of a classical group $G(V)$ as follows.

6.1 Unitary Groups with Respect to an Unramified Extension

In this subsection we assume that E is an unramified quadratic extension of F. Then $G(V)_L/G(V)_{L,0^+} \simeq \mathrm{U}(\mathbf{v}) \times \mathrm{U}(\mathbf{v}^*)$.

6.1.1 Suppose that $G(V) = \mathrm{U}_{2n}^+(F)$. From (4.1), we have $2n = \frac{1}{2}s(s+1) + \frac{1}{2}t(t+1)$ for some integers s and t. Moreover, since the anisotropic kernel of V is trivial, we know that both $\frac{1}{2}s(s+1)$ and $\frac{1}{2}t(t+1)$ are even. The complex dual group of $G(V)$ is $\mathrm{GL}_{2n}(\mathbb{C})$. Now we have the following two possible situations:

Suppose that $s-t$ is even. Define $d_1 = (\frac{s+t}{2})(\frac{s+t+2}{2})$ and $d_2 = (\frac{s-t}{2})^2$. Then $d_1 = \sum_{i=1}^{k} 2i$, $d_2 = \sum_{i=1}^{|\frac{s-t}{2}|}(2i-1)$ and $d_1 + d_2 = 2n$, where $k = \min(|\frac{s+t}{2}|, |\frac{s+t+2}{2}|)$. Then $y = (y^{(1)}, y^{(2)})$ is an element of $\mathrm{GL}_{d_1}(\mathbb{C}) \times \mathrm{GL}_{d_2}(\mathbb{C}) \subset \mathrm{GL}_{2n}(\mathbb{C})$ where

$$y^{(1)} = \mathrm{diag}(\gamma_2, \gamma_4, \dots, \gamma_{2k}) \quad \text{and} \quad y^{(2)} = \mathrm{diag}(\gamma_1, \gamma_3, \dots, \gamma_{|s-t|-1}),$$

γ_l is defined as in (5.1); $N = (N^{(1)}, N^{(2)})$ is an nilpotent element in $\mathfrak{gl}_{d_1}(\mathbb{C}) \times \mathfrak{gl}_{d_2}(\mathbb{C}) \subset \mathfrak{gl}_{2n}(\mathbb{C})$, where

$$N^{(1)} = \mathrm{diag}(\delta_2, \delta_4, \dots, \delta_{2k}) \quad \text{and} \quad N^{(2)} = \mathrm{diag}(\delta_1, \delta_3, \dots, \delta_{|s-t|-1}),$$

δ_l is defined as in (5.1).

Suppose that $s-t$ is odd. Define $d_1 = (\frac{s-t-1}{2})(\frac{s-t+1}{2})$ and $d_2 = (\frac{s+t+1}{2})^2$. Then $d_1 = \sum_{i=1}^{k} 2i$, $d_2 = \sum_{i=1}^{|\frac{s+t+1}{2}|}(2i-1)$ and $d_1 + d_2 = 2n$, where $k = \min(|\frac{s-t-1}{2}|, |\frac{s-t+1}{2}|)$. Then $y = (y^{(1)}, y^{(2)})$ is an element of $\mathrm{GL}_{d_1}(\mathbb{C}) \times \mathrm{GL}_{d_2}(\mathbb{C}) \subset \mathrm{GL}_{2n}(\mathbb{C})$, where

$$y^{(1)} = \mathrm{diag}(\gamma_2, \gamma_4, \dots, \gamma_{2k}) \quad \text{and} \quad y^{(2)} = \mathrm{diag}(\gamma_1, \gamma_3, \dots, \gamma_{|s+t+1|-1});$$

$N = (N^{(1)}, N^{(2)})$ is an nilpotent element in $\mathfrak{gl}_{d_1}(\mathbb{C}) \times \mathfrak{gl}_{d_2}(\mathbb{C}) \subset \mathfrak{gl}_{2n}(\mathbb{C})$ where

$$N^{(1)} = \mathrm{diag}(\delta_2, \delta_4, \dots, \delta_{2k}) \quad \text{and} \quad N^{(2)} = \mathrm{diag}(\delta_1, \delta_3, \dots, \delta_{|s+t+1|-1}).$$

The ordered pair (d_1, d_2) remains unchanged if s is replaced by $-s-1$ or t is replaced by $-t-1$ or both. For each ordered pair (s, t), there is a unique irreducible supercuspidal representation of $G(V)$, namely, $\pi_{s,t} := c\text{-Ind}_{G(V)_L}^{G(V)}(\zeta_s \otimes \zeta_t)$ where ζ_s, ζ_t are given in Subsection 4.1. This representation remains unchanged if s is replaced by $-s-1$ or t is replaced by $-t-1$ or both, because $\zeta_s = \zeta_{-s-1}$ and $\zeta_t = \zeta_{-t-1}$. In other words, there

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is a unique irreducible supercuspidal representation associated with the four ordered pairs of integers: (s,t), $(s,-t-1)$, $(-s-1,t)$, and $(-s-1,-t-1)$.

The same ordered pair (d_1,d_2) is associated with two ordered pairs (s,t) and (t,s), and there is a unique pair (y,N) associated with the ordered pair (d_1,d_2). Hence, if $s\neq t$ and $s\neq -t-1$, there are two distinct supercuspidal representations $\pi_{s,t},\pi_{t,s}$ with unipotent reductions associated with the Langlands parameter (y,N). If $s=t$ or $s=-t-1$, there is a single supercuspidal representation with unipotent reduction associated with the Langlands parameter (y,N).

6.1.2 Suppose that $G(V)=\mathrm{U}^-_{2n}(F)$. We have $2n=\frac{1}{2}s(s+1)+\frac{1}{2}t(t+1)$ for some integers s,t such that both $\frac{1}{2}s(s+1)$ and $\frac{1}{2}t(t+1)$ are odd. The situation is completely similar to Subsection 6.1.1.

6.1.3 Suppose that $G(V)=\mathrm{U}_{2n+1}(F)$. Now we have $2n+1=\frac{1}{2}s(s+1)+\frac{1}{2}t(t+1)$ for some integers s,t such that exactly one of $\frac{1}{2}s(s+1)$ and $\frac{1}{2}t(t+1)$ is even. The situation is similar to Subsection 6.1.1 except that $2n$ is replaced by $2n+1$.

6.2 Unitary Groups with Respect to a Ramified Extension

In this subsection we assume that E is a ramified quadratic extension of F.

6.2.1 Suppose that $G(V)=\mathrm{U}^+_{2n}(F)$. Then $G(V)_L/G(V)_{L,0^+}\simeq \mathrm{O}^+(\mathbf{v})\times \mathrm{Sp}(\mathbf{v}^*)$. From (4.1), we have $2n=2s^2+2t(t+1)$ for some even integer s and some integer t. Define $d_1=(s+t)(s+t+1)$ and $d_2=(s-t-1)(s-t)$. Then $d_1=\sum_{i=1}^{k}2i$, $d_2=\sum_{i=1}^{k'}2i$ and $d_1+d_2=2n$, where $k=\min(|s+t|,|s+t+1|)$ and $k'=\min(|s-t-1|,|s-t|)$. Then $y=(y^{(1)},y^{(2)})$ is an element of $\mathrm{GL}_{d_1}(\mathbb{C})\times\mathrm{GL}_{d_2}(\mathbb{C})\subset\mathrm{GL}_{2n}(\mathbb{C})$, where

$$y^{(1)}=\mathrm{diag}(\gamma_2,\gamma_4,\dots,\gamma_{2k})\quad\text{and}\quad y^{(2)}=\mathrm{diag}(-\gamma_2,-\gamma_4,\dots,-\gamma_{2k'})$$

if $t\geq 0$; and

$$y^{(1)}=\mathrm{diag}(-\gamma_2,-\gamma_4,\dots,-\gamma_{2k})\quad\text{and}\quad y^{(2)}=\mathrm{diag}(\gamma_2,\gamma_4,\dots,\gamma_{2k'})$$

if $t<0$; and $N=(N^{(1)},N^{(2)})$ is an nilpotent element in $\mathfrak{gl}_{d_1}(\mathbb{C})\times\mathfrak{gl}_{d_2}(\mathbb{C})\subset\mathfrak{gl}_{2n}(\mathbb{C})$, where

$$N^{(1)}=\mathrm{diag}(\delta_2,\delta_4,\dots,\delta_{2k})\quad\text{and}\quad N^{(2)}=\mathrm{diag}(\delta_2,\delta_4,\dots,\delta_{2k'}).$$

Note that d_1 is interchangeable with d_2 if t is replaced by $-t-1$; however, the pair (y,N) remains unchanged.

If $s\neq 0$ (*i.e.*, $d_1\neq d_2$), then $\mathrm{O}^+(\mathbf{v})$ is nontrivial. Hence, $\mathrm{O}^+(\mathbf{v})$ has two unipotent cuspidal representations, namely ζ_s and $\zeta_{-s}=\zeta_s\otimes\mathrm{sgn}$, and there is only one unipotent cuspidal representation for $\mathrm{Sp}(\mathbf{v}^*)$, namely $\zeta^*_t=\zeta^*_{-t-1}$. Therefore, we see that if (y,N) is the Langlands parameter associated with the irreducible supercuspidal representations $c\text{-}\mathrm{Ind}^{G(V)}_{G(V)_L}(\zeta_s\otimes\zeta^*_t)$, then the Langlands parameter associated with $c\text{-}\mathrm{Ind}^{G(V)}_{G(V)_L}(\zeta_{-s}\otimes\zeta^*_t)$ is $(-y,N)$.

If $s=0$ (*i.e.*, $d_1=d_2$), then $\mathrm{O}^+(\mathbf{v})$ is trivial. There is only one irreducible representation with unipotent reduction associated with the ordered pair $(0,t)$. And in this case (y, N) and $(-y, N)$ are in the same conjugacy class, and hence there is only one Langlands parameter.

6.2.2 Suppose that $G(V) = \mathrm{U}^-_{2n}(F)$. Then $G(V)_L/G(V)_{L,0^+} \simeq \mathrm{O}^-(\mathbf{v}) \times \mathrm{Sp}(\mathbf{v}^*)$. From (4.1), we have $2n = 2s^2 + 2t(t+1)$ for some odd integer s and some integer t. The situation is similar to Subsection 6.2.1.

6.3 Symplectic Groups

Suppose that $G(V) = \mathrm{Sp}_{2n}(F)$. Then $G(V)_L/G(V)_{L,0^+} \simeq \mathrm{Sp}(\mathbf{v}) \times \mathrm{Sp}(\mathbf{v}^*)$. From (4.1), we have $2n = 2s(s+1) + 2t(t+1)$ for some integers s and t. The complex dual group of $G(V)$ is $\mathrm{SO}_{2n+1}(\mathbb{C})$. Define $d_1 = (s+t+1)^2$ and $d_2 = (s-t)^2$. It is easy to check that $d_1 + d_2 = 2n+1$. The unordered pair $\{d_1, d_2\}$ remains unchanged if s is replaced by $-s-1$ or t is replaced by $-t-1$ or both. Note that exactly one of d_1, d_2 is even. Then $y = (y^{(1)}, y^{(2)})$ is a semisimple element of $\mathrm{SO}_{d_1}(\mathbb{C}) \times \mathrm{SO}_{d_2}(\mathbb{C}) \subset \mathrm{SO}_{2n+1}(\mathbb{C})$, where

$$y^{(1)} = \mathrm{diag}(\gamma_1, \gamma_3, \ldots, \gamma_{2|s+t+1|-1}), \qquad y^{(2)} = \mathrm{diag}(-\gamma_1, -\gamma_3, \ldots, -\gamma_{2|s-t|-1})$$

if $s+t+1$ is odd; and

$$y^{(1)} = \mathrm{diag}(-\gamma_1, -\gamma_3, \ldots, -\gamma_{2|s+t+1|-1}), \qquad y^{(2)} = \mathrm{diag}(\gamma_1, \gamma_3, \ldots, \gamma_{2|s-t|-1});$$

if $s+t+1$ is even; and $N = (N^{(1)}, N^{(2)})$ is a nilpotent element in $\mathfrak{so}_{d_1}(\mathbb{C}) \times \mathfrak{so}_{d_2}(\mathbb{C}) \subset \mathfrak{so}_{2n+1}(\mathbb{C})$ where

$$N^{(1)} = \mathrm{diag}(\delta_1, \delta_3, \ldots, \delta_{2|s+t+1|-1}) \quad \text{and} \quad N^{(2)} = \mathrm{diag}(\delta_1, \delta_3, \ldots, \delta_{2|s-t|-1}).$$

Note that from our definition, $\det(y^{(i)}) = 1$ for $i = 1, 2$, and hence $y^{(i)}$ is indeed in $\mathrm{SO}_{d_i}(\mathbb{C})$.

For such an unordered pair $\{d_1, d_2\}$ the pair (y, N) is uniquely determined. Note that the unordered pair $\{d_1, d_2\}$ remains the same if s and t are interchanged. For each ordered pair (s, t), as in Subsection 6.1.1, there is a unique irreducible supercuspidal representation associated with the four ordered pairs of integers: (s, t), $(s, -t-1)$, $(-s-1, t)$, and $(-s-1, -t-1)$. Hence a such pair (y, N) is associated with two irreducible supercuspidal representations if both d_1, d_2 are nonzero (*i.e.*, $s \neq t$ and $s \neq -t-1$), namely, $c\text{-}\mathrm{Ind}_{G(V)_L}^{G(V)}(\zeta_s \otimes \zeta_t)$ and $c\text{-}\mathrm{Ind}_{G(V)_L}^{G(V)}(\zeta_t \otimes \zeta_s)$; and (y, N) is associated with the unique irreducible supercuspidal representation if one of d_1, d_2 is zero (*i.e.*, $s = t$ or $s = -t-1$).

6.4 Orthogonal Groups

6.4.1 Suppose that $G(V) = \mathrm{O}^+_{2n}(F)$. Note that $\pi \otimes \mathrm{sgn}$ is also an irreducible supercuspidal representation with unipotent reduction. Now $G(V)_L/G(V)_{L,0^+} \simeq \mathrm{O}^+(\mathbf{v}) \times \mathrm{O}^+(\mathbf{v}^*)$. From (4.1), we have $2n = 2s^2 + 2t^2$ for some even integers s, t. From Subsection 4.1 we know that for each ordered pair (s, t) of even integers there is an irreducible supercuspidal representation $\pi_{s,t} := c\text{-}\mathrm{Ind}_{G(V)_L}^{G(V)}(\zeta_s \otimes \zeta_t)$. If both s, t are nonzero, then four representations $\pi_{s,t}$, $\pi_{s,-t}$, $\pi_{-s,t}$ and $\pi_{-s,-t}$ are all distinct. However, we have $\pi_{-s,-t} \simeq \pi_{s,t} \otimes \mathrm{sgn}$ and $\pi_{-s,t} \simeq \pi_{s,-t} \otimes \mathrm{sgn}$, since $\zeta_{-i} \simeq \zeta_i \otimes \mathrm{sgn}$.

The complex dual group of $G(V)$ is $\mathrm{O}_{2n}(\mathbb{C})$. Define $d_1=(s+t)^2$ and $d_2=(s-t)^2$. Then $2n=d_1+d_2$, and $y=(y^{(1)},y^{(2)})$ is a semisimple element of $\mathrm{O}_{d_1}(\mathbb{C})\times\mathrm{O}_{d_2}(\mathbb{C})\subset\mathrm{O}_{2n}(\mathbb{C})$, where

$$y^{(1)}=\mathrm{diag}(\gamma_1,\gamma_3,\ldots,\gamma_{2|s+t|-1})\quad\text{and}\quad y^{(2)}=\mathrm{diag}(-\gamma_1,-\gamma_3,\ldots,-\gamma_{2|s-t|-1});$$

$N=(N^{(1)},N^{(2)})$ is a nilpotent element in $\mathfrak{so}_{d_1}(\mathbb{C})\times\mathfrak{so}_{d_2}(\mathbb{C})\subset\mathfrak{so}_{2n}(\mathbb{C})$, where

$$N^{(1)}=\mathrm{diag}(\delta_1,\delta_3,\ldots,\delta_{2|s+t|-1})\quad\text{and}\quad N^{(2)}=\mathrm{diag}(\delta_1,\delta_3,\ldots,\delta_{2|s-t|-1}).$$

Note that the ordered pair (d_1,d_2) remains the same if s,t are interchanged, or if s is replaced by $-s$ and t is replaced by $-t$. Therefore, if both s,t are nonzero and $s\neq\pm t$, the pair (y,N) is associated with the four representations $\pi_{s,t}$, $\pi_{-s,-t}$, $\pi_{t,s}$, and $\pi_{-t,-s}$. Moreover, d_1 and d_2 are interchanged, and hence y is replaced by $-y$ if either s is replaced by $-s$ or t is replaced by $-t$ but not both. Then we see that the Langlands parameter associated with the representations $\pi_{-s,t}$, $\pi_{s,-t}$, $\pi_{t,-s}$, and $\pi_{-t,s}$ is $(-y,N)$.

If exactly one of s,t is zero, say $t=0$, then $d_1=d_2$, (y,N) and $(-y,N)$ are conjugate, and the four representations $\pi_{s,0},\pi_{-s,0},\pi_{0,s},\pi_{0,-s}$ have the same Langlands parameter; *i.e.,* they are in the same L-packet.

Remark 6.1 According to the above recipe, the Langlands parameters associated with the supercuspidal representations π and $\pi\otimes\mathrm{sgn}$ with unipotent reductions of $\mathrm{O}^+_{2n}(F)$ are the same; *i.e.,* π and $\pi\otimes\mathrm{sgn}$ are in the same L-packet.

Example 6.2 Let s be an even positive integer. Let η be a fixed unipotent cuspidal representation of $\mathrm{O}^+_{2s^2}(\mathbf{f})$ given in Subsection 4.1; *i.e.,* η is isomorphic to ζ_s or ζ_{-s}. Let V be an $4s^2$-dimensional quadratic space with trivial anisotropic kernel, and let L be a good lattice in V such that both L^*/L and $L/L^*\mathfrak{p}$ are $2s^2$-dimensional over $\mathbf{f}$. Define $y=\mathrm{diag}(\gamma_1,\gamma_3,\ldots,\gamma_{4s-1})$ and $N=\mathrm{diag}(\delta_1,\delta_3,\ldots,\delta_{4s-1})$. We have four irreducible supercuspidal representations with unipotent reduction of $\mathrm{O}(V)\simeq\mathrm{O}^+_{4s^2}(F)$ with which we associate the Langlands parameters (y,N) or $(-y,N)$:

representations	Langlands parameter
$c\text{-}\mathrm{Ind}_{G(V)_L}^{G(V)}(\eta\otimes\eta)$, $c\text{-}\mathrm{Ind}_{G(V)_L}^{G(V)}((\eta\otimes\mathrm{sgn})\otimes(\eta\otimes\mathrm{sgn}))$	(y,N)
$c\text{-}\mathrm{Ind}_{G(V)_L}^{G(V)}((\eta\otimes\mathrm{sgn})\otimes\eta)$, $c\text{-}\mathrm{Ind}_{G(V)_L}^{G(V)}(\eta\otimes(\eta\otimes\mathrm{sgn}))$	$(-y,N)$

6.4.2 Suppose that $G(V)=\mathrm{O}^-_{2n}(F)$. Then $G(V)_L/G(V)_{L,0^+}\simeq\mathrm{O}^-(\mathbf{v})\times\mathrm{O}^-(\mathbf{v}^*)$. By (4.1), we have $2n=2s^2+2t^2$ for some odd integers s and t. The situation is completely similar to Subsection 6.4.1 except for the parity of s,t.

6.4.3 Suppose that $G(V)=\mathrm{O}'_{2n}(F)$. Then $G(V)_L/G(V)_{L,0^+}\simeq\mathrm{O}^\epsilon(\mathbf{v})\times\mathrm{O}^{-\epsilon}(\mathbf{v}^*)$ where $\epsilon=+$ or $-$. By the assumption that the center E of the even Clifford algebra of V is an unramified quadratic extension of F, we know that the dimensions of $\mathbf{v}$ and $\mathbf{v}^*$ are both even. Now we have $2n=2s^2+2t^2$ for some integers s,t such that exactly one of them is even. The situation is completely similar to Subsection 6.4.1.

7 Proofs of the Main Theorems

Keep the setting of Subsection 2.3. We can write

$$\pi_0 = c\text{-Ind}_{G(V_0)_{L_0}}^{G(V_0)}(\eta_{s_0} \otimes \eta^*_{t_0})$$

for some cuspidal representation η_{s_0} (resp. $\eta^*_{t_0}$) of $G(\mathbf{v}_{s_0})$ (resp. $G(\mathbf{v}^*_{t_0})$) where $\mathbf{v}^*_{t_0} = L_0^*/L_0$, $\mathbf{v}_{s_0} = L_0/L_0^*\mathfrak{p}_D$ and L_0 is a good lattice in V_0, and η_{s_0} (resp. $\eta^*_{t_0}$) is the s_0-th (resp. t_0-th) term in its corresponding sequence $\{\eta_i \mid i \in \mathbb{Z}\}$ (resp. $\{\eta^*_i \mid i \in \mathbb{Z}\}$). By Proposition 4.6 we know that $\pi_i = c\text{-Ind}_{G(V_i)_{L_i}}^{G(V_i)}(\eta_{s_0+i} \otimes \eta^*_{t_0+i})$ for $i \in \mathbb{Z}$.

7.1 Unitary Case (I)

Proof of Theorem 5.3 First we assume that D is an unramified quadratic extension of F. Then both $G(\mathbf{v}_0)$ and $G(\mathbf{v}^*_0)$ are finite unitary groups. From Remark 4.2 we know that $\dim(\mathbf{v}_{s_0}) = (s_0(s_0+1))/2$ and $\dim(\mathbf{v}^*_{t_0}) = (t_0(t_0+1))/2$. By Proposition 4.6 we have

$$\begin{aligned}n_i := \dim(V_i) &= \tfrac{1}{2}(s_0+i)(s_0+i+1) + \tfrac{1}{2}(t_0+i)(t_0+i+1)\\ &= \left(\tfrac{s_0-t_0}{2}\right)^2 + \left(\tfrac{s_0+t_0}{2}+i\right)\left(\tfrac{s_0+t_0}{2}+i+1\right).\end{aligned} \tag{7.1}$$

Because we now assume that $\dim(V_0)$ and $\dim(V_1)$ are of the same parity, from Subsection 4.4.1 we see that $s_0 - t_0$ must be even. The sequence $\{n_i \mid i \in \mathbb{Z}\}$ achieves its minimum at $i = -1, 0$ under our assumption on the index, so $s_0 + t_0 = 0$, and hence $n_i = (\frac{s_0-t_0}{2})^2 + i(i+1)$. From Subsection 4.4.1 $\pi_i \otimes \mathrm{sgn}^{((i+1)(i+2)+t_0)/2}$ has unipotent reduction and hence it is isomorphic to $c\text{-Ind}_{G(V_i)_{L_i}}^{G(V_i)}(\zeta_{s_0+i} \otimes \zeta_{t_0+i})$. From Subsection 6.1.1, the Langlands parameter (y_i, N_i) for $\pi_i \otimes \mathrm{sgn}^{((i+1)(i+2)+t_0)/2}$ is

$$\begin{aligned}y_i &= \mathrm{diag}(\gamma_1, \gamma_3, \dots, \gamma_{|s_0-t_0|-1}, \gamma_2, \gamma_4, \dots, \gamma_{2i^\sharp}),\\ N_i &= \mathrm{diag}(\delta_1, \delta_3, \dots, \delta_{|s_0-t_0|-1}, \delta_2, \delta_4, \dots, \delta_{2i^\sharp}),\end{aligned}$$

where $i^\sharp = \min(|i|, |i+1|)$. Therefore, we have $y_0 = \mathrm{diag}(\gamma_1, \gamma_3, \dots, \gamma_{|s_0-t_0|-1})$ and $N_0 = \mathrm{diag}(\delta_1, \delta_3, \dots, \delta_{|s_0-t_0|-1})$, and hence

$$y_i = \mathrm{diag}(y_0, \gamma_2, \gamma_4, \dots, \gamma_{2i^\sharp}) \quad \text{and} \quad N_i = \mathrm{diag}(N_0, \delta_2, \delta_4, \dots, \delta_{2i^\sharp})$$

for any $i \in \mathbb{Z}$.

Next we assume that D is a ramified quadratic extension of F. As mentioned in Section 2.1 we assume that the dimensions n_i are all even. In this situation, one of $G(\mathbf{v}_{s_0})$ and $G(\mathbf{v}^*_{t_0})$ is a finite even orthogonal group and the other is a finite symplectic group. We can assume that $G(\mathbf{v}^*_{t_0})$ is symplectic. Now every π_i has unipotent reduction from Subsection 4.4.2. By Remark 4.5 we see that s_0 is even and $\dim(\mathbf{v}_{s_0}) = 2(\frac{s_0}{2})^2$; t_0 is odd and $\dim(\mathbf{v}^*_{t_0}) = 2(\frac{t_0-1}{2})(\frac{t_0+1}{2})$. It is easy to see that $\frac{s_0}{2}$ is even (resp. odd) if the Witt index of V_0 is half of the dimension of V_0 (resp. half of the dimension of V_0 minus 1). Now the i-th term after the term of dimension $2(\frac{s_0}{2})^2$ has dimension $2(\frac{s_0+i}{2})^2$ if i is even, and has dimension $2(\frac{s_0+i-1}{2})(\frac{s_0+i+1}{2})$ if i odd. By Proposition 4.6 we conclude that

$$n_i=\begin{cases}2(\frac{s_0+i}{2})^2+2(\frac{t_0+i-1}{2})(\frac{t_0+i+1}{2}), & \text{if } i \text{ is even;}\\ 2(\frac{s_0+i-1}{2})(\frac{s_0+i+1}{2})+2(\frac{t_0+i}{2})^2, & \text{if } i \text{ is odd}\end{cases}$$
$$=(\tfrac{s_0-t_0-1}{2})(\tfrac{s_0-t_0+1}{2})+(\tfrac{s_0+t_0-1}{2}+i)(\tfrac{s_0+t_0+1}{2}+i).$$

The sequence $\{n_i \mid i\in\mathbb{Z}\}$ achieves its minimum at $i=-1,0$, so $s_0+t_0-1=0$; in particular, s_0 and t_0 cannot be both positive. Hence, we have $n_i=(\frac{s_0-t_0-1}{2})(\frac{s_0-t_0+1}{2})+i(i+1)$. From Subsections 6.2.1 and 6.2.2, the Langlands parameter (y_i,N_i) for π_i is

$$y_i=\operatorname{diag}(-\gamma_2,-\gamma_4,\dots,-\gamma_{2k},\gamma_2,\gamma_4,\dots,\gamma_{2i^\sharp}),$$
$$N_i=\operatorname{diag}(\delta_2,\delta_4,\dots,\delta_{2k},\delta_2,\delta_4,\dots,\delta_{2i^\sharp}),$$

where $k=\min(|\frac{s_0-t_0-1}{2}|,|\frac{s_0-t_0+1}{2}|)$ and $i^\sharp=\min(|i|,|i+1|)$. Therefore,

$$y_i=\operatorname{diag}(y_0,\gamma_2,\gamma_4,\dots,\gamma_{2i^\sharp})\quad\text{and}\quad N_i=\operatorname{diag}(N_0,\delta_2,\delta_4,\dots,\delta_{2i^\sharp}),$$

and hence the theorem is proved. ∎

Example 7.1 Suppose that D is an unramified quadratic extension of F. Let η (resp. η^*) be the unipotent cuspidal representation of $G(\mathbf{v})$ (resp. $G(\mathbf{v}^*)$) with $\dim(\mathbf{v})=15=\frac{5\times(5+1)}{2}$ (resp. $\dim(\mathbf{v}^*)=3=\frac{2\times(2+1)}{2}$). We want to determine the position of π in its sequence by the preservation principle. Suppose the dimensions of V_0,V_1 are of the same parity. Consider the representation $\pi=c\text{-}\mathrm{Ind}_{G(V)_L}^{G(V)}(\eta\otimes\eta^*)$.

Now we have $\eta=\zeta_5=\zeta_{-6}$ and $\eta^*=\zeta_2=\zeta_{-3}$. Since $\zeta_5=\eta_5\otimes\mathrm{sgn}$ and $\zeta_{-3}=\eta_{-3}\otimes\mathrm{sgn}$, we have $s=-6=s_0-2$, $t=2=t_0-2$ with $s_0=-4=-t_0$, and then

i	$\cdots$	-3	-2	-1	0	1	$\cdots$
$\dim(\mathbf{v}_{s_i})$	$\cdots$	21	15	10	6	3	$\cdots$
$\dim(\mathbf{v}^*_{t_i})$	$\cdots$	1	3	6	10	15	$\cdots$
n_i	$\cdots$	22	18	16	16	18	$\cdots$

Hence, $\pi=\pi_{-2}$ in its sequence with $n_0=16$. Thus the semisimple elements y_i in the Langlands parameters (y_i,N_i) of $\pi_i\otimes\mathrm{sgn}^{\frac{(i+1)(i+2)+4}{2}}$ are:

i	s_i	t_i	n_i	$\pi_i\otimes\mathrm{sgn}^{\frac{(i+1)(i+2)+4}{2}}$	y_i
-2	-6	2	18	π_{-2}	$\operatorname{diag}(\gamma_1,\gamma_3,\gamma_5,\gamma_7,\gamma_2)$
-1	-5	3	16	π_{-1}	$\operatorname{diag}(\gamma_1,\gamma_3,\gamma_5,\gamma_7)$
0	-4	4	16	$\pi_0\otimes\mathrm{sgn}$	$\operatorname{diag}(\gamma_1,\gamma_3,\gamma_5,\gamma_7)$
1	-3	5	18	$\pi_1\otimes\mathrm{sgn}$	$\operatorname{diag}(\gamma_1,\gamma_3,\gamma_5,\gamma_7,\gamma_2)$
2	-2	6	22	π_2	$\operatorname{diag}(\gamma_1,\gamma_3,\gamma_5,\gamma_7,\gamma_2,\gamma_4)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

7.2 Unitary Case (II)

Note that from our assumption, D is an unramified quadratic extension of F in this case.

Proof of Theorem 5.5 Keep the notation from Subsection 7.1. Because we now assume that the dimensions of V_0, V_1 are of the opposite parity, from Subsection 4.4.1, $s_0 - t_0$ is odd. Then $s_0 + t_0 + 1$ and $s_0 - t_0 - 1$ are both even. Then from (7.1) we have

$$\begin{aligned} n_i &= \tfrac{1}{2}(s_0+i)(s_0+i+1) + \tfrac{1}{2}(t_0+i)(t_0+i+1) \\ &= \big(\tfrac{s_0-t_0-1}{2}\big)\big(\tfrac{s_0-t_0+1}{2}\big) + \big(\tfrac{s_0+t_0+1}{2}+i\big)^2. \end{aligned}$$

Because the sequence $\{n_i \mid i \in \mathbb{Z}\}$ achieves its minimum when $i = 0$, we have $s_0 + t_0 + 1 = 0$, and hence $n_i = \big(\frac{s_0-t_0-1}{2}\big)\big(\frac{s_0-t_0+1}{2}\big) + i^2$. Now assume i is odd; then the representation in (5.4) is isomorphic to $c\text{-Ind}_{G(V_0)_{L_0}}^{G(V_0)}(\zeta_{s_0+i}\otimes\zeta_{t_0+i})$, which has unipotent reduction and has Langlands parameter (y_i, N_i) given by

$$\begin{aligned} y_i &= \operatorname{diag}(\gamma_2,\gamma_4,\dots,\gamma_{2k},\gamma_1,\gamma_3,\dots,\gamma_{2|i|-1}), \\ N_i &= \operatorname{diag}(\delta_2,\delta_4,\dots,\delta_{2k},\delta_1,\delta_3,\dots,\delta_{2|i|-1}), \end{aligned}$$

where $k = \min(|\frac{s_0-t_0-1}{2}|, |\frac{s_0-t_0+1}{2}|)$ from Subsection 6.1.1. Moreover, we know that $y_0 = \operatorname{diag}(\gamma_2,\gamma_4,\dots,\gamma_{2k})$ and $N_0 = \operatorname{diag}(\delta_2,\delta_4,\dots,\delta_{2k})$. Therefore, we have

$$y_i = \operatorname{diag}(y_0,\gamma_1,\gamma_3,\dots,\gamma_{2|i|-1}) \quad\text{and}\quad N_i = \operatorname{diag}(N_0,\delta_1,\delta_3,\dots,\delta_{2|i|-1})$$

for any odd integer i. ∎

7.3 Split Orthogonal-symplectic Case

In this subsection we assume that V_0 is an even-dimensional quadratic space.

Proof of Theorem 5.7 By the assumption in Subsection 2.1, we know that both $\mathbf{v}_{s_0}$ and $\mathbf{v}^*_{t_0}$ are even-dimensional for this case , and hence we have $\dim(\mathbf{v}_{s_0}) = 2(\frac{s_0}{2})^2$ and $\dim(\mathbf{v}^*_{t_0}) = 2(\frac{t_0}{2})^2$ for some even integers s_0, t_0 from Subsection 4.1 and Remark 4.5. By Proposition 4.6 we have

$$n_i = \begin{cases} 2(\frac{s_0+i}{2})^2 + 2(\frac{t_0+i}{2})^2 & \text{if } i \text{ is even,} \\ 2(\frac{s_0+i-1}{2})(\frac{s_0+i+1}{2}) + 2(\frac{t_0+i-1}{2})(\frac{t_0+i+1}{2}) & \text{if } i \text{ is odd.} \end{cases}$$

This is now case (III) of Subsection 2.3, and the dimension n_i^* defined in (2.2) becomes

$$n_i^* = \big(\tfrac{s_0-t_0}{2}\big)^2 + \big(\tfrac{s_0+t_0}{2}+i\big)^2.$$

Because the sequence $\{n_i^* \mid i \in \mathbb{Z}\}$ achieves its minimum at $i = 0$, we have $s_0 + t_0 = 0$, *i.e.*, $n_i^* = (\frac{s_0-t_0}{2})^2 + i^2$. Now because $s_0 = -t_0$, we have $\dim(\mathbf{v}_{s_0}) = \dim(\mathbf{v}^*_{t_0})$, *i.e.*, $\mathrm{O}(\mathbf{v}_{s_0}) \simeq \mathrm{O}(\mathbf{v}^*_{t_0})$; in particular, from Subsections 6.4.1, 6.4.2, and 6.4.3, we know that the anisotropic kernel of V_0 (and hence every quadratic space in the sequence $\{V_i \mid i \in \mathbb{Z}\}$) is either trivial or four-dimensional. Moreover, we know that $\eta_{s_0} \simeq \eta^*_{t_0} \otimes \mathrm{sgn}$ from Subsection 4.1, and consequently $\pi_0 \simeq \pi_0 \otimes \mathrm{sgn}$. Therefore, from the requirement of $\{\pi_i \mid i \in \mathbb{Z}\}$ in Subsection 2.3, we have $\pi_{-i} \simeq \pi_i \otimes \mathrm{sgn}$ for any $i \in \mathbb{Z}$. Hence, from Subsection 6.4.1 (in particular, Example 6.2), the Langlands parameter (y_0, N_0) associated with π_0 is

$$y_0 = \operatorname{diag}(-\gamma_1,-\gamma_3,\dots,-\gamma_{|s_0-t_0|-1}) \quad\text{and}\quad N_0 = \operatorname{diag}(\delta_1,\delta_3,\dots,\delta_{|s_0-t_0|-1}). \tag{7.2}$$

If i is even, then V_i is an even quadratic space, $s_i = s_0 + i$, $t_i = t_0 + i$ and $s_0 = -t_0$. Hence $|\frac{s_i - t_i}{2}| = |\frac{s_0 - t_0}{2}|$, $|\frac{s_i + t_i}{2}| = |i|$, and, from Subsections 6.4.1 or 6.4.2, the pair (y_i, N_i) associated with π_i is

$$\begin{aligned} y_i &= \mathrm{diag}(-\gamma_1, -\gamma_3, \dots, -\gamma_{|s_0 - t_0| - 1}, \gamma_1, \dots, \gamma_{2|i|-1}) = \mathrm{diag}(y_0, \gamma_1, \dots, \gamma_{2|i|-1}), \\ N_i &= \mathrm{diag}(\delta_1, \delta_3, \dots, \delta_{|s_0 - t_0| - 1}, \delta_1, \dots, \delta_{2|i|-1}) = \mathrm{diag}(N_0, \delta_1, \dots, \delta_{2|i|-1}). \end{aligned} \tag{7.3}$$

If i is odd, then V_i is a symplectic space, $s_i = s_0 + i - 1$, $t_i = t_0 + i - 1$ and $s_0 = -t_0$. Then $|\frac{s_i - t_i}{2}| = |\frac{s_0 - t_0}{2}|$, $|\frac{s_i + t_i}{2} + 1| = |i|$ and, from Subsection 6.3, the pair (y_i, N_i) associated with π_i is the same as in (7.3). Thus, the theorem is proved. ∎

Remark 7.2 Because $\frac{s_0}{2}$, $\frac{t_0}{2}$ in the proof are of the same parity, $(\frac{s_0 - t_0}{2})^2$ is even, and hence we have $\det(y_0) = 1$ from (7.2). Then Theorem 5.7 is indeed a consequence of Conjecture 3.1.

7.4 Non-split Orthogonal-symplectic Case

In this subsection we assume that V_0 is a symplectic space.

Proof of Theorem 5.9 Now that V_0 is symplectic, so are $\mathbf{v}_{s_0}$ and $\mathbf{v}^*_{t_0}$. Hence, both s_0 and t_0 are odd and $\dim(\mathbf{v}_{s_0}) = 2(\frac{s_0 - 1}{2})(\frac{s_0 + 1}{2})$ (resp. $\dim(\mathbf{v}^*_{t_0}) = 2(\frac{t_0 - 1}{2})(\frac{t_0 + 1}{2})$). By Proposition 4.6, we have

$$n_i = \begin{cases} 2(\frac{s_0 + i - 1}{2})(\frac{s_0 + i + 1}{2}) + 2(\frac{t_0 + i - 1}{2})(\frac{t_0 + i + 1}{2}), & \text{if } i \text{ is even,} \\ 2(\frac{s_0 + i}{2})^2 + 2(\frac{t_0 + i}{2})^2, & \text{if } i \text{ is odd.} \end{cases}$$

We are now in case (IV) of Subsection 2.3. Then the dimension n_i^* defined in (2.2) is

$$n_i^* = (\tfrac{s_0 - t_0}{2})^2 + (\tfrac{s_0 + t_0}{2} + i)^2.$$

The sequence $\{n_i^* \mid i \in \mathbb{Z}\}$ achieves its minimum at $i = 0$; we have $s_0 + t_0 = 0$, *i.e.*, $n_i^* = (\frac{s_0 - t_0}{2})^2 + i^2$. Because $\frac{s_0 + i}{2}$ and $\frac{t_0 + i}{2}$ are now of opposite parity for any odd integer i, from Subsection 6.4.3 we see that all quadratic spaces in the sequence $\{V_i \mid i \in \mathbb{Z}\}$ must have two-dimensional anisotropic kernel.

Because $s_0 = -t_0$, we now have $\mathrm{Sp}(\mathbf{v}_{s_0}) \simeq \mathrm{Sp}(\mathbf{v}^*_{t_0})$ and $\pi_{-i} \simeq \pi_i \otimes \mathrm{sgn}$ for any $i \in \mathbb{Z}$. Now from Subsection 6.3, we know that the Langlands parameter (y_0, N_0) associated with π_0 is

$$y_0 = \mathrm{diag}(\gamma_1, \gamma_3, \dots, \gamma_{|s_0 - t_0| - 1}) \quad \text{and} \quad N_0 = \mathrm{diag}(\delta_1, \delta_3, \dots, \delta_{|s_0 - t_0| - 1}). \tag{7.4}$$

If i is even, then V_i is symplectic, $s_i = s_0 + i$, $t_i = t_0 + i$ and $s_0 = -t_0$. Hence, $|\frac{s_i - t_i}{2}| = |\frac{s_0 - t_0}{2}|$, $|\frac{s_i + t_i}{2}| = |i|$, and, from Subsection 6.3, the pair (y_i, N_i) associated with π_i is

$$\begin{aligned} y_i &= \mathrm{diag}(\gamma_1, \gamma_3, \dots, \gamma_{|s_0 - t_0| - 1}, -\gamma_1, -\gamma_3, \dots, -\gamma_{2|i|-1}), \\ N_i &= \mathrm{diag}(\delta_1, \delta_3, \dots, \delta_{|s_0 - t_0| - 1}, \delta_1, \delta_3, \dots, \delta_{2|i|-1}). \end{aligned}$$

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If i is odd, then V_i is quadratic, $s_i = s_0 + i$, $t_i = t_0 + i$, and $s_0 = -t_0$. Hence $|\frac{s_i - t_i}{2}| = |\frac{s_0 - t_0}{2}|$, $|\frac{s_i + t_i}{2}| = |i|$ and, from Subsections 6.4.3, the pair (y_i, N_i) associated with π_i is

$$y_i = \operatorname{diag}(-\gamma_1, -\gamma_3, \dots, -\gamma_{|s_0 - t_0|-1}, \gamma_1, \gamma_3, \dots, \gamma_{2|i|-1}),$$
$$N_i = \operatorname{diag}(\delta_1, \delta_3, \dots, \delta_{|s_0 - t_0|-1}, \delta_1, \delta_3, \dots, \delta_{2|i|-1}).$$

From the above and (7.4), we conclude that

$$y_i = \begin{cases} \operatorname{diag}(y_0, -\gamma_1, -\gamma_3, \dots, -\gamma_{2|i|-1}), & \text{if } i \text{ is even;} \\ \operatorname{diag}(-y_0, \gamma_1, \gamma_3, \dots, \gamma_{2|i|-1}), & \text{if } i \text{ is odd,} \end{cases}$$
$$N_i = \operatorname{diag}(N_0, \delta_1, \delta_3, \dots, \delta_{2|i|-1}).$$

Thus, Theorem 5.9 is proved. ∎

Example 7.3 Suppose that $\pi = c\text{-}\mathrm{Ind}_{G(V)_L}^{G(V)}(\eta \otimes \eta^*)$, where η (resp. η^*) is a unipotent cuspidal representation of $\mathrm{O}(\mathbf{v})$ (resp. $\mathrm{O}(\mathbf{v}^*)$) with $\dim(\mathbf{v}) = 18 = 2 \times 3^2$ (resp. $\dim(\mathbf{v}^*) = 8 = 2 \times 2^2$). Hence, $\dim(V) = \dim(\mathbf{v}) + \dim(\mathbf{v}^*) = 26$. We want to determine the position of π in the sequence given by the preservation principle. Because $\mathrm{O}(\mathbf{v})$ (resp. $\mathrm{O}(\mathbf{v}^*)$) has two unipotent cuspidal representations $\zeta_3, \zeta_{-3} = \zeta_3 \otimes \operatorname{sgn}$ (resp. $\zeta_2, \zeta_{-2} = \zeta_2 \otimes \operatorname{sgn}$), we have following two possibilities:

(i) Suppose $(\eta, \eta^*) = (\zeta_3, \zeta_{-2})$, *i.e.*, $(s, t) = (6, -4) = (5 + 1, -5 + 1)$. Then $\pi = \pi_1$ with $s_0 = 5$, $t_0 = -5$ and $n_0 = 2 \times (\frac{5-1}{2}) \times (\frac{5+1}{2}) + 2 \times (\frac{-5-1}{2}) \times (\frac{-5+1}{2}) = 24$. Hence, we have the following table:

G	$\cdots$	O'	Sp	O'	Sp	O'	Sp	O'	$\cdots$
i	$\cdots$	-3	-2	-1	0	1	2	3	$\cdots$
$\dim(\mathbf{v}_{s_i})$	$\cdots$	2	4	8	12	18	24	32	$\cdots$
$\dim(\mathbf{v}^*_{t_i})$	$\cdots$	32	24	18	12	8	4	2	$\cdots$
n_i	$\cdots$	34	28	26	24	26	28	34	$\cdots$
n^*_i	$\cdots$	34	29	26	25	26	29	34	$\cdots$

Thus, the semisimple elements y_i in Langlands parameters (y_i, N_i) of π_i are:

i	s_i	t_i	n^*_i	π_i	y_i
0	5	-5	25	π_0	$\operatorname{diag}(\gamma_1, \gamma_3, \gamma_5, \gamma_7, \gamma_9)$
1	6	-4	26	π_1	$\operatorname{diag}(-\gamma_1, -\gamma_3, -\gamma_5, -\gamma_7, -\gamma_9, \gamma_1)$
2	7	-3	29	π_2	$\operatorname{diag}(\gamma_1, \gamma_3, \gamma_5, \gamma_7, \gamma_9, -\gamma_1, -\gamma_3)$
3	8	-2	34	π_3	$\operatorname{diag}(-\gamma_1, -\gamma_3, -\gamma_5, -\gamma_7, -\gamma_9, \gamma_1, \gamma_3, \gamma_5)$
4	9	-1	41	π_4	$\operatorname{diag}(\gamma_1, \gamma_3, \gamma_5, \gamma_7, \gamma_9, -\gamma_1, -\gamma_3, -\gamma_5, -\gamma_7)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

(ii) Suppose $(\eta, \eta^*) = (\zeta_{-3}, \zeta_2)$. The situation is similar to (i) except we only need to replace i by $-i$, *i.e.*, $\pi = \pi_{-1}$ with $n_0 = 24$.

(iii) Suppose $(\eta, \eta^*) = (\zeta_3, \zeta_2)$. Then $(s, t) = (6, 4) = (1 + 5, -1 + 5)$, $\pi = \pi_5$ with $s_0 = 1$, $t_0 = -1$, and $n_0 = 2 \times (\frac{1-1}{2}) \times (\frac{1+1}{2}) + 2 \times (\frac{-1-1}{2}) \times (\frac{-1+1}{2}) = 0$. Hence, we have the following table:

G	$\dots$	Sp	O′	Sp	O′	Sp	O′	Sp	$\dots$
i	$\dots$	0	1	2	3	4	5	6	$\dots$
$\dim(\mathrm{v}_{s_i})$	$\dots$	0	2	4	8	12	18	24	$\dots$
$\dim(\mathrm{v}^*_{t_i})$	$\dots$	0	0	0	2	4	8	12	$\dots$
n_i	$\dots$	0	2	4	10	16	26	36	$\dots$
n^*_i	$\dots$	1	2	5	10	17	26	37	$\dots$

Thus, the semisimple elements y_i in the Langlands parameters (y_i, N_i) of π_i are:

i	s_i	t_i	n^*_i	π_i	y_i
0	1	−1	1	π_0	$\operatorname{diag}(\gamma_1)$
1	2	0	2	π_1	$\operatorname{diag}(-\gamma_1, \gamma_1)$
2	3	1	5	π_2	$\operatorname{diag}(\gamma_1, -\gamma_1, -\gamma_3)$
3	4	2	10	π_3	$\operatorname{diag}(-\gamma_1, \gamma_1, \gamma_3, \gamma_5)$
4	5	3	17	π_4	$\operatorname{diag}(\gamma_1, -\gamma_1, -\gamma_3, -\gamma_5, -\gamma_7)$
5	6	4	26	π_5	$\operatorname{diag}(-\gamma_1, \gamma_1, \gamma_3, \gamma_5, \gamma_7, \gamma_9)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

(iv) Suppose $(\eta, \eta^*) = (\zeta_{-3}, \zeta_{-2})$. The situation is similar to (iii) with i replaced by $-i$, and hence $\pi = \pi_{-5}$ with $n_0 = 0$.

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