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CORRIGENDUM

Rank monotonicity in centrality measures—Corrigendum

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We correct the statement of Theorem 5 of our paper Boldi et al. (2017): the original statement is not true in general, as a simple counterexample shows. We, however, show that the statement holds under a mild positivity condition.

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1. Corrected statement and counterexample

Given a nonnegative square matrix M with spectral radius[1] $\rho(M)$, its *damped spectral ranking* (Vigna, 2016) is given by[2]

$$\boldsymbol{r} = \boldsymbol{v} \sum_{n\geq 0} (\alpha M)^n = \boldsymbol{v}\big(1 - \alpha M\big)^{-1} \tag{1}$$

where $0 \leq \alpha < 1/\rho(M)$ is a *damping factor*, and $\boldsymbol{v}$ is a nonnegative *preference vector*.

In Boldi et al. (2017), we discussed under which conditions damped spectral rankings on graphs, such as Katz's index or PageRank, enjoy *strict rank monotonicity*, that is, if it happens that when the score of $z \neq y$ is smaller than or equal to the score of y, after adding an arc from x to y the score of z becomes smaller than the score of y. In this note we correct the statement of Theorem 5, which is not true without the condition $\boldsymbol{v} > 0$.

Adding the latter condition to Theorem 5 of Boldi et al. (2017), the statement becomes[3]:

Theorem 1. *Let M and M' be two nonnegative matrices, such that $M' - M = \boldsymbol{\chi}_x^T \boldsymbol{\delta}$ (i.e., the matrices differ only on the x-th row, and $\boldsymbol{\delta}$ is the corresponding row difference). Let also $\boldsymbol{v} > 0$ be a positive preference vector and $0 \leq \alpha < \min\big(1/\rho(M), 1/\rho(M')\big)$; let $\boldsymbol{r} = \boldsymbol{v} \sum_{n\geq 0} (\alpha M)^n$ and $\boldsymbol{r}' = \boldsymbol{v} \sum_{n\geq 0} (\alpha M')^n$ be the damped spectral rankings associated with M and M', respectively. Assume further that:*

1. *there is exactly one y such that $\delta_y > 0$;*
2. *$r_y < r'_y$.*

Then, if $r_z \leq r_y$ we have $r'_z - r_z < r'_y - r_y$, and in particular $r'_z < r'_y$.

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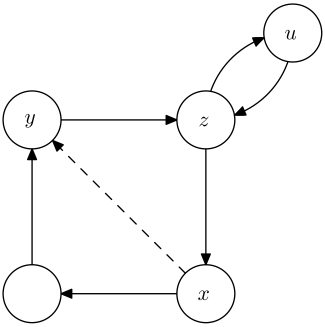

Figure 1. A counterexample.

The purpose of this note is to provide a proof of this corrected version of the theorem, highlighting the reasons behind the necessity of the condition $\boldsymbol{v} > 0$.

Let us start showing that the original statement was indeed false. Consider the graph in Figure 1 (without the dashed arrow): let M be its adjacency matrix and M' be the matrix of the graph obtained after the addition of the arc $x \to y$. M has dominant eigenvalue $\sqrt{\varphi}$, where $\varphi = (\sqrt{5} + 1)/2$ is the golden ratio. As usual, let $C = (1 - \alpha M)^{-1}$.

Let us first prove that $c_{yz} = c_{yy}$ iff $\alpha = 1/\varphi$. Note that every path from y to z can be seen as a (possibly empty) path from y to y, followed by a path from y to z which does not traverse again y. Hence the condition $c_{yz} = c_{yy}$ is equivalent to the fact that the sum s of the weights of the paths from y to z which do not traverse again y is exactly one.[4] Since we are considering the bare adjacency matrix M, a path of length n has weight α^n, and thus looking at the graph

$$s = \alpha\left(1 + \alpha^2 + \alpha^4 + \cdots\right) = \frac{\alpha}{1 - \alpha^2}$$

which is exactly 1 when $\alpha = 1/\varphi$. It is then easy to check that indeed, with this choice of α, we obtain[5]

$$c_{yy} = c_{yz} = \frac{1}{3 - \sqrt{5}}$$

Let us compute the damped spectral ranking of M (i.e., Katz's index of the graph in Figure 1). We observe that if we choose $\boldsymbol{v}$ so that $v_u = v_z = 0$, the condition $s = 1$ implies $r_y = r_z$, since all paths giving score to z pass through y. Thus, in particular, the condition $v_u > 0$ of Lemma 4 is necessary: we have $r_y \leq r_z$, there is a path from u to z not passing through y, and nonetheless $c_{yy} = c_{yz}$.

The same graph shows that in general it is not possible to prove strict rank monotonicity for a generic spectral ranking without additional conditions (strong connectivity or $\boldsymbol{r} > 0$ being not sufficient). The key observation is that the property $r_y = r_z$ remains true even after adding an arc toward y, because s does not change. Thus, in the case of Katz's index, when we add an arc from x to y in the graph of Figure 1, the score of y will increase (i.e., $r'_y > r_y$), as Katz's index is score monotone (Boldi and Vigna, 2014). However, also the score of z will increase exactly by the same amount (i.e., $r'_z - r_z = r'_y - r_y$), violating strict rank monotonicity.[6]

The case of PageRank is analogous, but since we have to normalize the rows of the adjacency matrix the sum of the weights of the paths from y to z which do not traverse again y is

$$\alpha\left(1 + \frac{1}{2}\alpha^2 + \frac{1}{4}\alpha^4 + \cdots\right) = \frac{\alpha}{1 - \frac{1}{2}\alpha^2}$$

which is exactly 1 when $\alpha = \sqrt{3} - 1$.

The counterexample of Figure 1 thus shows that Theorem 5 from Boldi et al. (2017) is not true in general, not even for strongly connected graphs; as a consequence, also Corollaries 1 and 2 of Boldi et al. (2017) are false in general.

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The two following corollaries are correct restatements of the same corollaries in Boldi et al. (2017), under the further assumption that the preference vector and damping factor be positive[7]:

Corollary 1. *PageRank satisfies the strict rank-monotonicity axiom, for every graph, positive damping factor and positive preference vector.*

Corollary 2. *Katz's index satisfies the strict rank-monotonicity axiom, for every graph, positive damping factor and positive preference vector.*

2. Proof of Theorem 1

Let us consider the *graph induced by a nonnegative matrix M* as the graph whose arcs correspond to the nonzero entries of M; one can also consider M as providing the weights of such arcs. Let us write $x \rightsquigarrow_{\neg y} z$ to mean that there exists, in the graph induced by M, a path from x to z not passing through y (note that, in particular, this implies $x, z \neq y$).

First of all, we need to extend a result from McDonald et al. (1995):

Theorem 2. *Let M be a nonnegative matrix, $0 < \alpha < 1/\rho(M)$ and $C = \left(1 - \alpha M\right)^{-1}$. Given indices x, y, z we have that*

- $c_{yz} = c_{yx}c_{xz}/c_{xx}$ *if* $y \not\rightsquigarrow_{\neg x} z$;
- $c_{yz} > c_{yx}c_{xz}/c_{xx}$ *if* $y \rightsquigarrow_{\neg x} z$.

The above statement extends Theorem 3.9 from McDonald et al. (1995) by removing the condition that x, y, and z be distinct.

Proof. We notice that if $x = y$ or $x = z$, the statement trivializes, as in the case $x = y$ we have $x \not\rightsquigarrow_{\neg x} z$, so the thesis reduces to the identity $c_{xz} = c_{xx}c_{xz}/c_{xx} = c_{xz}$ analogously if $x = z$.

The remaining case is $y = z$. In this case, the property

$$c_{xx}c_{zz} - c_{xz}c_{zx} > 0$$

is part of the statement of Theorem 1 of Willoughby (1977). □

Let us now recall Lemma 3 of Boldi et al. (2017):

Lemma 3. *Let M be a nonnegative matrix, $0 \le \alpha < 1/\rho(M)$ a damping factor and $\boldsymbol{v}$ a nonnegative preference vector. Let*

$$\boldsymbol{r} = \boldsymbol{v} \sum_{n \ge 0} (\alpha M)^n$$

be the associated damped spectral ranking and let $C = \left(1 - \alpha M\right)^{-1}$. Then, given y and z such that $c_{yz} > 0$ and letting $q = c_{yy}/c_{yz}$, we have $c_{wy} \le q \cdot c_{wz}$ for all w. In particular, if $r_y > 0$

- *if* $r_z \le r_y$, *then* $c_{yz} \le c_{yy}$;
- *if* $r_z < r_y$, *then* $c_{yz} < c_{yy}$.

Using Theorem 2, we can strengthen part of its statement under some additional conditions:

Lemma 4. *With the notation of Lemma 3, if $r_z \le r_y$ and there is some u with $v_u > 0$ and $u \rightsquigarrow_{\neg y} z$, then $c_{yz} < c_{yy}$.*

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Proof. We recall from Theorem 2 that if there is no path from w to z not passing through y we have

$$q \cdot c_{wz} = c_{wy}$$

otherwise

$$q \cdot c_{wz} > c_{wy}$$

Now, if $v_u > 0$, $u \rightsquigarrow_{\neg y} z$, and $c_{yy} \leq c_{yz}$, then as above $q \leq 1$, but

$$r_y = \sum_w v_w c_{wy} = \sum_{w \rightsquigarrow_{\neg y} z} v_w c_{wy} + \sum_{w \not\rightsquigarrow_{\neg y} z} v_w c_{wy} < \sum_{w \rightsquigarrow_{\neg y} z} v_w c_{wz} + \sum_{w \not\rightsquigarrow_{\neg y} z} v_w c_{wz} = r_z$$

as $0 \neq v_u c_{uy} < v_u\, q c_{uz} \leq v_u c_{uz}$ appears in the first summation. □

Based on these results, we can finally provide a proof of Theorem 1:

Proof of Theorem 1. The proof is the same as that of Theorem 3 of Boldi et al. (2017). However, we use first the score-monotonicity condition to prove the case in which $r'_z \leq r_z$. Then, when analyzing the case $c_{yz} > 0$ at the end of the proof, since $\boldsymbol{v} > 0$ the empty path from z to z makes it possible to apply Lemma 4, obtaining the stricter bound $q = c_{yy}/c_{yz} > 1$, which turns the last sequence of inequalities of the proof into a strict one. □

Notes

1 The *spectral radius* of a matrix is the largest absolute value of an eigenvalue.
2 In this paper we use row vectors.
3 The original statement of Theorem 5 of Boldi et al. (2017) also assumed that $r_x, r_y > 0$; this is not needed anymore, because it is implied by $\boldsymbol{v} > 0$.
4 Recall that the summation $\sum_{n \geq 0} (\alpha M)^n$ can be interpreted as providing in row i and column j the sum of the weights of the paths from i to j, where the weight of a path is the product of the weights of its arcs.
5 Of course, one can get to the same conclusion by inverting symbolically $1 - \alpha M$ and then solving the equation $c_{yy}(\alpha) = c_{yz}(\alpha)$, but we believe that reasoning geometrically on the paths makes much clearer how the counterexample was concocted.
6 We remark that this counterexample is conceptually similar to the one for Seeley's index (Boldi et al., 2017): the choice of α transfers exactly the score of y to z.
7 Note that also Theorem 4 from Boldi et al. (2017) needs the condition $\alpha > 0$ to work in the strict case.

Conflict of interest. The authors have nothing to disclose.

References

Boldi, P., Luongo, A., and Vigna, S. (2017). Rank monotonicity in centrality measures. *Network Science*, *5*(4), 529–550.
Boldi, P., and Vigna, S. (2014). Axioms for centrality. *Internet Mathematics*, *10*(3–4), 222–262.
McDonald, J. J., Neumann, M., Schneider, H., and Tsatsomeros, M. J. (1995). Inverse M-matrix inequalities and generalized ultrametric matrices. *Linear Algebra and its Applications*, *220*, 321–341.
Vigna, S. (2016). Spectral ranking. *Network Science*, *4*(4), 433–445.
Willoughby, R. A. (1977). The inverse M-matrix problem. *Linear Algebra and its Applications*, *18*(1), 75–94.

Reference: Boldi P., Luongo A., and Vigna S. (2017). Rank monotonicity in centrality measures. *Network Science* **5**(4), 529–550.

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