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GOING-DOWN UNDERRINGS

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Let R be an integral domain with quotient field K. Ten conditions equivalent to "Either R is algebraic over $\mathbb{Z}$ or $t.d.(R/\mathbb{F}_p) \leq 1$ for some p" are given. One of these conditions, referred to in the title, is "Each extension of subrings of R having quotient field K satisfies the going-down property." As consequences, other classes of rings are also characterised.

1. Introduction

Let R be a (commutative integral) domain with quotient field K. Numerous studies of R have proceeded in terms of its overrings (that is, the rings contained between R and K). More recently (see [5], [7], [8]), R has also been studied via its subrings and, more specifically, via its underrings (that is, the subrings of R having quotient field K). This article considers domains R each of whose subrings is "small." Beginning with [11], a number of papers have considered this question in the case where "small" means "Noetherian." Here, we shall interpret "small" as meaning variously "of (Krull) dimension at most 1," "a going-down domain" (in the sense of [3]), or "treed." Of course, these three interpretations of "small" need not be equivalent for a given ring. However, our main result, Theorem 2.1, shows that when any of these conditions

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(or a few other "smallish" ones) is imposed on *all* the subrings (or, for that matter, just on all the underrings) of R , it becomes equivalent to the correspondingly quantified version of any other "small" condition. Perhaps, the most interesting part of Theorem 2.1 is this: each extension of underrings of R satisfies *GD* (the going-down property) if and only if either R is algebraic over $\mathbb{Z}$ or $t.d.(R/\mathbb{F}_p) \leq 1$ for some p .

In Theorem 2.1, the quantified smallness conditions for underrings turn out to be equivalent to the corresponding assertions for subrings. While similar phenomena for "not necessarily small" conditions are wide-spread (see [5], [8]), [7] noted, interalia, that a domain integral over each underring need not be integral over each subring. In fact, [7] characterized the domains in question, and it is gratifying that, in Corollary 2.3, our present work leads to new characterizations of them. Section 2 also contains subring/underring characterizations of some other classes of domains.

Throughout, usage is standard, as in [1], [15]; in particular, the going-down, incomparability, and lying-over properties of extensions are denoted by *GD*, *INC* and *LO* , respectively. The integral closure of a domain A is denoted by A' , and $\dim(A)$ means the Krull dimension of A . For ease of reference, we assume that the reader has access to the going-down survey [9].

2. Results

By definition, a domain R is a going-down domain in the case that $R \subseteq T$ satisfies *GD* for each domain T containing R . It is known (see [9, (4.1)]) that the test rings T can be taken to be overrings of R . The main examples of going-down domains are arbitrary domains of dimension at most 1 , arbitrary Prüfer domains, and certain $D + M$ constructions. Examples in [3, Theorem 4.2 (ii)] show, in contrast to the case of a Prüfer domain, that not all the overrings of an arbitrary going-down domain need be going-down domains. Determining when all the underrings (or all the subrings) of a going-down domain are going-down domains is one of the motivations for Theorem 2.1. Since the survey [9] was written, most work on going-down domains has emphasized special cases, such as [13] and its various globalizations [6].

For our purposes, a key fact about domains A is

$$\dim(A) \leq 1 \Rightarrow A \text{ is a going-down domain } \Rightarrow A \text{ is treed.}$$

(The last implication was shown in [3, Theorem 2.2]. As usual, A is said to be treed in case $\mathrm{Spec}(A)$, as a poset under inclusion, is a tree.) In certain contexts, these implications are reversible. In fact, all three conditions are equivalent if A is Noetherian; or if A is a *GCD*-domain (see [18, Theorem 3.7] and [2, Corollary 4.3]). However, in general, the implications are not reversible. For instance, if A is a Prüfer domain of dimension at least 2, then A (and each of its overrings) is a (are) going-down domain(s). Moreover, a construction of Lewis (see [9, (4.4)]) shows that a treed domain need not be a going-down domain. This contrasts with underring/subring behaviour, for Theorem 2.1 will show that each subring of A has dimension at most 1 if (and only if) each underring of A is treed.

THEOREM 2.1. *For a domain R, the following conditions are equivalent:*

(1) $\dim(A) \leq 1$ for each underring A of R;

(2) $\dim(A) \leq 1$ for each subring A of R;

(3) each underring of R is a going-down domain;

(4) each subring of R is a going-down domain;

(5) each underring of R is treed;

(6) each subring of R is treed;

(7) $A \subseteq B$ satisfies GD for each inclusion $A \subseteq B$ of underrings of R;

(8) $A \subseteq B$ satisfies GD for each inclusion $A \subseteq B$ of subrings of R;

(9) A' is a Prüfer domain for each underring A of R;

(10) A' is a Prüfer domain for each subring A of R;

(11) either R is algebraic over $\mathbb{Z}$ or $\mathrm{t.d.}(R/\mathbb{F}_p) \leq 1$ for some p.

Proof. (11) $\Rightarrow$ (2): Deny. Then (11) holds and we can select a subring A of R such that $\dim(A) > 1$. Thus, there exist distinct nonzero prime ideals $P \subseteq Q$ of R. Choose $0 \neq a \in P$ and $b \in Q \backslash P$. With D denoting the prime ring of A, put $B = D[a,b]$. As $P \cap B \subseteq Q \cap B$ are distinct nonzero primes of B, we have $\dim(B) > 1$. There are now two cases.

If $\text{char}(R) = 0$, D is essentially $\mathbb{Z}$, and so B is a subring of the algebraic number field $\mathbb{Q}(a,b)$. Then by the Krull-Akizuki theorem (as in [1, Proposition 5, page 500]), $\dim(B) \leq 1$, the desired contradiction.

In the remaining case, D is essentially $\mathbb{F}_p$ for some p. This case also follows via Krull-Akizuki, but we shall give an alternate proof. As B is a finite-type $\mathbb{F}_p$-algebra, a well known corollary of Noether Normalization gives $\dim(B) = t.d.(B/\mathbb{F}_p)$. Hence, $\dim(B) \leq t.d.(R/\mathbb{F}_p) \leq 1$, the desired contradiction.

(2) $\Rightarrow$ (4): Any domain of dimension at most 1 is a going-down domain.

(4) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (5): Any going-down domain is treed [3, Theorem 2.2].

(5) $\Rightarrow$ (11): Let D denote the prime ring of R. Suppose the assertion fails. Now, R contains a transcendence basis $\{X_i\}$ for the field extension induced by $D \subset R$; moreover, $|\{X_i\}| \geq 1$ if $\text{char}(R) = 0$ and $|\{X_i\}| \geq 2$ if $\text{char}(R) > 0$. Observe that $B = D[\{X_i\}]$ is not treed. (The point is that if A is a domain but not a field and X is transcendental over A, then $A[X]$ is not treed.) Let E denote the integral closure of B in R. As R is algebraic over B, the "clearing denominators" trick shows that R is an overring of E. By (5), E is therefore treed.

Since B is integrally closed (essentially by Gauss' lemma), the going-down theorem of Krull [16] yields that $B \subseteq E$ satisfies GD. Moreover, integrality assures that $B \subseteq E$ satisfies LO (see [15, Theorem 44]). It now follows easily from GD, LO and the treedness of E that B is treed, the desired contradiction.

We have now shown the equivalence of (2), (3), (4), (5), and (11). Next, we bring (1), (7), and (8) into the fold.

(2) $\Rightarrow$ (1): Trivial.

(1) $\Rightarrow$ (3): See the above explanation for (2) $\Rightarrow$ (4).

(4) $\Rightarrow$ (8): If $A \subseteq B$ are domains and A is a going-down domain, then $A \subseteq B$ satisfies GD.

(8) ⇒ (7): Trivial.

(7) ⇒ (11): (*This is the key implication in the theorem.*) Suppose the assertion fails. Let D and $\{X_i\}$ be as in the proof that (5) ⇒ (11).

Suppose $\mathrm{char}(R) = 0$. Pick $X \in \{X_i\}$ and put $B = D[\{X_i : X_i \neq X\}]$. Again by Gauss, B is integrally closed. Moreover, viewing B as the direct limit of polynomial rings in finitely many variables, we infer from [1, Exercise 12(e), page 44] that B is a coherent domain. (To apply the cited exercise, invoke the Hilbert Basis Theorem and the fact that free modules are flat.) As $2X$ is transcendental over B, we see similarly that $B[2X]$ is integrally closed and coherent. Consequently, by a result of Papick (see [9, (3.14)]), in order to show that the overring extension $B[2X] \subset B[X]$ does not satisfy GD, it is enough to prove that $B[X]$ is not $B[2X]$-flat.

For simplicity, put $S = B[2X]$ and $T = B[X]$. As the prime $2S$ of S survives in T, the "nonflatness" assertion will follow by showing $X \notin S_{2S}$ (see [17, Theorem 2]). For this, notice that if $X \in S_{2S}$, then writing X as a fraction and cross-multiplying would lead to

$$X \sum a_i (2X)^i = \sum b_i (2X)^i ,$$

where all $a_i, b_i \in B$ and some $a_j \notin 2B$. However, by comparing coefficients of X^{j+1}, we would then find $a_j 2^j = b_{j+1} 2^{j+1}$, whence $a_j = 2b_{j+1} \in 2B$, which is absurd. Thus, $S \subset T$ does not satisfy GD.

Let V (respectively, W) denote the integral closure of S (respectively, T) in R. By the going-down theorem of Krull, $S \subset V$ satisfies GD. Moreover, since $\{X_i\}$ is a transcendence basis, R is algebraic over T (and S); hence, by "clearing denominators," R is an overring of V (and W). By (7), $V \subseteq W$ satisfies GD. Thus, by composing $S \subseteq V$ and $V \subseteq W$, we see that $S \subseteq W$ satisfies GD. In conjunction with the fact that integrality makes $T \subseteq W$ satisfy LO, this forces $S \subseteq T$ to satisfy GD. This contradiction (to the preceding paragraph) completes the characteristic zero case.

The case of positive characteristic is handled similarly. For this, pick distinct $X, Y \in \{X_i\}$ and put $B = D[\{X_i : X_i \neq X, Y\}]$. Also, take

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$S = B[XY, Y]$ and $T = B[X, Y]$. As XY is transcendental over $B[Y]$, one shows as above that S is integrally closed and coherent. To show that $S \subseteq T$ does not satisfy GD, the above reasoning reduces us to showing $X \notin S_{YS}$. By change of variable, this is equivalent to showing $XY^{-1} \notin T_{YT}$. If this failed, there would exist polynomials $f \in T$ and $g \in T \backslash YT$ such that $gX = fY$. As $Y \nmid g$ and $Y \nmid X$, this would lead to Y not being a prime element of T, which is absurd. Now, having shown that $S \subseteq T$ does not satisfy GD, one repeats *verbatim* all but the final sentence of the preceding paragraph. This gives the desired contradiction in the positive characteristic case, thus completing the proof that (7) ⇒ (11).

To this point, we have shown the equivalence of (1), (2), (3), (4), (5), (7), (8) and (11). To complete the proof, we shall show (4) ⇒ (6) ⇒ (5) and (11) ⇒ (10) ⇒ (9) ⇒ (11).

(4) ⇒ (6): See the above explanation for (3) ⇒ (5).

(6) ⇒ (5): Trivial.

(11) ⇒ (10): Assume (11). As (11) ⇒ (1), $\dim(R) \leq 1$. In the same way, we see that if a domain B is contained in the quotient field of R, then $\dim(B) \leq 1$. (The point is that B "inherits" (11) from R.) In particular, the valuative dimension of any subring A of R is at most 1. Then (10) follows via [14, Corollaire 3, page 61] (see also [10, Theorem 6]).

(10) ⇒ (9): Trivial.

(9) ⇒ (11): Suppose the assertion fails. Let D and $\{X_i\}$ be as in the proof that (5) ⇒ (11). As above, $B = D[\{X_i\}]$ is not treed; a fortiori, B is neither a going-down domain nor a Prüfer domain. Let E denote the integral closure of B in R. As in the proof that (5) ⇒ (11), R is an overring of E. By (9), E' is a Prüfer domain and, a fortiori, a going-down domain. As E' is integral over B (by transitivity of integrality) and B is integrally closed, a descent result of Heinzer (see [9, (4.5a)]) (respectively, of the author (see [9, (4.6a)])) yields that B is a Prüfer domain (respectively, a going-down domain). This contradiction establishes (9) ⇒ (11) and completes the proof of Theorem 2.1.

Apropos of conditions (9) and (10) in Theorem 2.1, it is well known that the integral closure of a domain A is a Prüfer domain if and only if $A \subseteq B$ satisfies *INC* for each overring B of A. We next use *INC* to characterise the domains algebraic over their prime rings.

COROLLARY 2.2. *For a domain R, the following conditions are equivalent:*

(1) $A \subseteq B$ satisfies INC for each inclusion $A \subseteq B$ of subrings of R;

(2) Either R is algebraic over $\mathbb{Z}$ or R is an algebraic field extension of some $\mathbb{F}_p$.

Proof. Let D denote the prime ring of R. Observe that (2) is equivalent to requiring that R be algebraic over D.

(1) => (2): It is enough to observe that if $X \in R$ is transcendental over D, then $D \subseteq D[X]$ does not satisfy *INC*. (For this, notice that if M is a maximal ideal of D, then M is lain over by the non-maximal prime $MD[X]$ of $D[X]$.)

(2) => (1): Assume (2), and let $A \subseteq B$ be subrings of R. By Theorem 2.1 [(11) => (2)], A and B each have dimension at most 1. To show that $A \subseteq B$ satisfies *INC*, it is therefore enough to show $P \cap A \neq 0$ for each nonzero (prime) ideal P of B. This, in turn, follows from (2), as B is algebraic over $(D \subseteq) A$. (In detail, choose nonzero $u \in P$, and notice that $P \cap A$ contains the necessarily nonzero constant term of any minimal-degree algebraicity equation of u over A.) The proof is complete.

We next use *GD*, *LO* (and, implicitly, *INC*) to get a new characterisation of the rings featured in [7].

COROLLARY 2.3. *For a domain R with quotient field K, the following conditions are equivalent:*

(1) $A \subseteq B$ satisfies both GD and LO for each inclusion $A \subseteq B$ of underrings of R;

(2) $A \subseteq B$ satisfies both GD and LO for each inclusion $A \subseteq B$ of subrings of R;

(3) R is integral over each of its underrings;

(4) Either (i) R is (isomorphic to) a subring of the ring of all algebraic integers; or

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(ii) $R = K$ *is an algebraic field extension of some* $\mathbb{F}_p$ *; or*

(iii) R *has positive characteristic and precisely one valuation ring of* K *does not contain* R *.*

Proof. (1) => (3): Assume (1), and let A be an underring of R. By Theorem 2.1 [(7) ⇒ (1)], A and R each have dimension at most 1. If $R = K$, then $A = K$ also, since $A \subseteq R$ satisfies *LO*; in particular, R is integral over A in this case. Thus, without loss of generality, $\dim(R) = 1 = \dim(A)$. We may now apply [4, Remark 3.12(b)] (which explicitly used *INC*-theoretic considerations), to conclude that R is integral over A.

(3) <=> (4): This is the principal result in [7].

(4) ⇒ (2): Assume (4), and hence (3). As for the assertion about *GD* in (2), Theorem 2.1 [(11) ⇒ (8)] shows that we need only prove that either R is algebraic over $\mathbb{Z}$ or $t.d.(R/\mathbb{F}_p) \leq 1$ for some p. This is clear if either (i) or (ii) obtains. Moreover, given (iii), we have $t.d.(R/\mathbb{F}_p) = 1$ (see [7, Theorem 2.3 (iii)]).

Next, we shall establish the assertion about *LO* in (2). As integrality implies *LO* (see [15, Theorem 44]), we may assume that (iii) holds. It will suffice to show that $A \subseteq R$ satisfies *LO* for each subring A of R. There are two cases.

Suppose first that R is not algebraic over A. Then $t.d.(R/A) = 1$, and so we may pick $X \in R$ transcendental over A. Let B denote the integral closure of $A[X]$ in R. As R is algebraic over $A[X]$, the "clearing denominators" trick shows that R is an overring of B. As each of the three extensions $A \subseteq A[X]$, $A[X] \subseteq B$, and $B \subseteq R$ satisfies *LO*, so does their composite, $A \subseteq R$, as asserted.

In the remaining case, R is algebraic over A. Let D denote the integral closure of A in R. By "clearing denominators," R is an overring of D. Then, as the composite of *LO*-extensions $A \subseteq D$ and $D \subseteq R$, the extension $A \subset R$ also satisfies *LO* (and, in fact, is integral in this case). This completes the proof that (4) ⇒ (2).

As (2) ⇒ (1) trivially, the proof of Corollary 2.3 is complete.

Remark 2.4. In the context of Theorem 2.1, consider the conditions

(a) $A \subseteq R$ satisfies GD for each underring A of R, and

(b) $A \subseteq R$ satisfies GD for each subring A of R.

Trivially, (b) => (a). In fact, (a) => (b) as well. To see this, assume (a) and let A be a subring of R. By [12, Proposition 1.1], $A \subseteq D \subseteq R$ for some underring D of R such that D is a free A-module. Then $A \subseteq D$ satisfies GD, essentially because of flatness (see [15, Exercise 37, page 44]). As (a) assures that $D \subseteq R$ also satisfies GD, it follows that $A \subseteq R$ satisfies GD. Thus, (a) ⇔ (b).

Clearly, (a) and (b) are implied by the equivalent conditions in Theorem 2.1. However, the converse is false. To see this, let R be a field such that either $t.d.(R/\mathbb{Q}) > 0$ or $t.d.(R/\mathbb{F}_p) > 1$ for some p. Evidently, R does not satisfy condition (11) in Theorem 2.1. Nevertheless, R satisfies (a) (and hence (b)), for any field is flat over each of its underrings (see [8, Proposition 2.1(a)]).

It seems natural to ask whether the domains R satisfying (a) and (b) can be characterised by augmenting condition (11) in Theorem 2.1 with "or R is a field." We do not know the answer to this question.

In Remark 2.4, we appealed twice to the fact that flatness implies GD. In our final result, we shall consider the flat-theoretic analogues of conditions (7) and (8) in Theorem 2.1. The upshot will be new characterisations of a much-studied class of domains. (For background on other characterisations of these domains, see the Corollary and the references cited in the first paragraph of [5].)

PROPOSITION 2.5. *For a domain* R, *the following conditions are equivalent:*

(1) B *is* A*-flat for each inclusion* $A \subseteq B$ *of underrings of* R;

(2) B *is* A*-flat for each inclusion* $A \subseteq B$ *of subrings of* R;

(3) Either R *is (isomorphic to) an overring of* $\mathbb{Z}$ *or* R *is an algebraic field extension of some* $\mathbb{F}_p$.

Proof. (3) => (2): Assume (3), and let $A \subseteq B$ be subrings of R. If $\mathrm{char}(R) = 0$, then A is a Prüfer (in fact, Euclidean) domain, and so the torsion-free A-module B is A-flat. If $\mathrm{char}(R) > 0$, then integrality forces both A and B to be fields (see [1, Lemma 2, page 326]) and so, being A-free, B is thus A-flat.

(2) ⇒ (1): Trivial.

(1) ⇒ (3): Assume (1), and let B be an underring of R. As B is flat over each of its underrings, [8, Theorem 2.4] yields that either B is (isomorphic to) an overring of $\mathbb{Z}$ or B is a field. Thus, without loss of generality, we may assume that each underring of R is a field. It now follows easily from the extension theorem for valuations (see [15, Theorem 56]) that R is algebraic over some $\mathbb{F}_p$.

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