**INDUCED REPRESENTATIONS AND ALTERNATING GROUPS**

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**Introduction.** This paper is based on part of the thesis of one of the authors (5), submitted at the University of Toronto in 1963. In the first part of the paper a result on induced representations (2, 4, 9) is generalized slightly and a number of corollaries are derived. In the rest of the paper a special case of this result is applied to put the representation theory of the alternating group on a par with that of the symmetric group. A knowledge of the representation theory of \( S_n \) (7) on the part of the reader is assumed.

1. Let \( G \) be a group of finite order \([G:1]\) and \( \Lambda \) be an algebraically closed field of characteristic not dividing the order of \( G \). We denote by \( \otimes \) the Kronecker product of two matrices. If \( T \) and \( S \) are any two irreducible representations of \( G \), consider the group \([T \otimes S]_G\) generated by all elements of the form \([Tu, Sv] = Tu \otimes Sv\), where \( u \) and \( v \) are elements of \( G \) and \( Tu \) is the image of \( u \) in \( T \) of \( G \). The group \([T \otimes S]_G\) is not a representation of \( G \) but of \( G \times G \). We call \([T \otimes S]_G\) the outer tensor product of \( T \) and \( S \) of \( G \). The diagonal subgroup \((T \otimes S)_G\) of \([T \otimes S]_G\), which consists of elements of the form \((Tu, Su) = (Tu \otimes Su)\), is a representation of \( G \) and is called the inner tensor product of \( T \) and \( S \) of \( G \). Such a representation is, in general, reducible over \( \Lambda \), since we assume \( \Lambda \) to be a splitting field for \( G \) and for all its subgroups.

Next if \( p \) and \( q \) are any two fixed elements of \( G \), consider the group \([T_p \otimes S_q]_G\) generated by all elements of the form \([T_p u, S_q v] = T(pup^{-1}) \otimes S(qvq^{-1})\). The subgroup of this group consisting of all elements of the form \((T_p u, S_q u) = T(pup^{-1}) \otimes S(qvq^{-1})\) may be denoted by \((T_p \otimes S_q)_G\). Then, since the characters are equal, we have the following lemma.

1.1. **Lemma.** \([T \otimes S]_G \cong [T_p \otimes S_q]_G\) and \((T \otimes S)_G \cong (T_p \otimes S_q)_G\).

The representation \( T \) of \( G \) restricted to a subgroup \( H \) will be denoted by \([T]_H\) and, conversely, if \( S \) is a representation of \( H \), then the representation of \( G \) induced by \( S \) will be denoted by \([S]_G\).

Let \( T \) be an irreducible representation of \( G \) over a field of characteristic 0. If \( I \) is the identity representation of the subgroup \( H \), then \([9]\)

1.2 \([\{I\}^G \otimes T]_G \cong \{[T]_H\}^G\).
A more general result is obtained by replacing the identity representation \( I \) by any representation \( S \) of \( H \) in the form \((2, 4)\)

\[ \langle \{S\}^a \otimes T \rangle G \cong \langle \{S \otimes \{T\}_H \}^a \rangle. \]

Robinson, Taulbee, and Littlewood made use of 1.2 and 1.3 to decompose the inner tensor product representation of two irreducible representations of the symmetric group. In the following we generalize 1.3 and obtain a number of corollaries.

1.4. Theorem. Let \( F, H, \) and \( K = F \cap H \) be subgroups of a finite group \( G \) such that \([G:F] = [H:K]\) and \( H \) contains a set of coset representatives of \( F \) in \( G \). If \( T \) and \( S \) are any representations of \( G \) and \( F \) respectively over the field \( \Lambda \), then

\[ \langle \langle \{S\}^a \otimes T \rangle \rangle_H \cong \langle \langle S \otimes \{T\}_F \rangle \rangle_K. \]

Proof. Let \([G:F] = n\) and \( g_1(=1), g_2, \ldots, g_n \) be a set of coset representatives of \( F \) in \( G \), which are also elements of \( H \). Without loss of generality we may assume that \( g_3, g_4, \ldots, g_n \) do not belong to \( K \).

Then we have the unique decomposition

\[ G = \bigcup_{r=1}^{n} F g_r, \]

into disjoint cosets of \( F \) in \( G \), so that

\[ H \cap G = H \cap \left( \bigcup_{r=1}^{n} F g_r \right) = \bigcup_{r=1}^{n} (H \cap F g_r) = \bigcup_{r=1}^{n} (H \cap F) g_r. \]

Since \( H \cap G = H \) and \( H \cap F = K \), we obtain the unique decomposition

\[ H = \bigcup_{r=1}^{n} K g_r, \]

of \( H \) into disjoint right cosets of \( K \) in \( H \). Moreover, if we now define a mapping \( \phi: F g_r \to K g_r \), then \( \phi \) is one-to-one and \( \phi^{-1}: K g_r \to F g_r \) exists, so that there is a one-to-one relation between the cosets \( F g_r \) of \( F \) in \( G \) and the cosets \( K g_r \) of \( K \) in \( H \). This correspondence ensures that the permutation (matrix) representation of \( G \) induced by the identity representation of \( F \), restricted to \( H \), is identical (not just isomorphic) with the permutation (matrix) representation of \( H \) induced by the identity representation of \( K \).

Denote by \( C^a(g) \) the conjugate class of \( G \) to which the element \( g \) belongs and by \([C^a(g) \cap F: C^a(g)]\) the ratio of the number of elements of \( C^a(g) \cap F \) to the number of elements of \( C^a(g) \). Then the permutation character of \( g \) in \([I]^a \) is

1.5

\[ [G:F][C^a(g) \cap F: C^a(g)]. \]

If we now restrict ourselves to elements of the subgroup \( H \), then 1.5 takes the form

1.6

\[ [G:F][C^a(g) \cap F: C^a(g)]_H = [G \cap H:F \cap H][C^a(g) \cap F \cap H:C^a(g)] \]

\[ = [H:K][C^a(g) \cap K: C^a(g)], \]
where \( g \) is an element of \( H \) and \( [G:F] = [H:K] \). If the character of an element \( h \) of \( H \) in \( S \) of \( F \) is \( \chi^S(h) \), then from 1.6 the character on each side of the desired relation is

\[
[H:K][C^H(h) \cap K:C^H(h)]\chi^S(h)\chi^T(h),
\]

where \( \chi^S(h) = 0 \) if \( h \) lies in \( (G - F) \cap H \), i.e. if \( h \) lies in \( H - K \), the complement of \( K \) in \( H \), which proves 1.4.

Next, if we set \( T = I \) in 1.4, the result reduces to

1.7

\[
\{\{S\}^q\}_H = \{\{S\}_K^q\}_H.
\]

If \( F \cap H = 1 \), the right side gives a regular representation of \( H \) with multiplicity equal to \( \deg(S) \).

Our next corollary is a counterpart of Mackey’s subgroup theorem (3) and gives information about the representation on the right side of 1.7, which may be stated as follows:

Let \( N \) be a \( \Lambda F \)-module, where \( \Lambda \) is any field and \( \Lambda F \) is the group algebra of \( F \) over \( \Lambda \). For each \((F,H)\)-double coset \( D = FaH \), \((a \otimes N)\) is a left \( \Lambda F \)-module for the subgroup \( \bar{F} = aFa^{-1} \cap H \) of \( F \) and \( N(D) = \{(a \otimes N)\}_F^H \) is a left \( \Lambda H \)-module which depends only on the double coset \( D \). Moreover,

\[
(1.8)
\]

\[
\{\{N\}^q\}_H = \sum_D N(D)
\]

is a left \( \Lambda H \)-module, where the sum is taken over all \((F,H)\)-double cosets \( D \) in \( G \).

From 1.7 it follows that \( \{\{S\}_K\}_H \) also depends on the double coset decomposition of \( G \) with respect to \( F \) and \( H \), which we may state in the form of the following corollary.

1.9. COROLLARY. Let \( N \) be a \( \Lambda F \)-module of \( F \) and

\[
G = \bigcup_{i=1}^t Fa_i H,
\]

where \( t \) is the number of double cosets. Then

\[
\{\{N\}_K\}_H = \sum_{i=1}^t N(D_i),
\]

where \( N(D_i) = \{(a_i \otimes N)\}_F^H \) with \( \bar{F} = a_i Fa_i^{-1} \cap H \).

From now on we explicitly assume that \( \Lambda \) is a field whose characteristic does not divide the order \([G:1]\) of \( G \). If \( \theta \) and \( \phi \) are any two class functions belonging to the algebra \( \text{cf}(G) \) of all class functions of \( G \), an inner product may be defined thus:

\[
(1.10)
\]

\[
\langle \theta, \phi \rangle = \frac{1}{[G:1]} \sum_g \theta(g) \phi^*(g),
\]

where \( \phi^* \) is the conjugate of \( \phi \). If \( \phi \) is an irreducible character of \( G \), then \( \langle \theta, \phi \rangle \) is the multiplicity of the character \( \phi \) in \( \theta \).
1.11. **Corollary.** If $\theta$ is a common irreducible character of $F$ and $K$, and $\phi$ is an irreducible character of $H$, then

$$\langle \langle \{\theta\}^\circ_H, \phi \rangle \rangle = \langle \langle \{\theta\}^\circ_K, \{\phi\}_K \rangle \rangle.$$

**Proof.** Since $\theta$ is an irreducible character of $K$, Frobenius’ Reciprocity Formula gives

$$\langle \langle \{\theta\}^\circ_K, \{\phi\}_K \rangle \rangle = \langle \langle \{\theta\}^\circ_H, \phi \rangle \rangle.$$

But $\{\theta\}^\circ_H = \{\theta\}^\circ_H$ from 1.7, where on the right side $\theta$ is viewed as a representation of $F$. Hence, we have

$$\langle \langle \{\theta\}^\circ_H, \phi \rangle \rangle = \langle \langle \{\theta\}^\circ_K, \{\phi\}_K \rangle \rangle.$$

2. In this section, let $\Lambda$ be the field of complex numbers and $\Lambda S_n$ and $\Lambda A_n$ be the group algebras of the symmetric group $S_n$ and the alternating group $A_n$. Each class of conjugate elements of $S_n$ is defined by means of a partition $(\lambda) = (\lambda_1\lambda_2\ldots\lambda_h)$; $n = \lambda_1 + \lambda_2 + \ldots + \lambda_h$ and $\lambda_i \geq \lambda_{i+1} > 0$. We use the symbol $(\lambda)$ to denote the conjugate class of $S_n$ determined by the corresponding partition. Corresponding to each class $(\lambda)$, there exists a uniquely defined irreducible representation $[\lambda]$ and a (left) irreducible $\Lambda S_n$-module defined in term of the Young diagram

$$\ldots \ldots (\lambda_1 \text{ nodes})$$

$$[\lambda] \ldots \ldots (\lambda_2 \text{ nodes})$$

$$\ldots (\lambda_h \text{ nodes})$$

where we use the same symbol $[\lambda]$ to denote the corresponding Young diagram. The Young diagram $[\lambda]$ can also be described by a Frobenius symbol

$$\begin{bmatrix}
a_1 & a_2 & \ldots & a_h \\
b_1 & b_2 & \ldots & b_h
\end{bmatrix},$$

where $a_i$ is the number of nodes to the right of the diagonal in the $i$th row and $b_i$ is the number of nodes below the diagonal in the $i$th column. The diagram $[\lambda']$, obtained from $[\lambda]$ by interchanging rows and columns, is the conjugate of $[\lambda]$ and corresponds to the conjugate representation of $[\lambda]$ and the dual of the irreducible $\Lambda S_n$-module defined by $[\lambda]$. If $[\lambda] = [\lambda']$, then $[\lambda]$ is a self-conjugate diagram and in this case we call the conjugate class $(2\lambda_1 - 1, 2\lambda_2 - 3, 2\lambda_3 - 5, \ldots)$ a splitting class of $S_n$. Frobenius (1) proved the following theorem.

2.1. **Theorem.** If the diagram $[\lambda] \neq [\lambda']$, then the representations $[\lambda]$ and $[\lambda']$ of $S_n$ restricted to $A_n$ are equivalent irreducible representations of $A_n$, while if the diagram $[\lambda] = [\lambda']$ the representation $[\lambda]$ of $S_n$ restricted to $A_n$ splits into two (conjugate) irreducible representations $[\lambda]^+$ and $[\lambda]^-$ of the same degree.

Thus, every irreducible character of a self-conjugate representation of $S_n$
is the sum of two characters of \( A_n \). The class \((2\lambda_1 - 1, 2\lambda_2 - 3, \ldots) = (p_1, p_2, \ldots, p_n)\) for which the character \( \chi^\lambda_{(p)} \) of \( S_n \) is

\[
\chi^\lambda_{(p)} = (-1)^{\frac{1}{2}\sum_j p_j (p_j - 1)}
\]

splits into two classes \((p)^+\) and \((p)^-\) in \( A_n \). The corresponding characters of the two classes \((p)^+\) and \((p)^-\) in \([\lambda]^+\) are

\[
\xi, \xi^* = \frac{1}{2}\{(-1)^{\frac{1}{2}\sum_j p_j (p_j - 1)} \pm \sqrt{p_1 p_2 p_3 \ldots (i)^{\frac{1}{2}\sum_j p_j (p_j - 1)}}\},
\]

which are interchanged between the two classes in \([\lambda]^+\).

2.3. Lemma. The characters of the representations \([\lambda]^+\) and \([\lambda]^−\) of \( A_n \) are real or complex according as the number of nodes above or below the diagonal of the Young diagram \([\lambda]\) is even or odd respectively.

Proof. From 2.2, the character of the representation \([\lambda]^+\) is real or complex according as \(\frac{1}{2}(p_1 p_2 p_3 \ldots - 1)\) is even or odd respectively. The proof turns out to be trivial if we express the Young diagram in terms of the Frobenius symbol

\[
\begin{bmatrix}
a_1 & a_2 & \ldots & a_h \\
a_1 & a_2 & \ldots & a_h
\end{bmatrix}.
\]

Then \(\xi^\lambda_{(p)^+}\) is real or complex according as

\[
\frac{1}{2}[\prod_{i=1}^h (2a_i + 1) - 1]
\]

is even or odd, i.e. according as

\[
a_1 + a_2 + a_3 + \ldots + a_h
\]

is even or odd. But this is precisely the number of nodes above or below the diagonal of the diagram so that the lemma is proved.

From now on we shall use the same symbols to denote the irreducible representations of \( A_n \) and we shall not distinguish between \([\lambda]\) and \([\lambda']\) as representations of \( A_n \). Consider the \( n! \) Young tableaux obtained by placing the symbols \( 1, 2, 3, \ldots, n \) in \([\lambda]\) in all possible ways. In \( f^\lambda \) of these tableaux the symbols appear in their natural order in both row and column, and these tableaux are called standard. Making the following construction, we obtain the actual matrices of the irreducible representation \([\lambda]\) of \( S_n \) (7).

2.4. Theorem (Young). To construct the matrix representing \((r, r + 1)\) in \([\lambda]\) of \( S_n \), arrange the \( f^\lambda \) standard tableaux \( P_{t^1}, \ldots, P_{t^s} \) in dictionary order and set

(i) \( 1 \) (and \(-1\)) in the leading diagonal where \( t^p \) has \( r \) and \( r + 1 \) in the same row (column),

(ii) a quadratic matrix
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\[
\begin{pmatrix}
-\rho & \sqrt{1-\rho^2} \\
\sqrt{1-\rho^2} & \rho
\end{pmatrix}
\]

at the intersection of rows and columns corresponding to \(v_u\) and \(v_e\), where \(u < v\) and \(v_e\) is obtainable from \(v_u\) by interchanging \(r\) and \(r + 1\). If \(r\) appears in the \((i, j)\) position and \(r + 1\) in the \((k, l)\) position of \(v_u\) with \(i < k, l < j\), then \(
\rho^{-1} = (j - i) - (l - k),
\)

(iii) zeros elsewhere.

If \(|\lambda| \neq |\lambda'|\) the matrices representing \((12)(r, r + 1)\) can be constructed according to 2.4 and these will generate the corresponding representation of \(A_n\). If \(|\lambda| = |\lambda'|\), the representation will split into \(|\lambda|^+\) and \(|\lambda|^-\), in which case we have the following construction \((5, 10)\).

2.5. Theorem. To construct the matrix representing \((12)(r, r + 1)\) in \(|\lambda|^+\) (\(\{\lambda|^{-}\) of \(A_n\) arrange the \(f^\lambda\) standard tableaux in dictionary order and assign the first \(\frac{1}{2}f^\lambda\) tableaux to \(|\lambda|^+\) and the remainder to \(|\lambda|^{-}\). Construct the matrix representing \((12)\) according to 2.4 and the matrix representing \((r, r + 1)\) in the same manner, as long as the two associated tableaux in 2.4 (ii) both belong to the same set \(|\lambda|^+\) (\(|\lambda|^{-}\).)

If \(v_u\) lies in \(|\lambda|^+\) and \(v_e\) in \(|\lambda|^{-}\), we must have \(r = n - 2\) or \(n - 1\). Denote by \(v_{u_1}\) that tableau in which the part of the tableau in \(n - 2\) letters is conjugate to the same part in \(v_u\), so that \(v_{u_1}\) also lies in \(|\lambda|^+\). Set the quadratic matrix

\[
\begin{pmatrix}
-\rho & \epsilon \sqrt{1-\rho^2} \\
\epsilon^{-1} \sqrt{1-\rho^2} & \rho
\end{pmatrix}
\]

at the intersection of the rows and columns corresponding to \(v_u\) and \(v_{u_1}\), where \(\rho\) has the same meaning as in 2.4 (ii) and \(\epsilon = (\epsilon)^{f^0(p_1)\ldots p_{n-1}}\) with \(p_i = 2\lambda_i - (2i - 1)\). Putting zeros elsewhere leads to the matrix representing \((12)(r, r + 1)\) in \(|\lambda|^+\).

A similar construction applied to \(v_e\) yields \(v_{e_1}\) and with \(\epsilon\) and \(\epsilon^{-1}\) interchanged leads to the matrix representing \((12)(r, r + 1)\) in \(|\lambda|^{-}\).

Proof. The standard tableaux of \(|\lambda|\) arranged in dictionary order are such that the second half of the \(f^\lambda\) tableaux are the conjugates of the first half, but in reverse order.

Consider the matrices representing \(A_n\) constructed according to 2.4. If the order of the tableaux in the second half of the set is reversed, the two representations obtained by restricting \(|\lambda|\) to \(A_{n-2}\) (or \(A_{n-1}\) as the case may be) on the symbols \(1, 2, \ldots, n - 2\) (or \(n - 1\)) will be equivalent by 2.1, so that a further transformation will make them identical if necessary. It follows from Schur’s lemma that there exists a commuting matrix which should enable us to reduce the representation \(|\lambda|\) of \(A_n\), so transformed, into \(|\lambda|^+\) and \(|\lambda|^{-}\) leaving the two identical representations of \(A_{n-2}\) (or \(A_{n-1}\)) unchanged.

Combining these three transformations leads to a matrix

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where
\[ \sigma_j a_j = \sigma_j d_j = 1/\sqrt{2}, \quad \bar{b}_j = c_j = (i)^{\frac{1}{2}(p_1 p_2 \ldots p_k)} / \sqrt{2}, \]
and \( a_j d_j + b_j c_j = -1 \) with \( \sigma_j = \pm 1 \) according as the \( j \)th tableau is obtained from the first by an even or an odd number of interchanges of symbols. The effect of transformation by \( P \) is as desired in the theorem.

**Example.** As an illustration, consider the construction of the matrices of \([31^2]^+\) of the alternating group \( A_5 \):

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
4 & 3 & 2 & 5 & 5
\end{bmatrix}
\]

The group \( A_5 \) can be generated by \((12)(23), (12)(34), \) and \((12)(45)\). We construct the matrices representing \((12)(23)\) or \((12)(34)\) according to \(2.4\), but for the matrix representing \((12)(45)\) we must use the construction of \(2.5\), obtaining

\[
(12)(23): \begin{pmatrix} 1 & \cdot & \cdot & -1/2 & \sqrt{3}/2 \\
\cdot & -\frac{1}{2} & \cdot & \sqrt{3}/2 & \cdot \\
\cdot & \cdot & \cdot & -\sqrt{3}/2 & -\frac{1}{2} \end{pmatrix},
(12)(34): \begin{pmatrix} -1/3 & \sqrt{8}/3 & \cdot \\
\sqrt{8}/3 & 1/3 & \cdot \\
\cdot & \cdot & -1 \end{pmatrix}
\]

and

\[
(12)(45): \begin{pmatrix} -1 & \cdot & \cdot \\
\cdot & -1/4 & \epsilon \sqrt{15}/4 \\
\epsilon^{-1} \sqrt{15}/4 & 1/4 \end{pmatrix}
\]

where \( \epsilon = (i)^{\frac{1}{2}(6-1)} = -1 \).

3. Next let us determine the minimal left ideals of \( \Lambda A_n \) over \( \Lambda \). For this purpose, it is necessary to consider the cases \([\lambda] \neq [\lambda']\) and \([\lambda] = [\lambda']\) separately. First let \([\lambda] \neq [\lambda']\).
Let \( \theta_i \) be any standard tableau belonging to \([\lambda]\) of \( A_n\). Denote by \( P_i \) the product of the symmetric groups of the symbols of the rows and by \( Q_i \) the product of the symmetric groups of the symbols of the columns of the tableau \( \theta_i \). Then clearly

\[ P_i \cap Q_i = 1 \quad \text{and} \quad P_i Q_i \neq Q_i P_i \]

in general. Define

\[ 2e^\lambda = \sum_i \sum_{p \in P_i, q \in Q_i} (\epsilon_q + \epsilon_p) pq, \]

where \( \epsilon_p = \pm 1 \) according as \( p \) is an even or an odd permutation. It is well known that the expressions

\[ \sum_{p \neq q} \epsilon_p pq \quad \text{and} \quad \sum_{q \neq q} \epsilon_q pq, \quad \left( \sum_{p \neq q} \epsilon_p pq \right) \left( \sum_{q \neq q} \epsilon_q pq \right) = 0, \]

are primitive characteristic units, i.e. non-zero idempotents of the group algebra \( \Delta S_n \) corresponding to \([\lambda]\) and \([\lambda']\), which do not belong to the centre of the algebra. Hence, \( e^\lambda \) is a central idempotent, but for a constant. Thus we may associate the minimal ideal \( \Delta A_n e^\lambda \) with \([\lambda]\) of \( A_n\). It can be shown that these ideals are the desired irreducible \( \Delta A_n\)-modules.

Secondly let \([\lambda] = [\lambda']\). Then define

\[ 2e^{\lambda^+} = \sum_i \sum_{p \in P_i, q \in Q_i} f(pq)(\epsilon_q + \epsilon_p) pq, \]

where \( f(pq) = 2\xi^* \) and \( 2\xi^* \) for elements of the split classes and \( f(pq) = 1 \) otherwise and \( \xi, \xi^* \) are defined by 2.2. From 2.1, 2.2, and the above argument we conclude that \( e^{\lambda^+} \) is also a central idempotent and the roles of \( \xi \) and \( \xi^* \) are interchanged for the conjugate representation.

4. We next show how the irreducible components in an induced permutation representation of \( A_n \) can be determined. For this purpose denote by \([\lambda_1], [\lambda_2], \ldots, [\lambda_h]\) the permutation representation of \( S_n \) induced by the \( I \)-representation of the subgroup \( F = S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_h} \) whose reduction into its irreducible components is given by (7)

\[ [\lambda_1]. [\lambda_2]. \ldots. [\lambda_h] = \prod_{i,j} (1 - R_{ij})^{-1} [\lambda_1 \lambda_2 \ldots \lambda_h]. \]

\( R_{ij} \) for \( i < j \) indicates the process of raising a node from the \( j \)-th row to the \( i \)-th row of \([\lambda]\) to yield a new diagram and \( \Pi R_{ij} \) indicates the successive raising of the nodes subject to the restrictions that the result is to be disregarded (i) if any row contains more symbols than a previous row or (ii) if two symbols from the same row appear in the same column.

If \( K \) is the subgroup of all even permutations of \( F \), then by 1.7, we have

\[ \{ (I) F \}^{S_n} = \{ (I) F \}^{A_n}, \]

which may also be written as

\[ \{ [\lambda_1], [\lambda_2], \ldots, [\lambda_h] \}^{A_n} = \{ (I) K \}^{A_n}, \]
where \((I)F\) denotes the identity representation of \(F\). If we use the same symbol \([\lambda_1],[\lambda_2], \ldots, [\lambda_n] \quad (\lambda_1 > 2)\) to denote the permutation representation of \(A_n\) induced by \((I)K\), then its reduction is given by 4.1, where along with the above restrictions on the raising operators, we have the further restrictions that (iii) \([\lambda]\) and \([\lambda']\) are equivalent if \([\lambda] \not\equiv [\lambda']\) and (iv) \([\lambda] = [\lambda]^+ + [\lambda]^-\) if \([\lambda] = [\lambda']\).

**Example.** If \([3],[2]\) is the permutation representation of \(A_5\) induced by the identity representation of the subgroup of all even permutations of the product \(S_3 \times S_2\), then

\[
[3],[2] = \prod (1 - R_{i\bar{i}})^{-1} [32] = (1 + R_{12} + R_{12}^2) [32] = [32] + [41] + [51]
\]

If \([\lambda] \not\equiv [\lambda']\), then we can invert 4.1 in the form

\[
[\lambda_1\lambda_2 \ldots \lambda_n] = \prod (1 - R_{i\bar{i}})[\lambda_1],[\lambda_2]. \ldots,[\lambda_n],
\]

thus expressing an irreducible representation in terms of permutation representations. Here no restrictions on the \(R_{i\bar{i}}\) are needed. The relations 4.1 and 4.2 can be extended to all subgroups of \(A_n\) (6).

**Example.**

\[
[3,2] = \prod (1 - R_{i\bar{i}})[3],[2] = (1 - R_{12})[3],[2] = [3],[2] - [4],[1].
\]

**5.** Let \(\Lambda\) be a field of characteristic \(p\) in which all the modular irreducible representations of a finite group \(G\) of order \([G:1] = g^p\) with \((g', p) = 1\) can be realized. Two modular irreducible representations of \(G\) over \(\Lambda\) are called mutually "\(p\)-conjugate" if they are obtained from each other by a different choice of the \(p^a\)-th primitive roots of unity. If a modular representation of \(G\) is its own \(p\)-conjugate, it may be called a self \(p\)-conjugate representation. In case of \(S_n\), a self \(p\)-conjugate representation appears as a component of an ordinary self-conjugate representation. In the following \(\Lambda\) is a finite extension field of the Galois field \(GF(p)\). Let \(Q\) be the field of rational numbers and \(Z\) be the ring of integers. If \([\lambda]\) is a \(Q\)-representation of \(A_n\), then \([\lambda]\) is \(Q\)-equivalent to a \(Z\)-representation. In this \(Z\)-representation we reduce the coefficients modulo \(p\) and denote the resulting representation by \([\lambda]\). Now, by 2.1, if \([\lambda] \not\equiv [\lambda']\), then \([\lambda]\) and \([\lambda']\) of \(A_n\) contain the same modular irreducible components. First let \(p \neq 2\). Then we have the following theorem.

**5.1. Theorem.** Two distinct \(p\)-conjugate representations of \(S_n\) restricted to \(A_n\) are \(Q\)-equivalent representations of \(A_n\) over \(GF(p)\), while a self \(p\)-conjugate representation of \(S_n\) restricted to \(A_n\) splits into two \(p\)-conjugate representations of \(A_n\) over an extension of \(GF(p)\).

**Proof.** As in the case of ordinary irreducible representations, the characters in the \(p\)-conjugate representations of \(S_n\) are equal for all even \(p\)-regular elements.
and different only for certain odd $p$-regular elements of $S_n$. From this the first part follows. A self $p$-conjugate representation of $S_n$ is necessarily a component of an ordinary self-conjugate representation of $S_n$. Then the second part follows from 2.1.

Next let $p = 2$. In this case $1 \equiv -1 \pmod{2}$, so that every GF(2)-representation of $S_n$ is a self 2-conjugate representation. The result 5.1 needs to be modified. If $[\lambda] = [\lambda']$ is any representation, then

$$[\lambda] = \sum_T m^T . T,$$

where $m^T$ is the multiplicity of the GF(2)-irreducible representation $T$ of $S_n$. In this equation we have to single out those components $T$ with odd multiplicity. Then a representation $T$ of $S_n$ over GF(2) restricted to $A_n$ is an irreducible representation of $A_n$ unless $T$ is a component of $[\lambda] = [\lambda']$ with odd multiplicity. In the latter case $T$ splits into two equivalent or two 2-conjugate representations of $A_n$ according as all $[\lambda] = [\lambda']$ in which $T$ appears as a component are such that $(\lambda)$ are 2-singular or 2-regular classes respectively.

**Example.** The 4-dimensional modular irreducible representation $T$ of $S_5$ over GF(2), contained in $[31^2]$, is a self 2-conjugate representation of $S_5$. This representation $T$, restricted to $A_5$, is irreducible over GF(2); but in the extended field GF($2^2$) consisting of four elements 0, 1, $w$, $w^*$, it reduces into two representations $T_1$ and $T_2$ of dimension 2 each. Moreover, the class $\{31^2\}$ which corresponds to $[31^2]$ is 2-regular and hence $T_1$ and $T_2$ are 2-conjugate. The matrices representing (123) and (345) in $T$ are

$$T: (123): \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 \end{bmatrix}, \quad (345): \begin{bmatrix} 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \end{bmatrix}.$$ 

If we now transform by

$$\begin{bmatrix} 1 & w^* & 0 & 0 \\
0 & 0 & w & 1 \\
1 & w & 0 & 0 \\
0 & 0 & w^* & 1 \end{bmatrix},$$

we obtain the two 2-conjugate representations of $A_5$:

$$T_1 + T_2: (123): \begin{bmatrix}
w & 0 & 0 & 0 \\
0 & w & 0 & 0 \\
w^* & 0 & 0 & 0 \\
0 & 0 & w & w^* \end{bmatrix}, \quad (345): \begin{bmatrix} 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 \end{bmatrix}.$$ 

As in the case of $S_n$, the block structure of $A_n$ depends on the $p$-core $[\bar{\lambda}]$ of $[\lambda]$. Let us consider first those diagrams which are themselves $p$-cores, i.e.
from which no \(p\)-hooks are removable. In general, the maximum power of \(p\) which divides the degree \(f^a\) is given by (7)

\[
e(f^a) = e(n!) - e((n - a)!) + e(f^a_p),
\]

which reduces to

\[
e(f^a) = e(n!)
\]

if \(a = n\). If \(p \neq 2\), and \([\lambda]\) is not self-conjugate,

\[
e(f^a) = e(\frac{1}{2}n!)
\]

so that \([\lambda]\) of \(A_n\) is modularly irreducible and constitutes a block by itself. If \([\lambda] = [\lambda']\), then the representation will split and

\[
e(\frac{1}{2}f^a) = e(\frac{1}{2}n!)
\]

so that the same statement applies to each of \([\lambda]^+\) and \([\lambda]^−\).

In the case \(p = 2\) the situation is simplified by the fact that every 2-core is self-conjugate so that 5.5 still holds and each of the 2-conjugate components \([\lambda]^+\) and \([\lambda]^−\) is modularly irreducible and constitutes a block by itself.

Example. Let \(p = 3\) and \(n = 5\). The 3-core \([3, 1^2]\) containing 5 nodes yields two modularly irreducible representations of degree 3. The matrices constructed according to 2.4, on transforming by an appropriate matrix and reducing module 3, take the form

\[
\begin{bmatrix}
1 & 1 & -1 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix}
\quad ,
\begin{bmatrix}
-1 & 1 & 0 \\
1 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

The representation \([3, 1^2]\) of \(A_5\) is irreducible in \(GF(3)\). But if we extend the field to \(GF(3^2)\) consisting of 9 elements

\[
0, 1, -1, j, -j, 1 + j, -1 + j, -1 - j,
\]

where \(j^2 + 1 = 0 \text{ (mod 3)}\), the representation reduces into two 3-conjugate modularly irreducible representations:

\[
\begin{bmatrix}
1 & 1 & -1 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix}
\quad ,
\begin{bmatrix}
-1 & 1 & j \\
-1 & -1 & 0 \\
j & j & -1
\end{bmatrix}
\]
The representations are irreducible in any extension field. Also since the degrees (i.e. 3) are divisible by the characteristic of the field, they form their own 3-blocks and indecomposable components of the regular representation of $A_n$.

We have thus disposed of the blocks determined by the $p$-cores of $A_n$ containing exactly $n$ nodes. For the blocks determined by $p$-cores containing less than $n$ nodes, we have the following theorem.

5.6. Theorem. All ordinary irreducible representations of $A_n$ which have $p$-cores $[\lambda]$ or $[\lambda']$ belong to the same $p$-block of $A_n$.

Proof. We deduce the result from that for $S_n$. The representations $[\lambda]$ and $[\lambda']$ of $A_n$ contain the same irreducible representations. From this remark, the result follows.

In enumerating the ordinary and modular irreducible representations of $A_n$ in a block, we have to consider the following three types:

(i) The block determined by the non-self-conjugate $p$-cores $[\lambda]$ and $[\lambda']$.

(ii) The block determined by a self-conjugate $p$-core $[\lambda]$, to which no ordinary split representations of $A_n$ belong.


Clearly no split representation can belong to a block of the first type. For if it does, it has to belong to blocks determined by $p$-cores $[\lambda]$ and $[\lambda']$ in the case of $S_n$, which implies $[\lambda] = [\lambda']$, contrary to the assumption. Moreover, half the number of representations of $S_n$ determined by the two $p$-cores $[\lambda]$ and $[\lambda']$ are also representations of $A_n$. Hence, in a block of type (i) of $A_n$, the number of ordinary or modular representations of $A_n$ is the same as that for $S_n$. For actual enumeration see (7, 8).

In a block of the second type, there are no split representations. Also the number of ordinary irreducible representations in a $p$-block of $S_n$ of the second type is even, for if $[\lambda]$ belongs to the block, so does its conjugate $[\lambda']$. Thus the number of ordinary and modular irreducible representations of $A_n$ in a block of the second type is equal to half the number of the representations of the block of $S_n$ determined by the $p$-core $[\lambda]$.

In the third case the situation is more complicated. If $n = a + bp$, and the number $b$ of removable $p$-hooks is even, then the block of $S_n$ characterized by the self-conjugate $p$-core $[\lambda]$ does contain at least one self-conjugate $[\lambda]$ and this condition is both necessary and sufficient when $p = 2$. If $p \neq 2$, however, we can have a self-conjugate $[\lambda]$ when $b$ is odd, e.g. $[2, 1]$ for $p = 3$ and $[3^3]$ for $p = 5$. Thus, the distinction between $p$-blocks of types (ii) and (iii) depends upon both $b$ and $p$ and enumeration is more difficult.

We conclude by giving the $D$-matrices (mod 2, 3) of $A_n$ for

$$n = 3, 4, 5, 6, 7, 8.$$
D-Matrices (mod 2) of $A_n$

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**D-Matrices (mod 3) of A_n**

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### References


*Carleton University and University of Toronto*