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UNITS OF THE GROUP RING

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ABSTRACT. If R is a ring such that $x, y \in R$ and $xy=0$ imply $yx=0$ and $G \neq 1$, an ordered group, then we show that $\sum \alpha_g g$ is a unit in RG if and only if there exists $\sum \beta_h h$ in RG such that $\sum \alpha_g \beta_{g^{-1}} = 1$ and $\alpha_g \beta_h$ is nilpotent whenever $gh \neq 1$. We also show that if R is a ring with no nilpotent elements $\neq 0$ and no idempotents $\neq 0, 1$ then RG has only trivial units. Some applications are also given.

Introduction. In the first section of this paper we determine the units of the group ring RG where R is a ring with identity and G an ordered group. For example, we show that RG has only trivial units if R has no non-zero nilpotent elements and no idempotents $\neq 0, 1$. Corresponding results for the group ring, with R commutative, have been obtained by Parmenter [4] and for polynomial rings by Coleman and Enochs [2]. In Section 2 we give some applications. For example, we show that if the set of nilpotent elements of R form an ideal $\mathcal{N}$, then $J(RG) = \mathcal{N}G$ where $J(RG)$ denote the Jacobson radical of RG. We also show that if R and S are local rings with no non-zero nilpotent elements and $\sigma: RG \to SG$ is an isomorphism, then $\sigma(R) = S$.

§1. **Units.** In this section, we find the units of the group ring RG where R is a ring with identity and G an ordered group. Let $U(RG)$ denote the units of RG.

PROPOSITION 1.1. *Let R be any ring with identity and let G be an ordered group $\neq 1$. Then the following are equivalent.*

(i) *$U(RG) = \{\sum \alpha_g g \mid$ there exists β_g in R with $\sum \alpha_g \beta_{g^{-1}} = 1$ and $\alpha_g \beta_h = 0$ whenever $gh \neq 1$.*

(ii) *R has no nonzero nilpotent elements.*

Proof. Assume (i) holds and let $\gamma \in R$ be nilpotent. Say $\gamma^t = 0$. Then

$$(1+\gamma g)(1-\gamma g+\gamma^2 g^2-\gamma^3 g^3+\cdots \pm \gamma^{t-1} g^{t-1}) = 1.$$

Hence $1+\gamma g$ is a unit in RG. If $\gamma \neq 0$, $1+\gamma g$ does not satisfy condition (i). Hence $\gamma = 0$ and (ii) holds. Conversely, assume (ii) holds and let $ab = 1$ where $a = \sum_{i=1}^n \alpha_i g_i$ and $b = \sum_{i=1}^m \beta_i h_i$. We will show that $\alpha_i \beta_j = 0$ whenever $g_i h_j \neq 1$. The other statement follows immediately. Suppose that $g_1 < g_2 < \cdots < g_n$ and $h_1 < h_2 < \cdots < h_m$. For $i \neq n$ or $j \neq m$ we have $g_i h_j < g_n h_m$ and, hence,

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$g_ih_j \neq g_nh_m$. We want to show $\alpha_i\beta_j = 0$ whenever $g_ih_j > 1$. If $g_nh_m \leq 1$ there is nothing to show. If $g_nh_m > 1$ we have $\alpha_n\beta_m = 0$ from the above. Assume that we know that $\alpha_r\beta_s = 0$ whenever $g_rh_s > g_{i_1}h_{k_1} = g_{i_2}h_{k_2} = \cdots = g_{i_p}h_{k_p} > 1$ (the $g_{i_s}h_{k_s}$ being a complete list of products equal to $g_{i_1}h_{k_1}$). We see that $\alpha_{i_1}\beta_{k_1} + \cdots + \alpha_{i_p}\beta_{k_p} = 0$ and we may assume that $i_1 < i_2 < i_3 < \cdots < i_p$. Since $\alpha_r\beta_s = 0$ whenever $g_rh_s > g_{i_1}h_{k_1}$, we have $(\beta_s\alpha_r)^2 = \beta_s\alpha_r\beta_s\alpha_r = 0$. From our assumption, that R has no non-zero nilpotent elements, it follows that $\beta_s\alpha_r = 0$ whenever $g_rh_s > g_{i_1}h_{k_1}$. Now, multiplying the above equation on the right by α_{i_p} we obtain:

$$\alpha_{i_1}\beta_{k_1}\alpha_{i_p} + \alpha_{i_2}\beta_{k_2}\alpha_{i_p} + \cdots + \alpha_{i_p}\beta_{k_p}\alpha_{i_p} = 0$$

For $t < p$, $g_{i_p}h_{k_t} > g_{i_t}h_{k_t}$. Hence by the remark above, $\beta_{k_t}\alpha_{i_p} = 0$. We conclude that $\alpha_{i_p}\beta_{k_p}\alpha_{i_p} = 0$. Hence $(\alpha_{i_p}\beta_{k_p})^2 = 0$ and $\alpha_{i_p}\beta_{k_p} = 0$ using (ii). Working back, we obtain $\alpha_{i_t}\beta_{k_t} = 0$ for $1 \leq t \leq p$. Therefore, we have shown that $\alpha_i\beta_j = 0$ whenever $g_ih_j > 1$. An identical argument to that given above, starting with g_1h_1 shows that $\alpha_i\beta_j = 0$ whenever $g_ih_j < 1$. This completes the proof. □

LEMMA 1.2. *Suppose R is a ring such that if $x, y \in R$ and $xy = 0$ then $yx = 0$. Then the set of nilpotent elements of R forms an ideal.*

Proof. [2], Lemma 2. □

THEOREM 1.3. *Suppose that R satisfies the hypothesis of Lemma 1.2. Then $\sum \alpha_g g$ is a unit in RG if and only if there exist $\sum \beta'_h h$ in RG such that $\sum \alpha_g \beta'_{g^{-1}} = 1$ and $\alpha_g \beta'_h$ is nilpotent whenever $gh \neq 1$.*

Proof. First assume $\sum \alpha_g g$ is a unit in RG. Let $\mathcal{N}$ denote the set of nilpotent elements of R. From Lemma 1.2, $\mathcal{N}$ is an ideal. Passing from RG to $(R/\mathcal{N})G$, $\sum \bar{\alpha}_g g$ is a unit in $(R/\mathcal{N})G$. Proposition 1.1 then says that there exists $\sum \bar{\beta}_h h$ in $(R/\mathcal{N})G$ such that $\sum \bar{\alpha}_g \bar{\beta}_{g^{-1}} = \bar{1}$ and $\bar{\alpha}_g \bar{\beta}_h = 0$ whenever $gh \neq 1$. Hence $\sum \alpha_g \beta_{g^{-1}} = 1 + n$ where $n \in \mathcal{N}$ and $\alpha_g \beta_h$ is nilpotent whenever $gh \neq 1$. If $n^s = 0$ we see that

$$\sum \alpha_g \beta_{g^{-1}}(1 - n + n^2 - \cdots \pm n^{s-1}) = (1+n)(1 - n + n^2 - \cdots \pm n^{s-1}) = 1$$

and $\alpha_g \beta_h(1 - n + n^2 + \cdots \pm n^{s-1})$ is nilpotent whenever $gh \neq 1$ since $\mathcal{N}$ is an ideal. Putting $\beta'_h = \beta_h(1 - n + n^2 - \cdots \pm n^{s-1})$, then $\sum \beta'_g g$ satisfy the required conditions.

Before proving the converse, we show that $\mathcal{N}G$ is a nil ideal. Let $a = \sum_{i=1}^n \alpha_i g_i \in \mathcal{N}G$. The elements α_i are nilpotent with exponents k_i. Let $m = \sum_{i=1}^n k_i$. Now, using the property, $\alpha_i\alpha_j = 0$ implies $\alpha_j\alpha_i = 0$ it is easy to show that $a^m = 0$. Hence $\mathcal{N}G$ is a nil ideal. Suppose $\sum \alpha_g g$ satisfy the conditions, then from Proposition 1.1 $\sum \bar{\alpha}_g g$ is a unit in $(R/\mathcal{N})G$. Since $\mathcal{N}G$ is a nil ideal and $(R/\mathcal{N})G \cong RG/\mathcal{N}G$, we conclude that $\sum \alpha_g g$ is a unit in RG. □

REMARK. The class of rings for which the condition $\alpha\beta = 0$ implies $\beta\alpha = 0$ holds, includes the class of reduced rings.

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COROLLARY 1.4. *Let R be a ring with identity satisfying hypothesis of Lemma 1.2 with no idempotents $\neq 0, 1$. Then $\sum \alpha_g g$ is a unit in RG if and only if for some g, α_g is a unit and all the other α_g's are nilpotent.*

Proof. Suppose $\sum \alpha_g g$ is a unit in RG. Then by Theorem 1.3 there exist elements β_g in R such that $\sum \alpha_g \beta_{g^{-1}} = 1$ and $\alpha_g \beta_h$ is nilpotent whenever $gh \neq 1$. Hence $(\sum_g \alpha_g \beta_{g^{-1}})\alpha_h = \alpha_h \beta_{h^{-1}} \alpha_h + \sum_{g \neq h} \alpha_g \beta_{g^{-1}} \alpha_h = \alpha_h$ for any α_h. For $h \neq g$ we have $\alpha_g \beta_{h^{-1}}$ nilpotent, say $(\alpha_g \beta_{h^{-1}})^p = 0$. Then, using the property that $xy = 0$ implies $yx = 0$ and the fact that $(\alpha_g \beta_{h^{-1}})^p = 0$, we get $(\beta_{h^{-1}} \alpha_g)^p = 0$. Hence $\beta_{h^{-1}} \alpha_g$ is also nilpotent and consequently, since the set of nilpotent elements is an ideal, $\alpha_h \beta_{h^{-1}} \alpha_h = \alpha_h + n$ where n is nilpotent. Furthermore, $(\alpha_h \beta_{h^{-1}})^2 = \alpha_h \beta_{h^{-1}} + m_1$ and $(\beta_{h^{-1}} \alpha_h)^2 = \beta_{h^{-1}} \alpha_h + m_2$ where $m_1, m_2 \in \mathcal{N}$. Therefore, $\alpha_h \beta_{h^{-1}}$ and $\beta_{h^{-1}} \alpha_h$ are idempotent modulo $\mathcal{N}$, and we conclude that $\alpha_h \beta_{h^{-1}} \in \mathcal{N}$ or $\alpha_h \beta_{h^{-1}} - 1 \in \mathcal{N}$ and $\beta_{h^{-1}} \alpha_h \in \mathcal{N}$ or $\beta_{h^{-1}} \alpha_h - 1 \in \mathcal{N}$, since idempotents can be lifted modulo $\mathcal{N}$ and R has no idempotents $\neq 0, 1$ (see [3], Proposition 1, p. 72). If $\alpha_h \beta_{h^{-1}} \in \mathcal{N}$, then $\alpha_h \beta_{h^{-1}} \alpha_h \in \mathcal{N}$ and by the above $\alpha_h = \alpha_h \beta_{h^{-1}} \alpha_h - n \in \mathcal{N}$. Since $\sum \alpha_g \beta_{g^{-1}} = 1$, all the $\alpha_g \beta_{g^{-1}}$ cannot be elements of $\mathcal{N}$ (this would imply that $1 \in \mathcal{N}$). Say $\alpha_h \beta_{h^{-1}} \notin \mathcal{N}$ for some specific h. Then $\alpha_h \beta_{h^{-1}} - 1 \in \mathcal{N}$ and hence we have $\alpha_h \beta_{h^{-1}} \alpha_k - \alpha_k \in \mathcal{N}$ for each α_k. If $k \neq h$, then $\alpha_k \beta_{h^{-1}} \in \mathcal{N}$ and consequently $\beta_{h^{-1}} \alpha_k \in \mathcal{N}$ as was shown above. Hence $\alpha_h \beta_{h^{-1}} \alpha_k \in \mathcal{N}$ and, therefore, $\alpha_k \in \mathcal{N}$. Furthermore, from the fact that $\alpha_h \beta_{h^{-1}} \notin \mathcal{N}$ we also have $\beta_{h^{-1}} \alpha_h \notin \mathcal{N}$, and consequently $\beta_{h^{-1}} \alpha_h - 1 \in \mathcal{N}$. Say $\alpha_h \beta_{h^{-1}} = 1 + n_1$ and $\beta_{h^{-1}} \alpha_h = 1 + n_2$, n_1, $n_2 \in \mathcal{N}$. From this it now follows easily that $\alpha_h \beta_{h^{-1}}$ and $\beta_{h^{-1}} \alpha_h$ are units in R. Hence α_h has a left as well as a right inverse in R. Hence in $\sum \alpha_g g$ we have α_h a unit in R for some $h \in G$ and all the other α_g's are nilpotent. The converse follows from Theorem 1.3. □

COROLLARY 1.5. *Let R be a ring with no nilpotent elements $\neq 0$ and no idempotents $\neq 0, 1$. Then the only units in RG are of the form ug where u is a unit of R and g is in G.*

Proof. Since R has no nilpotent elements $\neq 0$ the hypothesis of Lemma 1.2 is satisfied. The result follows from Corollary 1.4. □

§2. **Applications.** Let $J(R)$ denote the Jacobson radical of R. Amitsur [1] has proved that $J(R[x]) = N[x]$, where $N = J(R[x]) \cap R$) and N is a nil ideal in R. Thus the following proposition is of some interest.

PROPOSITION 2.1. *Suppose R is a ring with no idempotents $\neq 0, 1$, and whose nilpotent elements form an ideal $\mathcal{N}$. Then $J(RG) = \mathcal{N}G$ where G is an ordered group $\neq 1$.*

Proof. Let $x = \sum_{i=1}^n \alpha_i g_i \in J(RG)$. Since $G \neq 1$, there is a $g \in G$ such that $g \neq g_1, g_2, \ldots, g_n$. Furthermore, since $x \in J(RG)$, $g - x$ is a unit in RG. From Corollary 1.4 we have α_i, $i = 1, 2, \ldots, n$ nilpotent. Hence $x \in \mathcal{N}G$. Similarly we have from Corollary 1.4 that $x \in \mathcal{N}G$ implies $x \in J(RG)$. □

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We now study some isomorphic group rings. Recall that a ring R with 1 is called local if the non-units of R form an ideal.

PROPOSITION 2.2. *Let R and S be local rings with no non-zero nilpotent elements. Let $G \neq 1$, be ordered. If $\sigma: RG \to SG$ is a homomorphism, then $\sigma(R) \subseteq S$.*

Proof. It is clear that the hypothesis of Corollary 1.5 is satisfied for local rings with no non-zero nilpotent elements. We now use the same argument as in Proposition 4.4 of [4]. □

COROLLARY 2.3. *Let R, S be local rings with 1, and with no non-zero nilpotent elements. Let $G \neq 1$, be ordered. If $\sigma: RG \to SG$ is an isomorphism, then $\sigma(R) = S$.*

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