TAUBERIAN THEOREMS FOR STRONG AND ABSOLUTE BOREL-TYPE METHODS OF SUMMABILITY

BY

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1. Introduction. Suppose throughout that \( s, a_n (n = 0, 1, 2, \ldots) \) are arbitrary complex numbers, that \( \alpha > 0 \) and \( \beta \) is real and that \( N \) is a non-negative integer such that \( \alpha N + \beta \geq 1 \). Let

\[
\begin{align*}
S_n &= \sum_{n=0}^{\infty} a_n (n \geq 0), \\
S_{-1} &= 0, \\
S_{\alpha,\beta}(z) &= \sum_{n=-N}^{\infty} s_n z^{\alpha n + \beta - 1}, \\
a_{\alpha,\beta}(z) &= \sum_{n=-N}^{\infty} a_n z^{\alpha n + \beta - 1}, \\
S_{\alpha,\beta}(z) &= \alpha e^{-z} s_{\alpha,\beta}(z), \\
a_{\alpha,\beta}(z) &= \alpha e^{-z} a_{\alpha,\beta}(z)
\end{align*}
\]

where \( z = x + iy \) is a complex variable and the power \( z^r \) is assumed to have its principal value.

Borel-type methods are defined as follows:

(a) Summability: If \( S_{\alpha,\beta}(x) \) exists for all \( x \geq 0 \) and tends to \( s \) as \( x \to \infty \), we say that \( s_n \to s(B, \alpha, \beta) \) or \( \sum_{n=0}^{\infty} a_n = s(B, \alpha, \beta) \);

(b) Strong summability with index \( p > 0 \): If \( S_{\alpha,\beta-1}(x) \) exists for all \( x \geq 0 \) and

\[
\int_{0}^{\infty} e^t |S_{\alpha,\beta-1}(t) - s|^p dt = o(e^{\gamma}),
\]

we say that \( s_n \to s[B, \alpha, \beta]_p \);

(c) Absolute summability: If \( s_n \to s[B, \alpha, \beta] \) and \( S_{\alpha,\beta}(x) \in BV_x[0, \infty) \),\(^{(2)} \) we say that \( s_n \to s[B, \alpha, \beta] \);

(d) Boundedness: If \( S_{\alpha,\beta}(x) \) exists and is bounded on \([0, \infty)\), we say that \( s_n = 0(1)(B, \alpha, \beta) \);

(e) Strong boundedness with index \( p > 0 \): If \( S_{\alpha,\beta-1}(x) \) exists for all \( x \geq 0 \) and

\[
\int_{0}^{\infty} e^t |S_{\alpha,\beta-1}(t)|^p dt = o(e^{\gamma}),
\]

we say that \( s_n = 0(1)[B, \alpha, \beta]_p \).

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\(^{(2)}\) \( f(x) \in BV_x[0, \infty) \) means that \( f(x) \) is of bounded variation with respect to \( x \) on \([0, \infty)\).
The summability method \((B, 1, 1)\) is the Borel exponential method \(B\) (see [7]). The \((B, \alpha, \beta)\) method is due to Borwein (see [2]) and the \([B, \alpha, \beta]_p\) and \([B, \alpha, \beta]\) methods are due to Borwein and Shawyer (see [4], [3] respectively). Strong Borel-type summability \([B, \alpha, \beta]\) (see [3]) is the \([B, \alpha, \beta]\) method.

The actual choice of the integer \(N\) in the above definitions is clearly immaterial. We shall therefore tacitly assume whenever a finite number of methods, with \(\alpha\) fixed and \(\beta = \beta_1, \beta_2, \ldots, \beta_k\), are under consideration that \(N\) is such that \(\alpha N + \beta_r \geq 1\) \((r = 1, 2, \ldots, k)\).

The following known result establishes a natural scale for these summability methods. (Theorem A(i) is [1, (II)]. Theorem A(ii) is [3, Theorem 9] when \(p = 1\) and part of [4, Theorem 9*(ii)] when \(p \geq 1\). Theorem A(iii) is [8, Lemma].)

**Theorem A.** Let \(\beta > \mu\).

(i) If \(s_n \to s(B, \alpha, \mu)\), then \(s_n \to s(B, \alpha, \beta)\).
(ii) If \(p \geq 1\) and \(s_n \to s[B, \alpha, \mu]_p\), then \(s_n \to s[B, \alpha, \beta]_p\).
(iii) If \(s_n \to s[B, \alpha, \mu]\), then \(s_n \to s[B, \alpha, \beta]\).

In [5] we established a number of tauberian theorems for the \((B, \alpha, \beta)\) method. In this paper we investigate all the corresponding results for the \([B, \alpha, \beta]_p\) method with \(p \geq 1\) and either prove them or show, by means of counterexamples, that they are false. We also examine some of the corresponding results for the \([B, \alpha, \beta]\) method.

2. **Preliminary results.** We first state some known results.

**Lemma 1.**

(i) If \(p \geq 1\) and \(s_n \to s[B, \alpha, \beta]_p\), then \(a_n \to 0[B, \alpha, \beta]_p\).
(ii) If \(s_n \to s[B, \alpha, \beta]\), then \(a_n \to 0[B, \alpha, \beta]\).

**Lemma 2.**

(i) If \(p \geq 1\) and \(s_n \to s[B, \alpha, \beta]_p\), then \(s_n \to s(B, \alpha, \beta)\).
(ii) If \(p > 0\) and \(s_n \to s(B, \alpha, \beta)\), then \(s_n \to s[B, \alpha, \beta + 1]_p\).

Lemma 1(i) is included in [3, Theorem 15] when \(p = 1\) and in [4, Theorem 15*] when \(p > 1\). Lemma 1(ii) is included in [3, Theorem 14]. Lemma 2(i) is [3, Theorem 3] when \(p = 1\) while Lemma 2(ii) follows from [4, Theorem 3*] and Theorem A(i) when \(p > 1\). Lemma 2(ii) is [4, Theorem 5*].

Wherever it occurs in the following lemmas, we suppose that \(f(x)\) is bounded and Lebesgue measurable on every finite interval \([0, X]\) and we let \(f_\delta(x)\) be defined by

\[
f_\delta(x) = \frac{1}{\Gamma(\delta)} \int_0^x (x - t)^{\delta - 1} f(t) \, dt
\]

where \(\delta > 0\).
Lemma 3. If $\delta > 0$ and $\gamma > 0$, then

$$f_{\delta + \gamma}(x) = \frac{1}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} f_{\delta}(t) \, dt.$$

Lemma 4.

(i) Let $f(x) = s_{a,\beta}(x)$ and let $\delta > 0$. Then $s_{a,\beta + \delta}(x) = f_\delta(x)$.

(ii) $A_{a,\beta}(x) = s_{a,\beta}(x) - s_{a,\beta + \alpha}(x) - \alpha e^{-x}s_{N-1}(x^{\alpha N + \beta - 1}/\Gamma(\alpha N + \beta))$.

Lemma 3 is a well-known result the proof of which is straightforward. Lemma 4(i) is [2, Lemma 2]. The proof of Lemma 4(ii) is also straightforward.

Lemma 5. If $s_n = 0(1)(B, \alpha, \beta)$, then $s_n = 0(1)(B, \alpha, \beta + \delta)$ for every $\delta > 0$.

Lemma 5 is [3, Theorem 8].

Lemma 6. Let $p \geq 1$. If $s_n = 0(1)[B, \alpha, \beta]_p$, then

(i) $s_n = 0(1)[B, \alpha, \beta]$,

(ii) $s_n = 0(1)[B, \alpha, \beta + \delta]_p$ where $0 < \delta < 1$, and

(iii) $s_n = 0(1)[B, \alpha, \beta + \delta]$, where $r > 0$ and $\delta \geq 1$.

Proof. (i) When $p = 1$ the result is [3, Theorem 4]. Thus we suppose that $p > 1$ and we let $1/p + 1/q = 1$. Using Hölder's inequality and Lemma 4(i), we have that

$$|S_{a,\beta}(x)| \leq e^{-x} \left[ \int_0^x e^t |S_{a,\beta-1}(t)| \, dt \right]^{1/p} \left[ \int_0^x e^{t/q} \, dt \right]^{1/q} \leq e^{-x} \{Ke^{x}\}^{1/p} \{e^x\}^{1/q} = K^{1/p}$$

for some positive constant $K$ since $s_n = 0(1)[B, \alpha, \beta]$.

(ii) When $p = 1$ the result is included in [3, Theorem 10]. Thus we again suppose that $p > 1$ and we let $1/p + 1/q = 1$. Furthermore, we let $f(x) = as_{a,\beta-1}(x)$, $L = 2^p/\Gamma(\delta)^p$, and $M = [1/\Gamma(\delta)]^p \int_0^1 e^{(1-p)t} |f_\delta(t)| \, dt$. Then, using Lemma 4(i), Hölder's inequality, and part of the proof of (i), we have for $x \geq 1$ that

$$\int_0^x e^t |S_{a,\beta+\delta-1}(t)|^p \, dt = \frac{1}{\Gamma(\delta)^p} \int_0^x e^{(1-p)t} \left[ \int_0^t (t-u)^{\delta-1} |f(u)| \, du \right]^p \, dt$$

$$\leq L \int_0^x e^{(1-p)t} \left[ \int_0^t |f(u)| \, du \right]^p \, dt$$

$$+ L \int_1^x e^{(1-p)t} \left[ \int_{t-1}^t (t-u)^{\delta-1} |f(u)| \, du \right]^p \, dt$$

$$\times \left[ \int_{t-1}^t (t-u)^{\delta-1} \, du \right]^{p/q} \, dt + M.$$
This establishes the desired result.

(iii) If \( \delta \geq 1 \), then

\[
\int_0^x e^t |s_{n, \beta + \delta - 1}(t)|^p dt \leq K e^t dt \leq K e^x
\]

for some positive constant \( K \) by Lemma 6(i) and Lemma 5.

**Lemma 7.** If

\[
e^{-x} \int_0^x f(t) dt = o(1),
\]

then

\[
e^{-x} \int_0^x f_b(t) dt = o(1)
\]

for every \( \delta > 0 \).

The proof of Lemma 7 is essentially the same as the proof of [3, Lemma 5].

**Lemma 8.** Let \( p \geq 1 \). If

\[
e^{-x} \int_0^x e^{(1-p)q} |f(t)|^p dt = 0(1) \quad \text{and} \quad e^{-x} \int_0^x f(t) dt = o(1),
\]

then

(i) \( e^{-x} \int_0^x e^{(1-p)\delta} |f_b(t)|^p dt = o(1) \) where \( 0 < \delta < 1 \) and

(ii) \( e^{-x} \int_0^x e^{(1-r)\delta} |f_b(t)|^r dt = o(1) \) where \( r > 0 \) and \( \delta \geq 1 \).

**Proof.** (i) Let \( \epsilon > 0 \). By hypothesis, there exists a number \( Y \geq 0 \) such that

\[
\left| \int_0^x f(t) dt \right| \leq \epsilon e^x
\]
for all \( x \geq Y \). Let

\[
N(\varepsilon) = \sup_{0 \leq x < \infty} \left| \int_{0}^{x} f(t) \, dt \right| < \infty.
\]

Now

\[
\limsup_{x \to \infty} e^{-x} \int_{0}^{x} e^{(1-p)\varepsilon} |f_{\varepsilon}(t)|^p \, dt = \limsup_{x \to \infty} e^{-x} \int_{0}^{x} \left| \frac{1}{\Gamma(\delta)} \int_{0}^{t} (t-u)^{\delta-1} f(u) \, du \right|^p \, dt
\]

\[
\leq \frac{2^p}{(\Gamma(\delta))^p} \left\{ \limsup_{x \to \infty} I_1 + \limsup_{x \to \infty} I_2 \right\}
\]

where

\[
I_1 = e^{-x} \int_{0}^{x} e^{(1-p)\varepsilon} \left| \int_{0}^{t-\varepsilon} (t-u)^{\delta-1} f(u) \, du \right|^p \, dt
\]

and

\[
I_2 = e^{-x} \int_{0}^{x} e^{(1-p)\varepsilon} \left| \int_{t-\varepsilon}^{t} (t-u)^{\delta-1} f(u) \, du \right|^p \, dt
\]

But, using the Second Mean Value Theorem,

\[
\limsup_{x \to \infty} I_1 = \limsup_{x \to \infty} e^{-x} \int_{0}^{x} e^{(1-p)\varepsilon} \left| \int_{\mu(t)}^{t-\varepsilon} f(u) \, du \right|^p \, dt
\]

\[
\leq 2^p e^{(\delta-1)p} \limsup_{x \to \infty} e^{-x} \int_{0}^{x} e^{(1-p)\varepsilon}(N(\varepsilon) + \varepsilon f')^p \, dt
\]

\[
\leq 2^p e^{(\delta-1)p} \limsup_{x \to \infty} e^{-x} \int_{0}^{x} e^{(1-p)\varepsilon}(N(\varepsilon)^p + \varepsilon f f') \, dt
\]

\[
= 2^{2p} e^{\delta p}
\]

since

\[
\left| \int_{\mu(t)}^{t-\varepsilon} f(u) \, du \right| \leq 2 \sup_{0 \leq y \leq t-\varepsilon} \left| \int_{0}^{y} f(u) \, du \right| \leq 2(N(\varepsilon) + \varepsilon f')
\]

and

\[
\lim_{x \to \infty} e^{-x} \int_{0}^{x} e^{(1-p)\varepsilon} (N(\varepsilon))^p \, dt = 0.
\]

Also, by hypothesis there is a number \( K \geq 0 \) such that

\[
e^{-x} \int_{0}^{x} e^{(1-p)\varepsilon} |f(t)|^p \, dt \leq K
\]
for all \( x \geq 0 \), and therefore, when \( p = 1 \),

\[
\limsup_{x \to \infty} I_2 \leq \limsup_{x \to \infty} e^{-x} \int_{x}^{\infty} (t-u)^{\delta-1} |f(u)| \, du \\
\leq \limsup_{x \to \infty} e^{-x} \int_{x}^{\infty} |f(u)| \, du \int_{u}^{u+\varepsilon} (t-u)^{\delta-1} \, dt \\
\leq K \frac{e^\delta}{\delta},
\]

while, when \( p > 1 \),

\[
\limsup_{x \to \infty} I_2 \leq \limsup_{x \to \infty} e^{-x} \int_{x}^{\infty} e^{(1-p)t} \left\{ \int_{t-\varepsilon}^{t} (t-u)^{\delta-1} |f(u)|^p \, du \right\}^{p-1} \, dt \\
= \left\{ \frac{e^{\delta}}{\delta} \right\}^{p-1} \limsup_{x \to \infty} e^{-x} \int_{x}^{\infty} e^{(1-p)t} \int_{t-\varepsilon}^{t} (t-u)^{\delta-1} |f(u)|^p \, du \\
\leq \left\{ \frac{e^{\delta}}{\delta} \right\}^{p-1} \limsup_{x \to \infty} e^{-x} \int_{x}^{\infty} |f(u)|^p \, du \int_{u}^{u+\varepsilon} (t-u)^{\delta-1} e^{(1-p)t} \, dt \\
\leq \left\{ \frac{e^{\delta}}{\delta} \right\}^{p} \limsup_{x \to \infty} e^{-x} \int_{x}^{\infty} e^{(1-p)t} |f(u)|^p \, du \leq K \left\{ \frac{e^{\delta}}{\delta} \right\}^{p}.
\]

Thus for \( p \geq 1 \) we have that

\[
\limsup_{x \to \infty} e^{-x} \int_{0}^{x} e^{(1-p)t} |f_\delta(t)|^p \, dt \leq \frac{2^p (2^{2p} + K\delta^{-p})}{[\Gamma(\delta)]^p} e^{5p}
\]

from which it follows that

\[
\limsup_{x \to \infty} e^{-x} \int_{0}^{x} e^{(1-p)t} |f_\delta(t)|^p \, dt = 0
\]

since \( \varepsilon \) is arbitrary. This establishes the desired result.

(ii) Since \( e^{-x}f_1(x) = o(1) \) by hypothesis, we have, when \( \delta = 1 + \mu \) where \( \mu > 0 \), that

\[
e^{-x}f_{1+\mu}(x) = e^{-x} \int_{0}^{x} f_\mu(t) \, dt = o(1),
\]

using Lemma 3 and Lemma 7. Hence, for \( \delta \geq 1 \),

\[
e^{-x} \int_{0}^{x} e^{(1-p)t} |f_\delta(t)|^p \, dt = e^{-x} \int_{0}^{x} e^{t} |e^{-t}f_\delta(t)|^p \, dt \\
= e^{-x} \int_{0}^{x} e^{t} o(1) \, dt \\
= o(1).
\]
If $b$ is a real number, we let

$$H_b = \{ z \mid \text{Re } z \geq b \}.$$ 

A function $g(z)$ is said to be of exponential type in $H_b$ if $g(z)$ is analytic in $H_b$ and if there are positive numbers $A$, $a$ such that $|g(z)| \leq Ae^{a|z|}$ for all $z$ in $H_b$.

**Lemma 9.** If $g(z)$ is of exponential type in $H_0$ and if

$$\int_0^\infty |g(x)|^p \, dx < \infty \quad (p > 0),$$

then

$$\int_0^\infty |g'(x)|^p \, dx < \infty.$$ 

Lemma 9 is due to Gaier [6, Theorem 2].

**Lemma 10.** If $g(z)$ is of exponential type in $H_b$ and $g(x) \in BV_x [b, \infty)$, then $$g^{(k)}(x) \in BV_x [b, \infty)$$ for every non-negative integer $k$.

**Proof.** Suppose that $g^{(k)}(x) \in BV_x [b, \infty)$ where $k$ is a non-negative integer. Then

$$\int_0^\infty |g^{(k+1)}(x + b + 1)| \, dx < \infty$$

and

$$|g^{(k+1)}(z + b + 1)| \leq \frac{(k + 1)!}{2\pi} \int_0^{2\pi} |g(z + b + e^{i\theta})| \, d\theta \leq (k + 1)! A e^{a(|z| + |b| + 1)}$$

for all $z$ in $H_0$ where $A$, $a$ are positive constants. Hence, by Lemma 9,

$$\int_0^\infty |g^{(k+2)}(x + b + 1)| \, dx = \int_{b+1}^\infty |g^{(k+2)}(x)| \, dx < \infty$$

i.e. $g^{(k+1)}(x) \in BV_x [b+1, \infty)$. Since $g^{(k+1)}(x) \in BV_x [b, b+1]$, therefore $g^{(k+1)}(x) \in BV_x [b, \infty)$. The desired result now follows by induction.

3. **Tauberian theorems for strong Borel-type summability with index $p \geq 1$.**

We first show that the scale in Theorem A(ii) is proper. In [5] we showed that there is a sequence $\{ s_n \}$ which tends to a limit $(B, \alpha, \beta)$ but does not tend to a limit $(B, \alpha, \beta - 1)$. Hence, in view of Lemma 2, there is a sequence $\{ s_n \}$ which tends to a limit $[B, \alpha, \beta + 1]_p$ for every $p > 0$ but does not tend to a limit $[B, \alpha, \beta - 1]_p$ for any $p \geq 1$. 

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THEOREM 1. Let $p, r \geq 1$. If $s_n \to s[B, \alpha, \mu]_p$ and $a_n \to 0[B, \alpha, \beta]_p$, then $s_n \to s[B, \alpha, \beta]_p$.

Proof. By Lemma 2(i), $s_n \to s(B, \alpha, \mu)$. The result now follows by [9, Theorem 3] and the note following [9, Theorem 3].

THEOREM 2. Let $p \geq 1$. If $s_n \to s[B, \alpha, \beta + \varepsilon]_p$ for some $\varepsilon > 0$ and $s_n = 0(1)[B, \alpha, \beta]_p$, then $s_n \to s[B, \alpha, \beta + \delta]_p$ for every $\delta > 0$.

Proof. We can suppose without loss of generality that $s = 0$. Then $s_n \to 0(B, \alpha, \beta + \varepsilon)$ and $s_n = 0(1)(B, \alpha, \beta)$ by Lemma 2(i) and Lemma 6(i). Hence $s_n \to 0(B, \alpha, \beta + \delta)$ by [5, Theorem 2] for $\delta > 0$. Also $s_n = 0(1)[B, \alpha, \beta + \delta]_p$ by Lemma 6(ii) or (iii). Therefore, letting $f(x) = ax_{\alpha, \beta + \delta - 1}(x)$, we have that

$$e^{-x} \int_0^x f(t) \, dt = S_{\alpha, \beta + \delta}(x) = o(1)$$

and

$$e^{-x} \int_0^x e^{(1-p)t} |f(t)|^p \, dt = e^{-x} \int_0^x e^t |S_{\alpha, \beta + \delta - 1}(t)|^p \, dt = 0(1)$$

using Lemma 4(i), and consequently,

$$e^{-x} \int_0^x e^t |S_{\alpha, \beta + 2\delta - 1}(t)|^p \, dt = e^{-x} \int_0^x e^{(1-p)t} |f_s(t)|^p \, dt = o(1)$$

using Lemma 4(i) and Lemma 8, i.e. $s_n \to 0[B, \alpha, \beta + 2\delta]_p$. This establishes the desired result.

THEOREM 2*. Let $p \geq 1$. If $\sum_{n=0}^\infty a_n = s[B, \alpha, \beta + \varepsilon]_p$ for some $\varepsilon > 0$ and $a_n = 0(1)[B, \alpha, \beta]_p$, then $\sum_{n=0}^\infty a_n = s[B, \alpha, \beta + \delta]_p$ for every $\delta > 0$.

Proof. By Lemma 1(i), $a_n \to 0[B, \alpha, \beta + \varepsilon]_p$ and thus, by Theorem 2, $a_n \to 0[B, \alpha, \beta + \delta]_p$ for every $\delta > 0$. The result now follows by Theorem 1.

A real-valued function $g(x)$, with domain $[0, \infty)$, is slowly decreasing if for every $\varepsilon > 0$ there exist positive numbers $X$, $\delta$ such that $g(x) - g(y) > -\varepsilon$ whenever $x \geq y \geq X$ and $x - y \leq \delta$. The following result is [5, Theorem 3]: If $s_n \to s(B, \alpha, \beta + \varepsilon)$ for some $\varepsilon > 0$ and $S_{\alpha, \beta}(x)$ is slowly decreasing, then $s_n \to s(B, \alpha, \beta)$. We now show that there is no analogue to this result for the $[B, \alpha, \beta]_p$ method.

Let $\{s_n\}$ be the sequence defined by $\sum_{n=0}^\infty s_n(x^n/n!) = e^x \sin e^x$ (cf. [7, p. 183]). Then $S_{1,1}(x) = \sin e^x$ where we choose $N = 0$. Thus, using Lemma 4(i),

$$S_{1,2}(x) = e^{-x} \int_0^x e^t \sin e^t \, dt = e^{-x} (\cos 1 - \cos e^x) = o(1)$$
and therefore $s_n \to 0(B, 1, 2)$. (In fact, by [5, Theorem 2], $s_n \to 0(B, 1, 1 + \delta)$ for every $\delta > 0$.) Hence, by Lemma 2(ii), $s_n \to 0[B, 1, 3]$, for every $r > 0$. Furthermore,

$$e^{-x} \int_0^x e^t |S_{1,1}(t)| dt = e^{-x} \int_0^x e^t |\sin e^t| dt$$

$$= e^{-x} \int_0^r |\sin u| du \to \frac{L(r)}{\pi}$$

as $x \to \infty$ where $L(r) = \int_0^r |\sin u| du$. Therefore $s_n \to 0[B, 1, 2]$, $s_n \to 0[B, 1, 3]$, and both $e^{-x} \int_0^r e^t |S_{1,1}(t)| dt$ and $e^{-x} \int_0^r e^t |S_{1,1}(t)| dt$ are slowly decreasing (since they both tend to a limit as $x \to \infty$).

**Theorem 3.** Let $p \geq 1$. If $s_n \to s[B, \alpha, \mu]_p$ and

(i) $s_n \geq -K$ for all $n \geq 0$, or
(ii) $a_n \geq -K$ for all $n \geq 0$, or
(iii) $S_{\alpha,\mu}(z)$ is of exponential type in $H_\delta$, or
(iv) $A_{\alpha,\mu}(z)$ is of exponential type in $H_\delta$, or
(v) $|a_n| \leq K^n$ for all $n \geq 0$,

where $K$, $\delta$ are positive constants, then

$s_n \to s[B, \alpha, \beta]$,

for every $r > 0$.

**Proof.** By Lemma 2(i), $s_n \to s(B, \alpha, \mu)$. Hence, by [5, Theorem 5, 5*, 6, 6*, or 7], $s_n \to s(B, \alpha, \beta - 1)$. The result now follows by Lemma 2(ii).

4. Tauberian theorems for absolute Borel-type summability. We first show that the scale in Theorem A(iii) is proper in the sense that for each $\beta$ there is a sequence $\{s_n\}$ which is summable $|B, \alpha, \beta|$ but is not summable $|B, \alpha, \beta - 1|$.

Choose an integer $m$ such that $\alpha m > 1$ and let $P$ be the smallest integer such that $mP \geq N$. Let

$$x^ne^{-x} \sin e^x = \sum_{n=P}^{\infty} b_n x^n$$

and let

$$s_n = \begin{cases} \Gamma(\alpha n + \beta) b_k & \text{if } n = mk, \\ 0 & \text{otherwise}. \end{cases}$$

Then

$$S_{n,\beta}(x) = \alpha x^{\alpha m P + \beta - 1} e^{-x} e^{-x^m} \sin e^{x^m} = o(1)$$
and

\[
S_{\alpha,\beta}'(x) = \alpha (amP + \beta - 1)x^{amP + \beta - 2}e^{-x}e^{-x^m} \sin e^{x^m} \sin e^{x^{am}} \\
- \alpha x^{amP + \beta - 1}e^{-x}e^{-x^m} \sin e^{x^m} \\
- \alpha (am)x^{amP + \alpha m + \beta - 2}e^{-x}e^{-x^m} \sin e^{x^m} \\
+ \alpha (am)x^{amP + \alpha m + \beta - 2}e^{-x} \cos e^{x^m}
\]

so that \(S_{\alpha,\beta}'(x) = o(1)\) and \(S_{\alpha,\beta}'(x) \in L_1[0, \infty)\) since \(amP + \beta - 2 \geq \alpha N + \beta - 2 \geq 0\) by our choice of \(N\). Hence \(s_n \to 0 \mid B, \alpha, \beta \mid\). However

\[
S_{\alpha,\beta}''(x) = f(x) - \alpha (am)^2 x^{amP + 2\alpha m + \beta - 3}e^{-x}e^{-x^m} \sin e^{x^m}
\]

where \(f(x) \in L_1[0, \infty)\) and therefore \(S_{\alpha,\beta}''(x) \notin L_1[0, \infty)\) since \(am > 1\). Thus, since

\[
S_{\alpha,\beta-1}(x) = S_{\alpha,\beta}(x) + S_{\alpha,\beta}'(x)
\]

and

\[
S_{\alpha,\beta-1}'(x) = S_{\alpha,\beta}'(x) + S_{\alpha,\beta}''(x),
\]

we have that

\[
s_n \to 0(B, \alpha, \beta - 1) \quad \text{but} \quad s_n \not\to 0 \mid B, \alpha, \beta - 1 \mid.
\]

**Theorem 4.** If \(s_n \to s \mid B, \alpha, \mu \mid\) and \(a_n \to 0 \mid B, \alpha, \beta \mid\), then \(s_n \to s \mid B, \alpha, \beta \mid\).

**Proof.** By [5, Theorem 1], \(s_n \to s(B, \alpha, \beta)\). Thus it remains only to show that \(S_{\alpha,\beta}(x) \in BV_x[0, \infty)\). Let \(k\) be a positive integer. Then, in view of Theorem A(iii), \(A_{\alpha,\beta+(k-1)\alpha}(x) \in BV_x[0, \infty)\). Moreover, by Lemma 4(ii),

\[
S_{\alpha,\beta+(k-1)\alpha}(x) = A_{\alpha,\beta+(k-1)\alpha}(x) + S_{\alpha,\beta+k\alpha}(x) + \alpha e^{-x}s_{N-1} \frac{x^{\alpha N + \beta - 1}}{\Gamma(\alpha N + \beta)}.
\]

Therefore \(S_{\alpha,\beta+(k-1)\alpha}(x) \in BV_x[0, \infty)\) if \(S_{\alpha,\beta+k\alpha}(x) \in BV_x[0, \infty)\). Since, in view of Theorem A(iii), \(S_{\alpha,\beta+k\alpha}(x) \in BV_x[0, \infty)\) when \(\beta + k\alpha \geq \mu\), it readily follows that \(S_{\alpha,\beta}(x) \in BV_x[0, \infty)\).

If \(\{s_n\}\) is the sequence described in the paragraph preceding Theorem 3, then, using Lemma 4(i),

\[
S_{1,3}(x) = e^{-x} \int_0^x (\cos 1 - \cos e^t) \, dt
\]

and thus it is readily seen that \(s_n \to 0 \mid B, 1, 3 \mid\) and \(s_n \not\to 0 \mid B, 1, 2 \mid\). Hence there is also no immediate absolute summability analogue to [5, Theorem 3].

Our final results are extensions of a result due to Gaier (see [6]).

**Theorem 5.** If \(s_n \to s \mid B, \alpha, \mu \mid\) and \(S_{\alpha,\mu}(z)\) is of exponential type in \(H_\delta\) for some \(\delta > 0\), then \(s_n \to s \mid B, \alpha, \beta \mid\).
Proof. Let $k$ be a positive integer such that $\mu-k \leq \beta$. By [5, Theorem 6] we have that $s_n \to s(B, \alpha, \mu-k)$. Furthermore, since

$$S_{n,\mu-1}(z) = S_{n,\mu}(z) + S_{\alpha,\mu}^{(j)}(z),$$

it is readily seen that

$$S_{n,\mu-k}(z) = S_{\alpha,\mu}(z) + \sum_{j=1}^{k} \binom{k}{j} S_{\alpha,\mu}^{(j)}(z).$$

Since $S_{\alpha,\mu}(z)$ is of exponential type in $H_\delta$ and since $S_{\alpha,\mu}(x) \in BV_{\epsilon}(0, \infty)$ by hypothesis, we have, by Lemma 10, that $S_{\alpha,\mu}^{(j)}(x) \in BV_{\epsilon}(0, \infty)$ for $j = 1, \ldots, k$; also, since we choose $N$ so that $\alpha N + \mu - k \leq 1$, we have that $S_{\alpha,\mu}^{(j)}(x) \in BV_{\epsilon}(0, \delta)$ for $j = 1, \ldots, k$. Therefore, $S_{\alpha,\mu}^{(j)}(x) \in BV_{\epsilon}(0, \infty)$ for $j = 1, \ldots, k$ and, consequently, $S_{\alpha,\mu-k}(x) \in BV_{\epsilon}(0, \infty)$. Hence $s_n \to s |B, \alpha, \mu-k|$ and, by Theorem A(iii), $s_n \to s |B, \alpha, \beta|$.

Theorem 5*. If $s_n \to s |B, \alpha, \mu|$ and $A_{\alpha,\mu}(z)$ is of exponential type in $H_\delta$ for some $\delta > 0$, then $s_n \to s |B, \alpha, \beta|$.

Proof. By Lemma 1(ii), $a_n \to 0 |B, \alpha, \mu|$ and thus, by Theorem 5, $a_n \to 0 |B, \alpha, \beta|$. The result now follows by Theorem 4.

Theorem 6. If $s_n \to s |B, \alpha, \mu|$ and $|a_n| \leq K^n$ for all $n \geq 0$ where $K$ is a positive constant, then $s_n \to s |B, \alpha, \beta|$.

Proof. Since $|a_n| \leq K^n$ for all $n \geq 0$, we have that

$$|A_{\alpha,\mu}(z)| \leq Ae^{K|z|}$$

for some positive constant $A$. The desired result now follows by Theorem 5*.

References


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