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FIXED POINT THEOREMS OF DISCONTINUOUS INCREASING OPERATORS AND APPLICATIONS TO NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we obtain some new existence theorems of the maximal and minimal fixed points for discontinuous increasing operators in $C[\mathbb{I}, E]$, where $E$ is a Banach space. As applications, we consider the maximal and minimal solutions of nonlinear integro-differential equations with discontinuous terms in Banach spaces.

1. Introduction and preliminaries

For the sake of clarity, we first give some notations and concepts. Let $E$ be a real Banach space with norm $\| \cdot \|$, $\mathbb{I} = [a, b] \subset \mathbb{R}^1$ with $a < b$, and $C[\mathbb{I}, E]$ denote the set of all continuous functions defined on $\mathbb{I}$ with values in $E$. Clearly $C[\mathbb{I}, E]$ is a Banach space with the norm $\|x\|_C = \max_{t \in \mathbb{I}} \|x(t)\|$. For any $p \geq 1$, set

$$L_p[\mathbb{I}, E] = \left\{ x(t) : \mathbb{I} \to E \mid x(t) \text{ is strongly measurable and} \right. \int_\mathbb{I} \|x(t)\|^p \, dt < \infty \}\right\},$$

then $L_p[\mathbb{I}, E]$ is a Banach space with the norm $\|x\|_p = \left( \int_\mathbb{I} \|x(t)\|^p \, dt \right)^{1/p}$. Let a nonempty convex closed set $P$ be a cone in $E$. The cone $P$ defines an ordering in $E$ given by $x \leq y$ iff $y - x \in P$. The orderings in $C[\mathbb{I}, E]$ and $L_p[\mathbb{I}, E]$ are induced by the cone $P$ as follows, respectively, for $u, v \in C[\mathbb{I}, E]$, $u \leq v$ iff $u(t) \leq v(t)$ for any $t \in \mathbb{I}$; for $u, v \in L_p[\mathbb{I}, E]$, $u \leq v$ iff $u(t) \leq v(t)$ for almost all $t \in \mathbb{I}$. Obviously, $C[\mathbb{I}, E]$ is an ordered additive group which is additive by the common addition and the ordering induced by the cone of $P$ of $E$, i.e., $u_1, u_2, v_1, v_2 \in C[\mathbb{I}, E]$ and $u_1 \leq v_1$, $u_2 \leq v_2$ imply $u_1 + u_2 \leq v_1 + v_2$. For details on strongly measure functions and cone theory, see [9] and [4] respectively.

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It is common knowledge that fixed point theorems on increasing operators are used widely in nonlinear equations and other fields in mathematics (see [1]–[7]). But in most well-known documents, it is assumed generally that increasing operators possess stronger continuity and compactness (see [1]–[6]). In this paper, different from the increasing operators mapping ordering intervals of \( E \) into \( E \), \( A \) is an increasing operator from an ordering interval \( D \) of \( C[I,E] \) into \( C[I,E] \), and may be expressed as the form \( \sum_{i=1}^{m} K_i F_i \). We do not assume any continuity on \( A \). It is only required that \( (F_i D)(t) \) (almost all \( t \in I \)) and \( (K_i D_i)(t) \) \( (t \in I) \) possess very weak compactness, where \( (F_i D)(t) \) and \( (K_i D_i)(t) \) can be found in \( \S 2 \), \( i = 1, 2, \ldots, m \). In addition, if we use the results in [1]–[7] to study integral equations and differential equations in Banach spaces, we have to verify the compactness or weak compactness in such spaces as \( C[I,E] \) or \( L_p[I,E] \). But it is very difficult to examine the compactness type conditions in \( C[I,E] \) or \( L_p[I,E] \). So there is some difficulty in applying the results in [1]–[7] to nonlinear equations in Banach spaces. By using the conclusions of this paper, we may avoid the difficulty and only need to verify the compactness in \( E \) rather than \( C[I,E] \) or \( L_p[I,E] \), whereas the compactness in \( E \) is satisfied naturally in many cases (see \( \S 3 \)).

As applications, we show the existence of the maximal and minimal solutions of nonlinear integro-differential equations with discontinuous terms in Banach spaces.

\section*{\S 2. Fixed point theorems of increasing operators}

Let \( u_0, v_0 \in C[I,E] \), \( u_0 \leq v_0 \), \( D = [u_0, v_0] = \{ u \in C[I,E] \mid u_0 \leq u \leq v_0 \} \). For any \( i \in \{ i = 1, 2, \ldots, m \} \), \( 1 \leq p_1, p_2, \ldots, p_m < +\infty \), let \( F_i : D \rightarrow L_{p_i}[I,E] \) be an increasing operator, \( D_i = \{ w \in L_{p_i}[I,E] \mid F_i u_0 \leq w \leq F_i v_0 \} \), and \( K_i : D_i \rightarrow C[I,E] \) an increasing operator. Define operator \( A \) by \( A = \sum_{i=1}^{m} K_i F_i \), thus \( A \) is also an increasing operator from \( D \) into \( C[I,E] \).

In the following, for \( t \in I \), set

\[
(F_i D)(t) = \{ u(t) \in E \mid u \in F_i(D) \},
\]
\[
(K_i D_i)(t) = \{ u(t) \in E \mid u \in K_i(D_i) \},
\]

obviously,

\[
(F_i D)(t), (K_i D_i)(t) \subseteq E,
\]

here \( i = 1, 2, \ldots, m \).
LEMMA 1. Let $E$ be a Banach space, $P$ a cone in $E$, $x_n, y_n \in E$, and $x_n \leq y_n (n = 1, 2, \ldots)$. Then $x_n \overset{w}{\to} x^*$ and $y_n \overset{w}{\to} y^*$ imply $x^* \leq y^*$, where the notation $\overset{w}{\to}$ means that a sequence converges weakly to some element.

Proof. It is easy to follow from the assumptions that $y_n - x_n \in P (n = 1, 2, \ldots)$, $y_n - x_n \overset{w}{\to} y^* - x^*$. Since the convex closed set $P$ is weakly closed, $y^* - x^* \in P$, i.e., $x^* \leq y^*$. Thus Lemma 1 holds.

THEOREM 1. Let increasing operators $F_i : D \to L_{p_i}[I, E]$ $(i = 1, 2, \ldots, m$, which is the same sense in the following), increasing operators $K_i : D_i \to C[I, E]$ and $A = \sum_{i=1}^{m} K_i F_i$. Assume

(i) for almost all $t \in I$, any complete ordered subset of $(F_i D)(t)$ is relatively weakly compact in $E$; for any $t \in I$, any complete ordered subset of $(K_i D_i)(t)$ is also relatively weakly compact in $E$;

(ii) $F_i(D)$ are bounded sets in $L_{p_i}[I, E]$;

(iii) $u_0 \leq Au_0$, $Au_0 \leq v_0$.

Then $A$ has at least one fixed point in $D$.

Proof. It follows from the monotonicity of $A$ and condition (iii) that $A : D \to D$. Set $R = \{u \in A(D) \mid u \leq Au\}$. By $Au_0 \in R$, $R \neq \emptyset$. Taking any complete ordered set $N$ in $R$, we set $M = A(N)$, $M(t) = \{u(t) \in E \mid u \in M\}$. Clearly $M$ is also a complete ordered set in $R$ due to the definition of $R$ and the monotonicity of $A$, so is $M(t)$ in $E$ for any $t \in I$.

The following proof will be divided into cases: (a) there exists a $t^* \in I$ such that any element of $M(t^*)$ is not an upper bound of $M(t^*)$, and (b) for any $t \in I$, there exists an $x \in M(t)$ such that $x$ is an upper bound of $M(t)$.

In case of (a): Obviously $M(t^*) = (AN)(t^*) = \sum_{i=1}^{m} (K_i F_i(N))(t^*)$. Since $N \subset R \subset D$, and $N$ is a complete ordered set of $R$, $(K_i F_i(N))(t^*)$ are complete ordered sets of $(K_i D_i)(t^*) (i = 1, 2, \ldots, m)$. Now we show that $M(t^*)$ is relatively weakly compact in $E$. For any $\{z_n\} \subset M(t^*)$, it follows from $M(t^*) = \sum_{i=1}^{m} (K_i F_i(N))(t^*)$ that there exists a subsequence $\{w_n\} \subset N$ such that $z_n = \sum_{i=1}^{m} (K_i F_i w_n)(t^*)$. Let $y_{i,n} = (K_i F_i w_n)(t^*)$, clearly $y_{i,n} \subset (K_i F_i(N))(t^*) \subset (K_i D_i)(t^*)$ and $z_n = \sum_{i=1}^{m} y_{i,n}$, thus $\{y_{i,n}\}$ is complete ordered subset in $(K_i D_i)(t^*) (i = 1, 2, \ldots, m)$. By condition (i), $\{y_{1,n}\}$ has a weakly convergent subsequence $\{y_{1,n}^{(1)}\} \subset \{y_{1,n}\}$. Evidently $\{y_{1,n}^{(1)}\} \subset \{y_{1,n}\} (i = 1, 2, \ldots, m)$. Then we can choose a weakly convergent subsequence $\{y_{2,n}^{(2)}\}$ in $\{y_{2,n}^{(1)}\}$, and we have $\{y_{i,n}^{(2)}\} \subset \{y_{i,n}^{(1)}\} (i = 1, 2, \ldots, m)$. Using the same arguments and going on with the process, we can obtain a
weakly convergent subsequence \( \{y_{m,n}^{(m)}\} \) in \( \{y_{m,n}^{(m-1)}\} \), and \( \{y_{i,n}^{(m)}\} \subset \{y_{i,n}^{(m-1)}\} \) \( (i = 1, 2, \ldots, m) \). By above discussions we know that
\[
\{y_{i,n}^{(m)}\} \subset \{y_{i,n}^{(m-1)}\} \subset \cdots \subset \{y_{i,n}^{(1)}\} \subset \{y_{i,n}\}, \quad i = 1, 2, \ldots, m.
\]
and \( \{y_{i,n}^{(m)}\} \) is a weakly convergent sequence of \( \{y_{i,n}\} \). Obviously we may get \( z_n^{(m)} = \sum_{i=1}^{m} y_{i,n}^{(m)} \) corresponding to \( z_n = \sum_{i=1}^{m} y_{i,n} \), hence \( \{z_n^{(m)}\} \) is also a weakly convergent subsequence of \( \{z_n\} \). Observing that \( \{z_n\} \subset M(t^*) \) is arbitrary, we know that \( M(t^*) \) is relatively weakly compact.

Let \( \overline{M(t^*)}^w \) denote the closure of \( M(t^*) \) in \( E \) in the sense of weak topology. Then \( \overline{M(t^*)}^w \) is a compact set of \( M(t^*) \subset E \) in the sense of weak topology. For \( x \in M(t^*) \), set \( B(x) = \{ y \in \overline{M(t^*)}^w \mid x \leq y \} \). It is easy to know from Lemma 1 that \( \{ y \in E \mid x \leq y \} \) is weak closed in \( E \), thus \( B(x) = \overline{M(t^*)}^w \cap \{ y \in E \mid x \leq y \} \) is also weak closed in \( E \). Taking any finite members \( \{B(x_i) \mid x_i \in M(t^*), i = 1, 2, \ldots, k\} \), we set \( \overline{x} = \max\{x_i \mid i = 1, 2, \ldots, k\} \). Since \( M(t^*) \) is a complete ordered set, \( \overline{x} \in M(t^*) \) and \( x_i \leq \overline{x} (i = 1, 2, \ldots, k) \). Thus \( \overline{x} \in \bigcap_{i=1}^{k} B(x_i) \), that is, \( \bigcap_{i=1}^{k} B(x_i) \neq \emptyset \). Since \( \overline{M(t^*)}^w \) is a compact set in the sense of weak topology, it follows from the finite intersection property of compact set (see [10, Chapter 5]) that \( \bigcap_{x \in M(t^*)} B(x) \neq \emptyset \). Taking \( x^* \in \bigcap_{x \in M(t^*)} B(x) \), we know from the definition of \( B(x) \) and \( B(x) \subset \overline{M(t^*)}^w \) that \( x^* \in \overline{M(t^*)}^w \) and
\[
(2.1) \quad x \leq x^*, \quad \forall x \in M(t^*).
\]
Since any element of \( M(t^*) \) is not an upper bound of \( M(t^*) \),
\[
(2.2) \quad x \neq x^*, \quad \forall x \in M(t^*).
\]

By \( x^* \in \overline{M(t^*)}^w \) and on account of the famous Eberlein-Shmulian theorem, there exists a sequence \( \{x_n\} \) of \( M(t^*) \) such that
\[
(2.3) \quad x_n \overset{w}{\longrightarrow} x^*.
\]
It is clear to see from (2.1), (2.2) and (2.3) that for any \( x_{n_1} \in \{x_n\} \), there exists \( x_{n_2} \in \{x_n\} \) such that \( x_{n_1} \leq x_{n_2} \) and \( x_{n_1} \neq x_{n_2} \). Similarly, we can choose a subsequence \( \{x_{n_i}\} \subset \{x_n\} \) such that
\[
x_{n_1} \leq x_{n_2} \leq \cdots \leq x_{n_i} \leq \cdots, \quad x_{n_1} \neq x_{n_2} \neq \cdots \neq x_{n_i} \neq \cdots.
\]
Without loss of generality, we may assume that \( \{x_n\} \) satisfies

\[
(2.4) \quad x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots, \quad x_1 \neq x_2 \neq \cdots \neq x_n \neq \cdots
\]

Otherwise, we may replace \( \{x_n\} \) with \( \{x_{n_i}\} \). By (2.1) and (2.2),

\[
(2.5) \quad x_n \leq x^*, \quad x_n \neq x^*, \quad n = 1, 2, \ldots
\]

Take \( u_n \in M \) such that \( u_n(t^*) = x_n \). Obviously \( \{u_n\} \) is a complete ordered set of \( C[I, E] \), which, together with (2.4), implies

\[
(2.6) \quad u_1 \leq u_2 \leq \cdots \leq u_n \leq \cdots
\]

Letting \( v_{i,n} = F_i u_n \) for any \( n \), we know from the monotonicity of \( F_i \) that

\[
(2.7) \quad v_{i,1} \leq v_{i,2} \leq \cdots \leq v_{i,n} \leq \cdots, \quad i = 1, 2, \ldots, m.
\]

Thus for almost all \( t \in I \), we have

\[
(2.8) \quad v_{i,1}(t) \leq v_{i,2}(t) \leq \cdots \leq v_{i,n}(t) \leq \cdots
\]

By condition (i), there exist \( I_0 \subset I \) and \( \text{mes}(I \setminus I_0) = 0 \) such that for any \( t \in I_0 \), \( \{v_{n,i}(t)\} \) is relatively weakly compact and (2.8) holds. Thus there exists a subsequence \( \{v_{i,n_k}(t)\} \) of \( \{v_{i,n}(t)\} \) and \( v_{i,t} \in \{v_{i,n}(t)\} \) such that

\[
(2.9) \quad v_{i,n_k}(t) \overset{w}{\rightarrow} v_{i,t}, \quad t \in I_0.
\]

For any \( n_{k_0} \), by (2.8) we know that \( v_{i,n_{k_0}}(t) \leq v_{i,n_k}(t) \) when \( k_0 \leq k \). By Lemma 1 and (2.9), \( v_{i,n_{k_0}}(t) \leq v_{i,t} \). Hence we get

\[
(2.10) \quad v_{i,n}(t) \leq v_{i,t}, \quad n = 1, 2, \ldots, \quad t \in I_0
\]

since \( n_{k_0} \) is arbitrary. In view of standard arguments (such as the proof of Theorem 6.1 in [3]), by (2.8) and (2.9) we can prove

\[
(2.11) \quad v_{i,n}(t) \overset{w}{\rightarrow} v_{i,t}, \quad t \in I_0.
\]

Define \( v_i^*: I \rightarrow E \) as follows: when \( t \in I_0 \), \( v_i^*(t) = v_{i,t} \); when \( t \in I \setminus I_0 \), \( v_i^*(t) = 0 \). Then (2.10) and (2.11) imply that

\[
(2.12) \quad v_{i,n}(t) \leq v_i^*(t), \quad n = 1, 2, \ldots, \quad v_{i,n}(t) \overset{w}{\rightarrow} v_i^*(t), \quad \forall t \in I_0.
\]
Since \( v_{i,n} \) is strongly measurable because of \( v_{i,n} = F_i u_n \in L_{p_i}[I,E] \) \((i = 1,2,\ldots,m)\), by (2.12) and according to Pettis theorem and its proof (see Chapter V of [9]) \( v_i^*(t) \) is also strongly measurable. In view of the second formula of (2.12) and the weakly lower semi-continuity of norm, we have

\[
\|v_i^*(t)\| \leq \lim_{n \to \infty} \|v_{i,n}(t)\|, \quad \forall t \in I_0.
\]

By Fatou Lemma, we get

\[
\int_I \|v_i^*(t)\|^{p_i} \, dt \leq \int_I \lim_{n \to \infty} \|v_{i,n}(t)\|^{p_i} \, dt \leq \lim_{n \to \infty} \int_I \|v_{i,n}(t)\|^{p_i} \, dt,
\]

which, by \( v_{i,n} = F_i u_n \in F_i(D) \subset L_{p_i}[I,E] \) and condition (ii), implies \( v_i^* \in L_{p_i}[I,E] \). By (2.12) and according to the weak closeness of the cone \( P \), \( v_i^* \in D_i = \{w \in L_{p_i}[I,E] | F_i u_0 \leq w \leq F_i v_0\} \). Let \( u^* = \sum_{i=1}^m K_i v_i^* \). Clearly \( K_i v_i^* \in C[I,E] \), i.e., \( u^* \in C[I,E] \). Now we prove

\[
\begin{align*}
(2.13) \quad u_n &\leq u^*, \quad n = 1,2,\ldots; \\
(2.14) \quad u^* &\leq Au^*.
\end{align*}
\]

For any \( n_0 \), by (2.7) \( v_{i,n_0} \leq v_{i,n} \) when \( n_0 \leq n \). Hence

\[
F_i u_{n_0} = v_{i,n_0} \leq v_{i,n} \leq v_i^*,
\]

due to the first formula of (2.12). Since \( u_{n_0} \leq Au_{n_0} \) because of \( u_{n_0} \in M \subset R \), it follows from (2.15) and the monotonicity of \( K_i \), that

\[
u_{n_0} \leq Au_{n_0} = \sum_{i=1}^m K_i F_i u_{n_0} \leq \sum_{i=1}^m K_i v_{i,n} \leq \sum_{i=1}^m K_i v_i^* = u^*,
\]

thus (2.13) holds. By (2.13), \( v_{i,n} = F_i u_n \leq F_i u^* \), that is, \( v_{i,n}(t) \leq (F_i u^*)(t) \) for almost all \( t \in I \). Letting \( n \to \infty \) and observing the second formula of (2.12), by Lemma 1 we know \( v_i^*(t) \leq (F_i u^*)(t) \) for almost all \( t \in I \), i.e., \( v_i^* \leq F_i u^* \). So, by the definition of \( u^* \), \( u^* = \sum_{i=1}^m K_i v_i^* \leq \sum_{i=1}^m K_i F_i u^* = Au^* \), i.e., (2.14) holds.

For any \( u \in M \), if \( u_n \leq u \) holds for any \( n \), we have \( x_n = u_n(t^*) \leq u(t^*) \). Observing (2.3) and using Lemma 1, we get \( x^* \leq u(t^*) \), which contradicts (2.1) and (2.2). The contradiction and (2.13) mean that for \( \forall u \in M \), there exists some \( n_0 \) such that

\[
(2.16) \quad u \leq u_{n_0} \leq u^*.
\]
By (2.14), $Au^* \leq A(Au^*)$, thus $Au^* \in R$. (2.14) and (2.16) imply
\[(2.17) \quad u \leq u^* \leq Au^*, \quad \forall u \in M.\]

For any $v \in N$, it is clear that $v \leq Av$ and $Av \in M$ because of $N \subset R$ and $M = A(N)$. Thus, by (2.17) we get $v \leq Av \leq Au^*$ ($\forall v \in N$). Therefore $Au^*$ is an upper bound of $N$ in $R$, that is, $N$ has an upper bound in $R$.

In case of (b): Take $\{t_n\} \subset I$ such that $\{t_n\}$ is dense in $I$. In this case, there must exist an $x_1 \in M(t_1)$ such that $x_1$ is an upper bound of $M(t_1)$. Then we can select $u_1 \in M$ such that $u(t_1) = x_1$. If $u_1(t_2)$ is an upper bound of $M(t_2)$, let $u_2 = u_1$; if $u_1(t_2)$ is not an upper bound of $M(t_2)$, select $u_2 \in M$ such that $u_2(t_2)$ is an upper bound of $M(t_2)$. Since $M$ is a complete ordered set, it is obvious that $u_1 \leq u_2$ and $u_2(t_1) = u_1(t_1)$. Using the same arguments, we can select a sequence $\{u_n\}$ such that
\[u_1 \leq u_2 \leq \cdots \leq u_n \leq \cdots,\]

$u_n(t_n)$ is an upper bound of $M(t_n)$ and $u_n(t_i) = u_i(t_i)$ ($1 \leq i \leq n - 1$). Let $v_{i,n} = F_i u_n$ ($i = 1, 2, \ldots, m$). Evidently (2.7) holds and there exists $v_i^* \in L_{p_i}[I, E]$ such that (2.12) holds. Let $u^* = \sum_{i=1}^{m} K_i v_i^*$. Then (2.13) and (2.14) hold. In the following, we shall show $u \leq u^*$ for any $u \in M$. If otherwise, there exists some $u \in M$ such that $u \not\leq u^*$, i.e., there exists $\bar{t} \in I$ such that $u(\bar{t}) \not\leq u^*(\bar{t})$. Since $u, u^* \in C[I, E]$, there exists $\delta > 0$ such that when $t \in I$ and $|t - \bar{t}| < \delta$, $u(t) \not\leq u^*(t)$ holds. Selecting $t_n = \{t_n\}$ such that $|t_n - \bar{t}| < \delta$, we can get $u(t_n) \not\leq u^*(t_n)$. By (2.13), $u_n(t_n) \leq u^*$, that is, $u_n(t_n) \leq u^*(t_n)$. Hence $u(t_n) \not\leq u_n(t_n)$, which contradicts that $u_n(t_n)$ is an upper bound of $M(t_n)$. The contradiction means that for any $u \in M$, $u \leq u^*$. Using the same arguments as in the final proof of (a), we know that $N$ has an upper bound in $R$.

By the above discussions, we know that $N$ has one upper bound in $R$ under various conditions. It follows from Zorn’s lemma that $R$ has a maximal element. It is clear that any maximal element of $R$ is a fixed point of $A$. The proof is completed.

**Theorem 2.** If the conditions in Theorem 1 are satisfied, then $A$ has the minimal fixed point and the maximal fixed point in $D$.

**Proof.** Set $\text{Fix} A = \{u \in D \mid u = Au\}$. By Theorem 1, $\text{Fix} A \neq \emptyset$. Set
\[S = \{u \in A(D) \mid u \leq Au \text{ and } u \leq \pi, \forall \pi \in \text{Fix} A\}.\]
Obviously $S \neq \emptyset$ due to $Au_0 \in S$. Take any complete ordered set $N$ in $\mathbb{R}$ and let $M = A(N)$. It is clear that $M \subseteq S$. In the same way as in the proof of Theorem 1, we need to consider two cases separately. In the first case, by the same method of proving Theorem 1 we may find $\{u_n\}, \{v_{i,n}\}, v_i^*$ and $u^*$. Thus (2.6), (2.7), (2.8), (2.12), (2.13), (2.14) and (2.16) still hold. For any $\overline{u} \in \text{Fix} \, A$, it follows from $u_n \in M \subset S$ that $u_n \leq \overline{u}$. Letting $\overline{v_i} = F_i \overline{u}$ and observing $v_{i,n} = F_i u_n$, we know that $v_{i,n} \leq \overline{v_i}$ ($i = 1, 2, \ldots$), thus $v_{i,n}(t) \leq \overline{v_i}(t)$ for almost all $t \in I$. By (2.12) and in view of Lemma 1, $v^*_i(t) \leq \overline{v_i}(t)$ for almost all $t \in I$, i.e., $v^*_i \leq \overline{v_i}$. Since $\overline{u}$ is a fixed point of $A = \sum_{i=1}^m K_i F_i$,

$$u^* = \sum_{i=1}^m K_i v^*_i \leq \sum_{i=1}^m K_i \overline{v_i} = \sum_{i=1}^m K_i F_i \overline{u} = A \overline{u} = \overline{u},$$

thus $Au^* \leq A \overline{u} = \overline{u}$. By (2.14) and (2.16), we get

$$Au^* \leq A(Au^*), \quad u \leq u^* \leq Au^*, \quad \forall u \in M. \quad (2.18)$$

The above discussions show that $Au^* \in S$. For any $v \in N$, by $M = A(N)$, we know $Av \in M$. Observing $v \leq Av$ due to $N \subset S$, by (2.18) we have

$$v \leq Av \leq Au^*, \quad \forall v \in N,$$

which implies that $N$ has an upper bound in $S$. In the second case, we can use similar arguments to show that $N$ has an upper bound in $S$. Hence it follows from Zorn’s lemma that $S$ has a maximal element $w \in S$. Clearly

$$w \leq Aw, \quad w \leq \overline{u}, \quad \forall \overline{u} \in \text{Fix} \, A, \quad (2.19)$$

which means $Aw \leq A(Aw)$ and $Aw \leq A \overline{u} = \overline{u}$ ($\forall \overline{u} \in \text{Fix} \, A$). So $Aw \in S$. Since $w$ is a maximal element of $S$, by (2.19) we get $w = Aw$. Observing (2.19) again, we know that $w$ is a minimal fixed point of $A$ in $D$. Similarly, $A$ has a maximal fixed point in $D$. The proof is completed.

**Remark 1.** It is clear to see from the proof of Theorem 1 and Theorem 2 that if $I$ is a measurable closed subset of non-zero measure in $\mathbb{R}^n$, the two theorems still hold.

**Remark 2.** Comparing with some results in [1]–[7], we easily see that Theorem 1 and Theorem 2 are their generalizations and improvements.
§3. Applications

We first list for convenience the following assumptions:

(H₁) $E$ is sequentially weakly complete, $P$ a normal cone in $E$.

(H₂) $f_i(t, x) : J \times E \to E$ ($i = 1, 2, 3$, $J = [0, 1]$), we do not suppose that $f_i(t, x)$ are continuous), and the Nemytskii operators

\begin{equation}
(3.1) \quad f u = f_1(t, u(t)), \quad F u = f_i(t, u(t)), \quad i = 2, 3
\end{equation}

map continuous functions into strongly measurable functions.

(H₃) There exists $M > 0$ such that for $x, y \in E$, $y \leq x$,

\[ f_1(t, x) - f_1(t, y) \geq -M(x - y), \]

and $f_i(t, x)$ ($i = 2, 3$) are increasing on $x$ for $t \in J$.

Consider the nonlinear integro-differential equation

\[
\begin{aligned}
&u'(t) = f_1(t, u(t)) + \int_0^t k_1(t, s)f_2(s, u(s)) \, ds \\
&\quad + \int_J k_2(t, s)f_3(s, u(s)) \, ds,
\end{aligned}
\]

\[ u(0) = x_0, \]

where $t \in J$, $k_1(t, s) : \{(t, s) \in J \times J \mid s \leq t\} \to R^1$ and $k_2(t, s) : J \times J \to R^1$ are nonnegative and continuous. By the direct proof, it is easy to follow that the initial value problem (3.2) is equivalent to the equation

\begin{equation}
(3.3) \quad u(t) = e^{-Mt}x_0 + \int_0^t e^{-M(t-s)}\left[(f_1(s, u(s)) + M u(s))
\right.
\end{equation}

\[ \left. + \int_0^s k_1(s, \tau)f_2(\tau, u(\tau)) \, d\tau + \int_J k_2(s, \tau)f_3(\tau, u(\tau)) \, d\tau \right] ds, \]

if $f_1(t, x)$ is continuous, where $M$ is a constant given by (H₃) (also see Theorem 1.5.1 in [1]). Hence, when $f_1(t, x)$ is not continuous, we define the solutions of integral equation (3.3) as the solutions of the equation (3.2).

**Theorem 3.** Suppose that the assumptions (H₁)–(H₃) are fulfilled and there exist $u_0, v_0 \in C^1[J, E] = \{u \in C[J, E] \mid u(t) \text{ is differentiable}\}$, $u_0 \leq v_0$, $1 \leq p_1, p_2, p_3 < \infty$, such that

\begin{equation}
(3.4) \quad f_1 u_0, f_1 v_0 \in L_{p_1}[J, E], \quad F_i u_0, F_i v_0 \in L_{p_i}[J, E], \quad i = 2, 3,
\end{equation}
By the nonnegativity of increasing from (3.9)
\[ L \] has the maximal solution and minimal solution in
\[ D = [u_0, v_0] = \{ u \in C[J, E] | u_0 \leq u \leq v_0 \}. \]

**Proof.** For any \( u \in C[J, E] \), by (3.3) we can define the mapping

\[
Au = e^{-Mt}x_0 + \int_0^t e^{-M(t-s)} \left[ (f_1(s, u(s)) + Mu(s))
+ \int_0^s k_1(s, \tau)f_2(\tau, u(\tau)) d\tau + \int_J k_2(s, \tau)f_3(\tau, u(\tau)) d\tau \right] ds.
\]

(3.7)

\[
K_1h_1 = e^{-Mt}x_0 + \int_0^t e^{-M(t-s)}h_1(s) ds, \quad \forall h_1 \in L_{p_1}[J, E],
\]

(3.8)

\[
K_2h_2 = \int_0^t ds \int_0^s e^{-M(t-s)}k_1(s, \tau)h_2(\tau) d\tau, \quad \forall h_2 \in L_{p_2}[J, E],
\]

(3.9)

\[
K_3h_3 = \int_0^t ds \int_J e^{-M(t-s)}k_2(s, \tau)h_3(\tau) d\tau, \quad \forall h_3 \in L_{p_3}[J, E].
\]

(3.10)

By the nonnegativity of \( k_1(t, s) \) and \( k_2(t, s) \), it is easy to show that \( K_i \) are increasing from \( L_{p_i}[J, E] \) into \( C[J, E] \) \((i = 1, 2, 3)\). Set

\[
F_1u = f_1u + Mu, \quad u \in C[J, E].
\]

(3.11)

By \( (H_2) \), \( F_1 \) maps elements of \( C[J, E] \) into strongly measurable functions. For any \( u \in [u_0, v_0] \), by \( (H_3) \) we get \( F_1u_0 \leq F_1u \leq F_1v_0 \). Hence for almost all \( t \in J, 0 \leq (F_1u)(t) - (F_1u_0)(t) \leq (F_1v_0)(t) - (F_1u_0)(t) \). On account of the normality of \( P \), there exists a constant \( L > 0 \) such that

\[
\|(F_1u)(t) - (F_1u_0)(t)\| \leq L\|(F_1v_0)(t) - (F_1u_0)(t)\|,
\]
which, by (3.4), (3.11), implies $F_1u \in L_{p_1}[J,E]$. So $F_1$ is an increasing operator from $[u_0,v_0]$ into $L_{p_1}[J,E]$. Similarly, by (3.1), (3.4) and $(H_2)$, we can prove that $F_i : [u_0,v_0] \to L_{p_i}[J,E]$ $(i = 2, 3)$ are increasing. So by (3.1), (3.11) and (3.7)–(3.10), we can get

\[ A = \sum_{i=1}^{3} K_i F_i. \tag{3.12} \]

In view of the above discussions, we may have that $A$ is an increasing operator from $C[J,E]$ into $C[J,E]$.

Let $D_1 = \{ w \in L_{p_1}[J,E] \mid F_1u_0 \leq w \leq F_1v_0 \}$, it is clear to see from the monotonicity of $F_1$ that

\[ F_1(D) \subset D_1, \tag{3.13} \]

and $F_1u_0 \leq w \leq F_1v_0$ for any $w \in D_1$. By using the normality of $P$, we can get

\[ \|w(t)\| \leq \|(F_1u_0)(t)\| + L\|(F_1v_0)(t) - (F_1u_0)(t)\| \tag{3.14} \]

for almost all $t \in J$, here $L$ is a normal constant. For $t \in J$, set $D_1(t) = \{ w(t) \mid w \in D_1 \}$. By (3.4), (3.11) and (3.14), there exist $J_0 \subset J$ and $\text{mes} J_0 = \text{mes} J$ such that for $t \in J_0$, $D_1(t)$ is a bounded set in $E$. Now we show that any complete ordered set of $D_1(t)$ $(t \in J_0)$ is relatively weakly compact. Let $N \subset D_1(t)$ $(t \in J_0)$ be a complete ordered set and $\{x_n\}$ a sequence in $N$. We consider two cases:

(a) There exists an infinite set $\{x^{(k)}\} \subset \{x_n\}$ such that

\[ x^{(1)} = \inf\{x_n\}, \quad x^{(k)} = \inf\{\{x_n\} \setminus \{x^{(1)}, x^{(2)}, \ldots, x^{(k-1)}\}\}, \quad k = 1, 2, \ldots. \]

Thus

\[ (F_1u_0)(t) \leq x^{(1)} \leq x^{(2)} \leq \cdots \leq x^{(k)} \leq \cdots \leq (F_1v_0)(t), \quad t \in J_0. \tag{3.15} \]

Since the cone $P$ is normal, $P$ is reproduced by Proposition 19.4 in [2], that is, for any $\phi \in E^*$, there exist $\phi_i \in P^*$ $(i = 1, 2)$ such that $\phi = \phi_1 - \phi_2$. By (3.15), we have

\[ \phi_i((F_1u_0)(t)) \leq \phi_i(x^{(1)}) \leq \phi_i(x^{(2)}) \leq \cdots \leq \phi_i(x^{(k)}) \leq \cdots \leq \phi_i((F_1v_0)(t)), \quad i = 1, 2, \quad t \in J_0, \]
which, together with the boundedness of \( \{x^{(k)}\} \subset D_1(t) \) \((t \in J_0)\), shows that 
\( \{\phi_i(x^{(k)})\} \) \((i = 1, 2)\) are Cauchy sequence in \(R^1\). Hence \(\{x^{(k)}\}\) is weakly Cauchy sequence in \(E\) since \(\phi \in E^*\) is arbitrary. Since \(E\) is sequentially weakly complete, \(\{x^{(k)}\}\) converges weakly to some element in \(E\).

(b) There exists no \(x \in \{x_n\}\) such that \(x = \inf \{x_n\}\), or there exists a finite set \(\{\overline{x}^{(k)}\} \subset \{x_n\}\) such that

\[
\overline{x}^{(1)} = \inf \{x_n\}, \quad \overline{x}^{(k)} = \inf \{\{x_n\} \setminus \{\overline{x}^{(1)}, \overline{x}^{(2)}, \ldots, \overline{x}^{(k-1)}\}\}, \quad k = 2, 3, \ldots, k_0,
\]

and \(x \neq \inf M_1\) for any \(x \in M_1\), here \(M_1 = \{x_n\} \setminus \{\overline{x}^{(1)}, \overline{x}^{(2)}, \ldots, \overline{x}^{(k_0)}\}\). So we can obtain an infinite set \(\{x^{(k)}\} \subset M_1\) such that

\[
(3.16) \quad (F_1u_0)(t) \leq \cdots \leq x^{(k)} \leq \cdots \leq x^{(2)} \leq x^{(1)} \leq (F_1v_0)(t), \quad t \in J_0.
\]

Using the same method as in the proof of (a), we know that \(\{x^{(k)}\}\) given by (3.16) converges weakly to some element in \(E\).

By above discussions, any sequence \(\{x_n\}\) of the complete ordered set \(N \subset D_1(t) \) \((t \in J_0)\) has a convergent subsequence of \(\{x_n\}\), that is, any complete ordered set of \(D_1(t) \) \((t \in J_0)\) is relatively weakly compact. Observing (3.13) and the boundedness of \(D_1(t) \) \((t \in J_0)\), we know that for almost all \(t \in J\), any complete ordered set \((F_1D)(t) = \{w(t) \mid w \in F_1(D)\}\) \(\subset D_1(t)\) is relatively weakly compact, and \(F_1(D)\) is a bounded set in \(L_{p_1}\) \([J, E]\). Using the similar arguments, we can show that for almost all \(t \in J\), any complete ordered set of \((F_iD)(t) = \{w(t) \mid w \in F_i(D)\}\) \((i = 2, 3)\) is relatively weakly compact in \(E\) and \(F_i(D)\) are bounded sets in \(L_{p_i}\) \([J, E]\) \((i = 2, 3)\); for any \(t \in J\), any complete ordered set of \((K_iD_i)(t) = \{u(t) \mid u \in K_i(D_i)\}\) \((i = 1, 2, 3)\) is also relatively weakly compact in \(E\). Thus condition (i) and (ii) in Theorem 1 are satisfied.

We now show that condition (iii) in Theorem 1 is fulfilled. By (3.7) and (3.5), we have

\[
(Au_0)(t) - u_0(t) = \sum_{i=1}^{3} K_iF_iu_0(t) - u_0(t)
\]

\[
= e^{-Mt}x_0 + \int_0^t e^{-M(t-s)} \left[ (f_1(s, u_0(s)) + M u_0(s))
\right.
\]

\[
+ \int_0^s k_1(s, \tau)f_2(\tau, u_0(\tau))d\tau + \int_J k_2(s, \tau)f_3(\tau, u_0(\tau))d\tau \right] ds - u_0(t)
\]

\[
\geq e^{-Mt}x_0 + e^{-Mt} \int_0^t e^{Ms}[u_0'(s) + M u_0(s)]ds - u_0(t)
\]
\[ e^{-Mt}x_0 + e^{-Mt}(e^{Mt}u_0(t) - u_0(0)) - u_0(t) = e^{-Mt}(x_0 - u_0(0)) \geq \theta, \]

which means \( u_0 \leq Au_0 \). Similarly we can prove that \( Av_0 \leq v_0 \).

Since all conditions in Theorem 1 are satisfied, by Theorem 1 and Theorem 2, \( A \) has the maximal fixed point and the minimal fixed point in \( D \). Noting that fixed points of \( A \) are equivalent to solutions of Eq. (3.3), and Eq. (3.3) is equivalent to Eq. (3.2), the conclusions of Theorem 3 hold. The proof is completed.

**Remark 3.** In Theorem 1 and Theorem 2, the increasing operator \( A \) is divided into \( \sum_{i=1}^{m} K_i F_i \) such that \( (F_i D)(t) \) (almost all \( t \in I \) and \( (K_i D_i)(t) \) \( (t \in I) \) need only weak compact conditions in \( E \) \((i = 1, 2, \ldots, m) \). It is clear to see from Theorem 3 that these conditions are examined easily. Moreover, some concrete problems possess the form \( \sum_{i=1}^{m} K_i F_i \) originally. Hence this is very convenient in applications.

**Remark 4.** In order to study nonlinear equations in Banach spaces, the compactness type conditions and the dissipative type conditions are widely used (see [1]–[5]). But we do not use any condition of the aspects in Theorem 3 of this paper.

**Remark 5.** Since many widely used spaces such as Hilbert spaces, reflexive spaces and \( L_1 \) spaces are all sequentially weakly complete, Theorem 3 still holds in these spaces.

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