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CHEBYSHEV SETS IN $C[0, 1]$ WHICH ARE NOT SUNS

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Consider approximation of elements of $C[0, 1]$ with respect to the sup-norm by a non-empty subset V of $C[0, 1]$. Of interest in recent years are subsets V called suns. As $C[0, 1]$ is an MS-space [1, 5], the suns V of $C[0, 1]$ are precisely those subsets V for which each local best approximation is a global best approximation.

DEFINITION. V is called a *Chebyshev set* if each $f \in C[0, 1]$ has exactly one best approximation.

The question asked by Brosowski and Deutsch [1; 5, top 976] is whether every Chebyshev set must be a sun. In the case V is a subset of a finite dimensional linear space, V being a Chebyshev set does imply that V is a sun since V is boundedly compact, see [6]. However, there do exist Chebyshev sets which are not suns.

THEOREM 1. *Let F be an approximating function with non-negative parameter such that:*

(i) $0<a<b$ *implies* $0<F(a, .)<F(b, .)$
(ii) $a \to \infty$ *implies* $F(a, 0) \to \infty$
(iii) $\{a_k\} \to a > 0$ *implies* $\|F(a_k, .)-F(a, .)\| \to 0$
(iv) $\{a_k\} \to 0$ *implies* $F(a_k, x) \to 0$, $x>0$
(v) $F(0, .)=0$
(vi) $F(a, .) \subset C[0, 1]$ *for* $a>0$
(vii) $F(a, 0)>1$ *for* $a>0$.

Then $V=\{F(a, .): a \geq 0\}$ *is a Chebyshev set which is not a sun.*

Proof. Let $f \in C[0, 1]$. Let $\|f-F(a_k, .)\|$ be a decreasing sequence with limit $\rho(f)=\inf\{\|f-F(a, .)\|: a \geq 0\}$. By (ii) $\{a_k\}$ is a bounded sequence and has an accumulation point a_0, assume $\{a_k\} \to a_0$. By (iii, iv, v), $F(a_k, .) \to F(a_0, .)$ on $(0, 1]$, hence

$$|f(x)-F(a_0, x)| \leq \limsup_{k \to \infty} |f(x)-F(a_k, x)| \leq \rho(f) \qquad x \in (0, 1]$$

and by continuity, $\|f-F(a_0, .)\| \leq \rho(f)$. This existence argument is a special case of that of [2].

By (i), we have for all $f \in C[0, 1]$ and $0 \leq a<b<c$

$$f-F(c,.) < f-F(b,.) < f-F(a,.),$$

hence

$$\|f-F(b,.)\| < \max\{\|f-F(a,.)\|, \|f-F(c,.)\|\}$$

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and f cannot have two best approximations from V. By (vii), 0 is an isolated element of V.

Two examples of F satisfying (i) to (vii) are

$$F(a, x) = (1+a)\exp(-x/a) \qquad a > 0$$

$$F(a, x) = (1+a)/(1+x/a) \qquad a > 0,$$

the latter of which comes from [3, 383].

THEOREM 2. *Let V be as in the above theorem. Let W be a subset of $C[0, 1]$ such that*

(i) *all elements of W are <0*
(ii) *best approximations from W are unique*
(iii) *a best approximation by $W \cup \{0\}$ exists to all $f \in C[0, 1]$*
(iv) *for any $w \in W$, there exists $w_0 \in W$ such that $w<w_0<0$.*

Then $V \cup W$ is a Chebyshev set which is not a sun.

Proof. Existence of best approximations is obvious. The uniqueness results for V and W imply that the only possible case of non-uniqueness is to have one best approximation in each of V and W. (iv) and the uniqueness argument of Theorem 1 then give a contradiction. 0 is locally best to $F(a, .)$ for any $a>0$.

THEOREM 3. *Let F be as in Theorem 1 and T, U be sets satisfying the conditions on W in Theorem 2. For $\alpha>0$, let $V(\alpha)=T \cup \{F(a, .):0\leq a\leq \alpha\} \cup \{F(\alpha, .)-u:u \in U\}$. Then $V(\alpha)$ is a Chebyshev set that is not a sun.*

This is proven by arguments similar to those of Theorem 2.

T, U, W can be arbitrarily complicated, but the part of the family generated by F remains one-dimensional. Theorem 3 shows that this part of the family can be arbitrarily small. The author has been unable to construct a Chebyshev set which is not a sun and is not "one-dimensional" anywhere.

It should be noted that if we are merely interested in V for which best approximations (if they exist) are unique, such V can easily fail to be suns. Examples are given in [4, bottom 381, top 385].

In Chebyshev approximation of complex continuous functions, there also exist Chebyshev sets which are not suns. It can be shown that the families of Theorem 1 are complex Chebyshev sets on [0, 1] and the examples given still apply.

The author has since constructed a "two dimensional" Chebyshev set which is not a sun, and D. Braess has constructed an "n dimensional" Chebyshev set which is not a sun.

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