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PRINCIPAL IRREDUCIBLE LIE-ALGEBRA MODULES

BY

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Let V be a finite dimensional vector space over k, a field of characteristic $0, L$ be an algebraic Lie-subalgebra of $End_k(V)$, with the latter a Lie algebra in the canonical way, and let V be an L-module in the canonical way. For $X \in V$, let $LX = \{AX \mid A \in L\}$. Call V a *principal* L-module if $\exists X \in V$ such that $LX = V$; X will be called a *principal generator* of the L-module V.

EXAMPLES. 1. Let $L = \mathfrak{sl}(V)$. V is a principal $\mathfrak{sl}(V)$-module with $\{X \in V \mid X \neq 0\}$ as the set of principal generators when the dimension of V is greater than 1.

2. (a) Let $V = k^n$, $L = \mathfrak{o}(n)$ be the orthogonal Lie-algebra for a non-degenerate quadratic form P. If $n \geq 3$, V is an irreducible L-module, but V is not a principal L-module. For, let Q be the non-degenerate symmetric bilinear form associated to P; then $\forall A \in L$, $\forall X, Y \in V$, $Q(AX, Y) + Q(X, AY) = 0$. If V is principal with principal generator X, then for each $Y \neq 0$ in V, $\exists A \in L$ such that $Q(Y, AX) \neq 0$. Take $Y = X$ with $Q(X, AX) \neq 0$; but $Q(X, AX) = -Q(AX, X) = -Q(X, AX)$ by symmetry and this is a contradiction.

2. (b) Let $V = k^n$, $L = kI_V \otimes \mathfrak{o}(n)$ be the Lie-algebra where $P(X) = \sum x_i^2$. Clearly, V is an irreducible L-module with principal generator $X = (1, 0, 0, \ldots, 0)$.

Assume throughout that V is an irreducible L-module. This assumption entails that $L = Z(L) \oplus L'$, a direct sum decomposition into ideals where L' is the commutator subalgebra which is a semi-simple algebraic Lie-algebra and where $Z(L)$ is the center of L which is either O or is of dimension 1 over k and consists of semi-simple endomorphisms of V, [1].

Next we define the notion of semi-invariant for L. Let V^* be the k-dual of V and let $S_k(V^*)$ be the symmetric algebra on V^* over k. L acts canonically as a Lie-algebra of k-derivations on $S_k(V^*)$ with action D completely determined by its effect on $V^* = S_k(V^*)^1$; namely, $D(A)(Y) = -A^*(Y) = -Y \circ A$, $\forall A \in L$, $\forall Y \in V^*$. The *k-derivations of* $S_k(V^*)$ form a Lie-algebra; D is a homomorphism of Lie-algebras over k from L into *k-derivations of* $S_k(V^*)$.When L is an algebraic Lie-algebra, the Lie-algebra of $G \subseteqq GL(V)$, the homomorphism D is just the derivative of the homomorphism of algebraic groups

λ: *$G \to k$-automorphisms of $S_k(V^*)$ as graded k-algebra* where λg on

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$S_k(V^*)^1$ is given by $\lambda g(Y)=Y\circ g^{-1}$ for all $Y\in V^*$. D on L is just the morphism of tangent spaces at the identity elements induced by λ on G; see [2] and [3]. A *semi-invariant P for L* is a $P\in S_k(V^*)$, $P\notin S_k(V^*)^0$ such that $\forall A\in L$, $\exists c_A\in k$ with $D(A)(P)=c_AP$. Such a P is found in example 2 above.

Lemma 1. *If $kI_V\subseteqq L$ and P is a semi-invariant for L, then P is a form, i.e. $P\in S_k(V^*)^r$ for some $r\geq 1$.*

Proof. $D(cI_V)(Y)=-cY$, for all $Y\in V^*$. Hence, $\forall m\geq 0$, $\forall Q\in S_k(V^*)^m$, $D(cI_V)(Q)=-m\cdot cQ$. Thus, if P is a semi-invariant for L, with $P=\sum_{i=0}^r P_i$, where $P_i\in S_k(V^*)^i$ and $P_r\neq 0$,

$$D(cI_V)(P)=\sum_{i=0}^{r} D(cI_V)P_i=\sum_{i=0}^{r}-icP_i=c\sum_{i=0}^{r}-iP_i.$$

Since char $k=0$, $P_i=0$ if $i<r$.

The following gives a criterion for the existence of semi-invariants.

Theorem 1. *Let V be a principal irreducible L-module. There exists a semi-invariant P for L if and only if V is not a principal L′-module.*

Examples of the theorem are

i. V is any finite dimensional vector space over k. Take $L=End_k(V^*)$. When dim $V=1$, any basis element of $S_k(V^*)^j$ is a semi-invariant for L and $L'=0$; both clauses are true. When dim $V\geq 2$, each $S_k(V^*)^j$ is an irreducible $End_k(V)$-module and has dimension >1 when $j>0$. Thus $\nexists$ a semi-invariant P for $L\cdot L'=\mathfrak{sl}(V)$ and example 1 states that V is a principal L'-module. Thus both clauses are false.

ii. Example 2 illustrates the theorem with both clauses true.

iii. An example of Mikio Sato. Let $V=k^{4\times 3}$, the vector space of 4 by 3 matrices over k. Take $L=kI_V\otimes L'$ where L' is the semi-simple algebraic Lie algebra isomorphic to $\mathfrak{sp}(4)\times\mathfrak{o}(3)$ where $\mathfrak{sp}(4)$ is the symplectic Lie-algebra and $\mathfrak{o}(3)$ is the semi-simple lie algebra of example 2 above. Take a monomorphism of L into $End_k(V)$ by the mapping:

$$\mathfrak{sp}(4)\times\mathfrak{o}(3)\xrightarrow{T} End_k(V)$$

$$(A,B)\longrightarrow T(A,B):k^{4\times 3}\to k^{4\times 3}$$
$$X\to AX-XB,$$

with the skew-symmetric bilinear form on k^4 defining $\mathfrak{sp}(4)$ being $\langle Z,U\rangle=z_1u_3+z_2u_4-z_3u_1-z_4u_2$, and the quadratic form defining $\mathfrak{o}(3)$ being $R(Y)=y_1^2+y_2^2+y_3^2$. L is the algebraic Lie-algebra of the affine algebraic group $G=kI_V\cdot Sp(4)\times O(3)$. The point $X=$(column $(1,0,0,0)$, column $(0,1,0,0)$, column $(0,0,1,0)$) is a principal generator of V as an L-module. It can be shown that V is not a principal L'-module by an argument extending the ideas

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in 2(a). Let $X=(X_1, X_2, X_3)$ where $X_i \in k^{4\times 1}$. Consider $P_1=\langle X_2, X_3\rangle$, $P_2=\langle X_3, X_1\rangle$ and $P_3=\langle X_1, X_2\rangle$ in $S_k(V^*)^2$, three quadratic forms on V. Let their polarizations or associated bilinear forms on V be Q_1, Q_2, and Q_3. For Z with Z_1, Z_2, and Z_3 linearly independent over k in k^4, $Q_1(\ ,Z)$, $Q_2(\ ,Z)$ and $Q_3(\ ,Z)$ will be linearly independent over k in $S_k(V^*)^1$; obtain a surjective linear mapping

$$V \xrightarrow{q(Z)} k^3$$

$$X \mapsto (Q_1(X, Z), Q_2(X, Z), Q_3(X, Z)).$$

Assume that V is a principal L'-module with principal generator X; X_1, X_2, X_3 must be linearly independent and we have $f(X)=q(X)\circ T(\ ,\ )(X)|_{L'}$ a surjective mapping from L' to k^3. However, a straightforward computation gives the following result for all

$$B=\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \in \mathscr{o}(3).$$

$$f(X)(A, B)=(-bP_3(X)-aP_2(X),\ aP_1(X)-cP_3(X),\ cP_2(X)+bP_1(X)).$$

The image of $f(X)$ is not onto k^3, since $\forall B \in \mathscr{o}(3)$, the matrix product $f(X)(A, B)\cdot(P_1(X),\ P_2(X),\ P_3(X))^{\text{transpose}}=0$ in k contradicting that $f(X)$ is surjective. It is easily checked that the quadric form $P(X)=P_1(X)^2+P_2(X)^2+P_3(X)^2$ is semi-invariant for L. Both clauses in Theorem 1 are true.

The proof of Theorem 1 is a verification that Corollary (3, 4) of [4] applies to give the desired statement. In that context, V is a finite dimensional vector space over K, an algebraically closed field of characteristic 0. Let $G(\ )$ be the functor from fields to groups. G is a connected algebraic subgroup of $GL(V)$, and the basic assumption in [4] is that G is of the form $KId_V\cdot G'(K)$ a semi-direct product of affine algebraic groups, where G' is the commutator subgroup, that G acts irreducibly on V and that there exists a Zariski open dense orbit $o(G)$ for the action of G in the affine variety Specmax $(S_k(V^*))$ canonically associated to V. The assertions of the stated corollary are that the following three statements are equivalent:

(a) G has a semi-invariant in $S_k(V^*)$.

(b) G' does not have a Zariski open dense orbit in V.

(c) G^0_X, the connected component of the isotropy subgroup of X in $o(G)$, is a subgroup of G'.

Take a fixed algebraic closure K of k and use the terminology of [2], Ch. AG, §11, to observe that V is a k-structure on $V(K)=V\otimes_k K$. Identify $V(K)$ with Specmax $(S_k(V^*))$ and V with those points of the form $(S_k(V^*)\to k)\otimes I_K$.

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Let G be the smallest algebraic subgroup of $GL(V)$ whose Lie-algebra contains L; since L is algebraic, the Lie-algebra of G equals L. Let k_1 be a finite extension of k in K containing the eigenvalues of $Z(L)$. Then $G(k_1)$, and a fortiori, $G(K)$ is of the form required in the Corollary (3, 4). G is a k-group and has a k-morphic action on V. The G orbit of X in V is the image of G under the orbit k-morphism $or_X(G): G \to V$ given as the composite $G \cong G \times \{X\} \to V$. If $k[G]$ is the affine k-algebra of G, then the comorphism of $or_X(G)$, $or_X(G)^0: S_k(V^*) \to k[G]$ is given by $k[V] \to k[G] \underset{k}{\otimes} k[V] \text{—} \cdots \xrightarrow{Id \times eval(X)} k[G]$. X has open dense orbit under G if and only if $or_X(G)^0$ is a monomorphism of k-algebras; this latter maintains if and only if X has open dense orbit in $V(K)$ under $G(K)$. The following key lemma relates the action of G with that of L.

Lemma 2. *The G orbit of X in V is open dense in V if and only if X is a principal generator for the L-module V.*

Proof. $or_X(G)^0: S_k(V^*) \to k[G]$, when localized at $\mathfrak{m}_I$, the maximal ideal of the identity element of G, induces a morphism of the regular local rings $or_X(G)^0: S_k(V^*)_{\mathfrak{m}_X} \to k[G]_{\mathfrak{m}_I}$ and consequently a mapping of finite dimensional k-vector spaces $\alpha_X: \mathfrak{m}_X/\mathfrak{m}_X^2 \to \mathfrak{m}_1/\mathfrak{m}_1^2$. α_X is injective if and only if $or_X(G)^0$ is injective. $\mathfrak{m}_X/\mathfrak{m}_X^2$ is the cotangent space to V at X and $\mathfrak{m}_1/\mathfrak{m}_1^2$ is the cotangent space to G at I_V, canonically the k-duals of the respective spaces $T_X(V) \cong (\mathfrak{m}_X/\mathfrak{m}_X^2)^*$ and $L \cong (\mathfrak{m}_1/\mathfrak{m}_1^2)^* \cdot \alpha_X{}^*: L \to T_X(V)$ is surjective if and only if α_X is injective. $\mathfrak{m}_X/\mathfrak{m}_X^2$ is canonically isomorphic to V^* via $\tau_X: \mathfrak{m}_X/\mathfrak{m}_X^2 \to V^*$, $\tau_X: Y - Y(X)$ mode $\mathfrak{m}_X^2 \mapsto Y$ and hence, $T_X(V)$ is canonically isomorphic to V via τ_X^{*-1}. τ_X^{*-1}. α_X^* is precisely $L \to V$, $A \mapsto AX$. This proves the lemma.

The theorem follows by the observations that $P \in S_k(V(K)^*)$ is a semi-invariant for $L \underset{k}{\otimes} K$ if and only if P is a semi-invariant for $G(K)$ in $S_K(V(K)^*)$ under the action λ defined above. Since G acts k-morphically on V, i.e. all varieties have k-structure, P will belong to $S_k(V^*)$, and conversely. This justifies the following result, adding the condition c' which makes the computing of whether V is a principal L'-module convenient.

Theorem 1′. *Let V be a principal irreducible L-module. The following four conditions are equivalent*:

(a) *There exists a semi-invariant P for L in $S_k(V^*)$.*

(b) *V is not a principal L'-module.*

(c) *For a principal generator X in V, L_X, the isotropy subalgebra of X in L is a subalgebra of L'.*

(c′) *For a principal generator X in V, $L'X \subsetneq V$.*

From the discussion above, it is clear that the set of principal generators, $\mathcal{P}(L, V)$, of the L-module V is a Zariski open dense set in V. A reasonable

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inquiry is on the nature of this set. When $\nexists P$, no general result is known to the author. However, when there is a semi-invariant P, under certain conditions the set $\mathcal{P}(L, V)$ equals $V-Z(P)$, the complement of the hypersurface of zeros of P in V. The result is due to Mikio Sato [5], [6]. Define the gradient mapping of P; $\operatorname{Grad} P: V \to V^*$, with $\operatorname{Grad} P(X)(Z)=(D_Z P)(X)$ where D_Z is the k-derivation of degree -1 on $S_k(V^*)$ requiring $D_Z(Y)=Y(Z)$ for all $Y \in V^*=S_k(V^*)^1$ and all $Z \in V$. We will need the following proposition which has a straightforward proof.

Proposition. *Let V^* be an L-module via the contragredient action,*

$$L \times V^* \to V^*$$
$$(A, Y) \mapsto \hat{A} Y=-A^*(Y)=-Y \cdot A.$$

Then V^* is a principal irreducible L-module if and only if V is a principal irreducible L-module.

Proof. V^* is irreducible if and only if V is irreducible. Let k_1 be an extension of k in K over which $Z(L)$ is diagonalizeable and such that $L'_{k_1}=L' \otimes_k k_1$ splits over k_1. There exists a unique automorphism l of L_{k_1} over k_1 mapping canonical generators of L_{k_1} to canonical generators for the inverse root system by Theorem 3, p. 127 in [1]. Let T be the k_1 linear mapping of $V_{k_1}=V \otimes_k k_1$ to $V^*_{k_1}$ sending a basis of weight vectors in V_{k_1} each to its correspondent in a dual basis for $V^*_{k_1}$; obtain a commutative diagram of k_1 linear automorphisms with $l^2=I_{L_{k_1}}$ and $T^*=T$

$$\begin{array}{ccc} L_{k_1}\times V_{k_1} & \xrightarrow[\text{action}]{} & V_{k_1} \\ {\scriptstyle l\times T}\downarrow & & \downarrow{\scriptstyle T} \\ L_{k_1}\times V^*_{k_1} & \xrightarrow[\text{action}]{\text{contragred}} & V^*_{k_1} \end{array}$$

From this the equivalence of principality for the L-modules V and V^* follows.

Theorem 2. *Let V be a principal irreducible L-module with a semi-invariant P. The following are equivalent.*

1. *$\mathcal{P}(L, V)=V-Z(P)$, the Zariski open complement of the zeros of P.*
2. *The isotropy subalgebra L_X of a principal generator X is a reductive Lie-algebra.*
3. *The mapping* $\operatorname{Grad} P: V \to V^*$ *sends principal generators in V to principal generators in V^*.*

As examples of the theorem we refer to $2b$) where each of the three statements is clearly true. More interesting is example iii. The point $U=(\text{col}\,(1,0,0,0), \text{col}\,(0,1,0,0), \text{col}\,(0,0,0,0))$ is in $V-Z(P)$ but U is not a principal

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generator. $X=(\text{col }(1,0,0,0), \text{ col }(0,1,0,0), \text{ col }(0,0,1,0))$ is a principal generator but L_X is not reductive. Define an isomorphism $\mathcal{P}: V^* \to k^{3\times 4}$ by requiring $Y(Z)=\text{trace}\left(\mathcal{P}(Y)\begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} Z\right)$ for all $Y \in V^*$ and all $Z \in V$. For $c \in k$, $A \in \mathfrak{sp}(4)$, $B \in \mathfrak{o}(3)$, we have the diagram

$$\begin{array}{ccccc} V^* & \xrightarrow{\ P\ } & k^{3\times 4} & & W \\ {\scriptstyle cI'V^*+T(A,B)}\downarrow & & \downarrow & & \downarrow \\ V^* & \longrightarrow & k^{3\times 4} & & -cW+BW-WA \end{array}$$

commuting. $((\text{Grad } P)(X))=(\text{col }(0,0,0), \text{ col }(0,0,-2), \text{ col }(2,0,0), \text{ col }(0,0,0))$ is not principal since $\hat{L}$ Grad $P(X)$ has dimension 9.

The proof of Theorem 2 follows the pattern of that of Theorem 1. The basic result used is a theorem of Mikio Sato, [5], [6], which can be formulated as

THEOREM. *Let K be algebraically closed of characteristic* 0 *with $G(K)=KId_V \cdot G'(K)$ a semi-direct product of algebraic groups. Assume that G has an open dense orbit in $V(K)$. The following conditions are equivalent.*

(1) *The open dense orbit is $V(K)-Z(P)(K)$ the Zariski open complement of the zeros of P, a semi-invariant for $G(K)$.*

(2) *For X in the open dense orbit of $G(K)$ in $V(K)$, $G(K)_X$, the isotropy subgroup of X in $G(K)$ is a reductive group.*

(3) *There exists a semi-invariant P for $G(K)$ such that* Grad $P: V(K)\to V(K)^*$ *is a dominant morphism.*

The hypothesis of Theorem 2 and the Lemma 2 above apply to give the equivalence of 1 and of (1). For $X \in V$, G_X has k-structure. $G_X(k)$ is reductive if and only if $G_X(K)$ is reductive, and L_X is reductive if and only if $G_X(k)$ is reductive. Hence 2 and (2) are equivalent. Finally, the morphism Grad P is compatible with the action of G, namely for all $g \in G(K)$, Grad $P \circ g = c_g I_{V^*} \circ g \cdot$ Grad P for some $c_g \in K^\cdot$; thus, G orbits in V are sent to G orbits in V^*. The proposition above gives that V^* is a principal irreducible L-module. Grad P sends principal generators of V to principal general generators of V^* if and only if the image of an element in the open dense G-orbit in V under Grad P is in the open dense G-orbit in V^*. This is equivalent to Grad P being dominant.

Theorems 1 and 2 and the Lemma 2 relating principality with the existence of a Zariski dense orbit in V under the action of the associated algebraic group $G \subseteq GL(V)$, make a classification or complete enumeration of these L-modules desirable. Listings of such L-modules have been started. A table appears in [7] from which the L-modules V with L' simple and the necessary (but not sufficient) condition $\dim L'+1 \geq \dim V$ may be written down when $k=C$. In [4], the consequent list of such modules appears with indication of

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those which are principal, those for which a semi-invariant exists together with information on $\mathcal{P}(L, V)$ when available. When k is algebraically closed, $L = kI_V \oplus L'$ where $L' = S_1 \oplus S_2 \oplus \cdots \oplus S_m$ a direct sum of semi-simple ideals S_i and $V = V(1) \otimes V(2) \otimes \cdots \otimes V(m)$ where $V(i)$ is an irreducible S_i-module and the action of S_j, $j \neq i$, on $V(i)$ is trivial. The necessary condition $\dim L \geq \dim V$ implies that

$$* \text{ for at least one } i = 1, 2, \ldots, m$$
$$\dim S_i + 1 \geq \dim V(i).$$

Denote $\dim V(i)$ by n_i for each $i = 1, 2, \ldots, m$; since $S_i \hookrightarrow \mathfrak{sl}(n_i, k)$, the boundary condition $\dim S_i \leq n_i^2 - 1$ for each i, translates to the "Diophantine" inequality

$$\prod_{i=1}^{m} n_i \leq 1 + \sum_{i=1}^{m} (n_i^2 - 1) = (1-m) + \sum_{i=1}^{m} n_i^2.$$

Hence, when $m \geq 3$, not all dimensions n_i can grow large simultaneously. However for any $m \geq 1$, we have the principal irreducible L-module $V \cong k^{2^m} \otimes k^2 \otimes k^2 \otimes \cdots \otimes k^2$ for $L = kI_V \oplus \mathfrak{sl}(2^m, k) \oplus \mathfrak{sl}(2, k) \oplus \cdots \oplus \mathfrak{sl}(2, k)$ with m factors k^2 and $\mathfrak{sl}(2, k)$. For this L-module a semi-invariant P, a determinant form, exists and Grad P is a dominant morphism. A method for "generating" all principal irreducible L-modules from the basic building blocks S_i, and $V(i)$ or of characterizing them from their highest weights relative to a Cartan Subalgabra is being sought. See [8].

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